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Decomposition of primes in number fields defined by trinomials.

par P. LLORENTE, E. NART AND N. VILA

Abstract — *In this paper we deal with the problem of finding the prime-ideal decomposition of a prime integer in a number field K defined by an irreducible trinomial of the type $X^{p^m} + AX + B \in \mathbb{Z}[X]$, in terms of A and B . We also compute effectively the discriminant of K .*

1. Introduction

Let K be the number field defined by an irreducible trinomial of the type :

$$X^{p^m} + AX + B, \quad A, B \in \mathbb{Z}, \quad p \text{ prime}, \quad m \geq 1.$$

In this paper we study the prime-ideal decomposition of the rational primes in K . Our results extend those of Vélez in [6], where he deals with the decomposition of p in the case $A = 0$. However, the methods are different, ours being based on Newton's polygon techniques. The results are essentially complete except for a few special cases which can be handled by an specific treatment (see section 2.3). This is done explicitly for $p^n = 4$ or 5, so that there are no exceptions at all for quartic and quintic trinomials.

Let us remark that the main aim of the paper is to give a complete answer in the case $p|A, p \nmid B$ (Theorems 3 and 4). The results concerning the other cases are easily obtained applying the ideas of [2], where we dealt with the computation of the discriminant of K , whereas the case $p|A, p \nmid B$ was not even considered. We give also the p -valuation of the discriminant of K in all cases including those not covered by [2].

2. Results

Let $K = \mathbb{Q}(\theta)$, where θ is a root of an irreducible polynomial of the type :

$$f(X) = X^n + AX + B,$$

where $n, A, B \in \mathbb{Z}, n > 3$. For the case $n = 3$ see [1]. Let us denote by d and

$$D = (-1)^{\frac{n(n-1)}{2}} (n^n B^{n-1} + (-1)^{n-1} (n-1)^{n-1} A^n),$$

the respective discriminants of K and θ . For simplicity we shall write in the sequel N for the ideal norm $N_{K/\mathbb{Q}}$.

For any prime $q \in \mathbb{Z}$ and integer $u \in \mathbb{Z}$ (or q -adic integer $u \in \mathbb{Z}_q$) we shall denote by $v_q(u)$ the greatest exponent s such that $q^s | u$ and we shall write $u_q := u/q^{v_q(u)}$.

It is well-known that we can assume that the conditions :

$$v_q(A) \geq n-1, \quad v_q(B) \geq n,$$

are not satisfied simultaneously for any prime integer q . We shall make this assumption throughout the paper.

Let $F(X) \in \mathbb{Z}[X]$ be a polynomial, $q \in \mathbb{Z}$ a prime integer and let

$$F(X) \equiv \Phi_1(X)^{e_1} \cdot \dots \cdot \Phi_s(X)^{e_s} \pmod{q},$$

be the decomposition of $F(X)$ as a product of irreducible factors $(\text{mod } q)$. An integer ideal $\mathfrak{a}$ of any number field L will be called “ q analogous to the polynomial $F(X)$ ” if the decomposition of $\mathfrak{a}$ into a product of prime ideals of L is of the type :

$$\mathfrak{a} = \mathfrak{q}_1^{e_1} \cdot \dots \cdot \mathfrak{q}_s^{e_s}, \quad N_{L/\mathbb{Q}}(\mathfrak{q}_i) = q^{\deg(\Phi_i(X))} \text{ for all } i.$$

2.1. Decomposition of the primes q not dividing n .

THEOREM 1. *Let $q \in \mathbb{Z}$ be a prime number such that $q \nmid n$. Let us denote $a = (n-1, v_q(A))$ and $b = (n, v_q(B))$. The decomposition of q into a product of prime ideals of K is as follows :*

If $v_q(B) > v_q(A)$ and $q \nmid a$,

$$(2.1.1) \quad q = \mathfrak{q} \mathfrak{a}^{(n-1)/a}, \quad N(\mathfrak{q}) = q, \quad \mathfrak{a} \text{ } q\text{-analogous to } X^a - A_q.$$

If $v_q(B) \leq v_q(A)$ and $v_q(A) > 0$,

$$(2.1.2) \quad q = \mathfrak{a}^{n/b}, \quad \mathfrak{a} \text{ } q\text{-analogous to } X^b - B_q.$$

If $q \nmid AB$ and $q|D$, the decomposition of $f(X)$ into a product of irreducible factors (mod q) is of the type :

$$(2.1.3) \quad f(X) \equiv (x - u)^2 \cdot \Phi_1(X) \cdot \dots \cdot \Phi_s(X) \pmod{q},$$

and we have

$$(2.1.4) \quad q = \mathfrak{q}_1 \cdot \dots \cdot \mathfrak{q}_s \cdot \mathfrak{a}, \quad N(\mathfrak{q}_i) = q^{\deg(\Phi_i(X))} \text{ for all } i, \quad N(\mathfrak{a}) = q^2,$$

where

$$\mathfrak{a} = \begin{cases} \mathfrak{q} \cdot \mathfrak{q}', & N(\mathfrak{q}) = N(\mathfrak{q}') = q, \text{ if } v_q(D) \text{ even and } \left(\frac{D_q}{q}\right) = (-1)^{n-s} \\ \mathfrak{q}, & N(\mathfrak{q}) = q^2, \text{ if } v_q(D) \text{ even and } \left(\frac{D_q}{q}\right) = (-1)^{n-s+1} \\ \mathfrak{q}^2, & N(\mathfrak{q}) = q, \text{ if } v_q(D) \text{ odd.} \end{cases}$$

If $q \nmid ABD$, q is q -analogous to $f(X)$. (2.1.5)

$$v_q(d) = \begin{cases} n - 1 - a + \inf\{(n-1)v_q(B) - nv_q(A), (n-1)v_q(n-1)\}, & \text{if } v_q(B) > v_q(A) \text{ and } q \nmid a, \\ n - b, & \text{if } v_q(B) \leq v_q(A) \text{ and } v_q(A) > 0, \\ 0, & \text{if } q \nmid AB \text{ and } v_q(D) \text{ even,} \\ 1, & \text{if } q \nmid AB \text{ and } v_q(D) \text{ odd.} \end{cases}$$

2.2. Decomposition of the primes p dividing n

THEOREM 2. If $p \nmid A$, then p is p -analogous to $f(X)$ and $v_p(d) = 0$.

If $v_p(B) > v_p(A) > 0$, then

$$p = \mathfrak{a}^{(n-1)/a} \mathfrak{p}, \quad \mathfrak{a} \text{ } p\text{-analogous to } X^a + A_p, \quad N(\mathfrak{p}) = p$$

and $v_p(d) = n - a - 1$, where we have denoted $a = (n-1, v_p(A))$.

If $0 < v_p(B) \leq v_p(A)$ and $p \nmid v_p(B)$,

$$p = \mathfrak{p}^n, \quad N(\mathfrak{p}) = p \quad \text{and} \quad v_p(d) = n - 1 + \inf\{nv_p(A) - (n-1)v_p(B), nm\}.$$

From now on we assume that $n = p^m > 3$ for some prime $p \in \mathbb{Z}$ and integer $m \geq 1$.

THEOREM 3. Suppose that $p > 2$, $p|A$ and $p \nmid B$. Let us denote :

$$r_0 = v_p(f(-B)), r_1 = v_p(f'(-B)), r = \inf\{m+1, r_1, r_0\}, s_0 = v_p(D) - mn ; \\ e = p^{m-r+1}, e_k = p^{m-k}(p-1), 1 \leq k < m, e_m = p-2 ; J = (n-e)/(p-1), \\ I = \frac{1}{2}(v_p(D) - v_p(d)).$$

Then we have :

$$(2.2.1) \quad p = \begin{cases} p_1^{e_1} \cdot \dots \cdot p_{r-1}^{e_{r-1}} \cdot a, & N(p_k) = p \text{ for all } k, \text{ if } r \leq m, \\ p_1^{e_1} \cdot \dots \cdot p_{m-1}^{e_{m-1}} \cdot b, & N(p_k) = p \text{ for all } k, \text{ if } r = m+1, \end{cases}$$

where

$$a = \begin{cases} p^e, & N(p) = p, \text{ if } r_0 \leq r_1, \end{cases} \quad (2.2.2)$$

$$p^{e-1} \cdot p', \quad N(p) = N(p') = p, \text{ if } r_0 > r_1. \quad (2.2.3)$$

If $p = 3$ and $s_0 \leq m+2$,

$$b = \begin{cases} p^3, & N(p) = 3, \text{ if } s_0 = m+1 \end{cases} \quad (2.2.4)$$

$$p, \quad N(p) = 27, \quad (2.2.5)$$

$$\text{if } s_0 = m+2 \text{ and } D_3 \equiv (-1)^{m-1} \pmod{3}$$

$$p \cdot p', \quad N(p) = 3, N(p') = 9, \quad (2.2.5)$$

$$\text{if } s_0 = m+2 \text{ and } D_3 \equiv (-1)^m \pmod{3}.$$

If $p > 3$ or $p = 3$ and $s_0 > m+2$,

$$b = \begin{cases} p_m^{e_m} \cdot p^2, & N(p_m) = N(p) = p, \text{ if } v_p(D) \text{ odd} \\ p_m^{e_m} \cdot p, & N(p_m) = N(p) = p^2, \text{ if } v_p(D) \text{ even} \\ \text{and } \left(\frac{(-1)^{\frac{n(n-1)}{2}} 2D_p}{p} \right) = -1 \\ p_m^{e_m} \cdot p \cdot p', & N(p_m) = N(p) = N(p') = p, \text{ otherwise} \end{cases} \quad (2.2.6)$$

Moreover $I = J$ in cases (2.2.2) and (2.2.4), $I = J+1$ in case (2.2.3) and $I = J + [(s_0 - m)/2] + 1$ in the rest of the cases.

THEOREM 4. Suppose that $2|A$, $2 \nmid B$ and let r_0, r_1, r, s_0, e, e_k ($1 \leq k < m$), J and I be as in Theorem 3. Let $u = [(s_0 - m + 1)/2]$. Then we have

$$(2.2.7) \quad 2 = \begin{cases} p_1^{e_1} \cdot \dots \cdot p_{r-2}^{e_{r-2}} \cdot a, & N(p_k) = 2 \text{ for all } k, \text{ if } r \leq m, \\ p_1^{e_1} \cdot \dots \cdot p_{m-2}^{e_{m-2}} \cdot b, & N(p_k) = 2 \text{ for all } k, \text{ if } r = m+1, \end{cases}$$

where

$$a = \begin{cases} p^e, N(p) = 4, & \text{if } r_0 \leq r_1 & (2.2.8) \\ p_{m-1}^{e_{m-1}} p, N(p_{m-1}) = 2, N(p) = 4, & \text{if } r_1 = m \text{ and } r_0 = m + 1 & (2.2.9) \\ p_{r-1}^{e_{r-1}} p^{e-1} p', N(p_{m-1}) = N(p) = N(p') = 2, & \text{otherwise} & (2.2.9) \end{cases}$$

$$b = \begin{cases} p^2, N(p) = 2, & \text{if } v_2(D) - m \text{ even or} \\ & D_2 \equiv 1 + 2^n \pmod{4} & (2.2.10) \\ p, N(p) = 4, & \text{if } v_2(D) - m \text{ odd and} \\ & D_2 \equiv 3 + n^n + 2^{n^2} \pmod{8} & (2.2.11) \\ p \cdot p', N(p) = N(p') = 2, & \text{if } v_2(D) - m \text{ odd and} \\ & D_2 \equiv 7 + n^n + 2^{n^2} \pmod{8} & (2.2.11) \end{cases}$$

Moreover $I = J$ in cases (2.2.8), $I = J + 1$ in cases (2.2.9), $I = J + u - 1$ in cases (2.2.10) and $I = J + u$ in cases (2.2.11).

2.3. Quartic and quintic trinomials

In this section we complete the general theorems above in the cases $n = 4$ and 5. Let $n = p^m$. Theorems 2, 3 and 4 give the decomposition of p in all cases except for the following :

$$(2.3.1) \quad p | v_p(B) \text{ and } 0 < v_p(B) \leq v_p(A).$$

For the primes $q \neq p$ the only case not covered by Theorem 1 is :

$$(2.3.2) \quad q | (n - 1, v_q(A)) \text{ and } 0 < v_q(A) < v_q(B).$$

Equations satisfying (2.3.1) or (2.3.2) can be handled by an specific treatment but the results are too disperse to fit them into a reasonable theorem. For instance, for $n = 4$, (2.3.2) is not possible and (2.3.1) occurs only for $p = 2$ and equations :

$$(2.3.3) \quad X^4 + 2^{2+e}AX + 2^2B, \quad 2 \nmid AB, \quad e \geq 0.$$

For $n = 5$, (2.3.1) is not possible and (2.3.2) occurs only for $q = 2$ and equations :

$$(2.3.4) \quad X^5 + 2^2BX + 2^{3+e}C, \quad 2 \nmid BC, \quad e \geq 0.$$

THEOREM 5. *The decomposition of 2 in the number field defined by (2.3.3) or (2.3.4) is*

$$2 = \begin{cases} \mathfrak{a}, & \text{if } n = 4, \\ \tau \mathfrak{a}, & N(\tau) = 2, \tau \nmid \mathfrak{a}, \text{ if } n = 5, \end{cases}$$

where $\mathfrak{a}$ is an integer ideal having the following decomposition :

$$\mathfrak{a} = \mathfrak{p}^4, \quad \text{if } e = 0 \text{ or } 1.$$

For $e \geq 2$ and $B \equiv 1 \pmod{4}$:

$$\mathfrak{a} = \begin{cases} \mathfrak{p}^4, & \text{if } e = 2, B \equiv 1 \pmod{8} \text{ or } e \geq 3, B \equiv 5 \pmod{8}, \\ \mathfrak{p}^2 \mathfrak{p}_1^2, & \text{if } e = 2, B \equiv 13 \pmod{16} \text{ or } e \geq 3, B \equiv 1 \pmod{16}, \\ \mathfrak{p}_2^2, & \text{if } e = 2, B \equiv 5 \pmod{16} \text{ or } e \geq 3, B \equiv 9 \pmod{16}. \end{cases} \quad (2.3.5)$$

Whereas for $e \geq 2$ and $B \equiv 3 \pmod{4}$:

$$\mathfrak{a} = \begin{cases} \mathfrak{p}^2 \mathfrak{p}_1^2, & \text{if } B \equiv 7 \pmod{8}, \\ \mathfrak{p}_2^2, & \text{if } B \equiv 3 \pmod{8}. \end{cases}$$

In all cases $N(\mathfrak{p}) = N(\mathfrak{p}_1) = 2$ and $N(\mathfrak{p}_2) = 4$. Moreover, $v_2(d) = 4$ when $e = 0$ and in the cases (2.3.5), (2.3.6) and $v_2(d) = 6$ in the rest of the cases.

3. Proofs

The proofs of the Theorems of Section 2 are essentially based on an old technique developed by Ore concerning Newton's polygon of the trinomial $f(X)$ (cf. [3] and [4]). For commodity of the reader we sum up the results we need of [3] and [4] in Theorem 6 below.

We recall first some definitions about Newton's polygon. Let $F(X) = X^n + a_1 X^{n-1} + \dots + a_n \in \mathbb{Z}[X]$ and $p \in \mathbb{Z}$ be a prime number. The lower convex envelope Γ of the set of points $\{(i, v_p(a_i)), 0 \leq i \leq n\}$ ($a_0 = 1$) in the euclidean 2-space determines the so-called "Newton's polygon of $F(X)$ with respect to p ". Let $S_1, \dots, S_t$ be the sides of the polygon and ℓ_i, h_i the lenght of the projections of S_i to the X -axis and Y -axis respectively. Let $\varepsilon_i = (\ell_i, h_i)$ and $\ell_i = \varepsilon_i \cdot \lambda_i$ for all i . If S_i begins at the point $(s, v_p(a_s))$ let $s_j = s + j\lambda_i$ and :

$$b_j = \begin{cases} (a_{s_j})_p & \text{if the point } (s_j, v_p(a_{s_j})) \text{ belongs to } S_i, \\ 0 & \text{otherwise,} \end{cases}$$

for all $0 \leq j \leq \varepsilon_i$. The polynomial :

$$F_i(Y) = b_0 Y^{\varepsilon_i} + b_1 Y^{\varepsilon_i-1} + \dots + b_{\varepsilon_i},$$

is called the “associated polynomial of S_i ”. We define $F(X)$ to be “ S_i -regular” if p does not divide the discriminant of $F_i(Y)$. $F(X)$ will be called “ Γ -regular” if it is S_i -regular for all i .

THEOREM 6. (Ore [4], Theorems 6 and 8). *Let $F(X) \in \mathbb{Z}[X]$ be a monic irreducible polynomial and let $L = \mathbb{Q}(\alpha)$, α a root of $F(X)$. Let $p \in \mathbb{Z}$ be a prime ; with the above notations about Newton’s polygon Γ of $F(X)$ with respecto to p , we have the following decomposition of p into a product of integer ideals of L :*

$$p = \mathfrak{a}_1^{\lambda_1} \cdot \dots \cdot \mathfrak{a}_t^{\lambda_t}.$$

For each i , the ideal $\mathfrak{a}_i$ is p -analogous to $F_i(Y)$ if $F(X)$ is S_i -regular. Moreover, if $F(X)$ is Γ -regular we have :

$$v_p(i(\alpha)) = \sum_{i=2}^t \ell_i \left(\sum_{j=1}^{i-1} h_j \right) + \frac{1}{2} \sum_{i=1}^t (\ell_i h_i - \ell_i - h_i + \varepsilon_i),$$

where $i(\alpha)$ denotes the index of α . This expression for $v_p(i(\alpha))$ also coincides with the number of points with integer coordinates below the polygon except for the points on the X -axis and on the last ordinate.

For the proof of theorem 1 we need a well-known lemma (cf.[5]) :

LEMMA 1. *Let L be a number field of degree $[L : \mathbb{Q}] = n$. Let q be a prime integer unramified in L and let s be the number of prime ideals of L lying over q . Then, the discriminant d of L satisfies*

$$\left(\frac{d}{q} \right) = (-1)^{n-s}.$$

Proof of theorem 1. The assertions (2.1.1) and (2.1.2) are a straightforward application of Theorem 6. For (2.1.3) see the proof of [2 Theorem 2]. (2.1.4) is consequence of Lemma 1 and the fact that in this case $v_q(d) = 1$ if q ramifies [2, Theorem 2]. (2.1.5) is obvious and the assertions concerning the computation of $v_q(d)$ are contained in [2, Theorem 1].

Theorem 2 follows from Theorem 6 and [2, Theorem 1]. We shall deal with the proof of Theorems 3 and 4 altogether. The proof of Theorem 5 is similar to those of the general theorems.

Proof of Theorem 3 and 4. Since $p|A$ and $p \nmid B$, we have $f(X) \equiv (X+B)^n \pmod{p}$. Let Γ be the Newton's polygon of the polynomial :

$$F(X) := f(X-B) = \sum_{i=0}^n A_i X^{n-i},$$

where $A_0 = 1$, $A_i = \binom{n}{i} (-B)^i$ for $1 \leq i \leq n-2$, $A_{n-1} = f'(-B)$ and $A_n = f(-B)$.

It is easy to see that :

$$(3.2.1) \quad v_p(A_i) = v_p\left(\binom{n}{i}\right) = m - v_p(i), \quad 1 \leq i \leq n-2.$$

Let us determine first which would be the partial shape of Γ if the two final points $(n-1, r_1), (n, r_0)$ were omitted. By (3.2.1) we find that in that case Γ would have $m-1$ sides $S_1, \dots, S_{m-1}$ if $p=2$ and one more side S_m if $p>2$, each side S_k ending at the point (e_k, k) (see figure 1). In fact, $i = e_k$ is the greatest subindex with $v_p(A_i) = k$ and the slope of S_k is $1/e_k$ so that these slopes are strictly increasing. Now, when we consider the two final points of Γ we find that we can always assure that Γ contains the sides $S_1, \dots, S_{m-1}$ if $r > m$, the sides $S_1, \dots, S_{r-1}$ if $r \leq m$ and $p > 2$, and the sides $S_1, \dots, S_{r-2}$ if $r \leq m$ and $p = 2$.

Figure 1

Let Γ' denote, in each case, the rest of the sides of Γ . By Theorem 6, the assertions (2.2.1) and (2.2.7) are proved. In order to find the further decomposition of the respective ideals $\mathfrak{a}$ and $\mathfrak{b}$ of Theorem 3 and 4 we shall study the shape and associated polynomials of Γ' . We must distinguish several cases. Before, note that for each $1 \leq k \leq m$, the number of points with integer coordinates below the sides $S_1 \cup \dots \cup S_k$ except for the points on the X -axis and on the last ordinate is

$$I_k = p^{m-k} \left(\frac{p^k - 1}{p - 1} - k \right) \quad \text{for } 1 \leq k < m,$$

and

$$I_m = \frac{n-1}{p-1} - 2m + 1.$$

Case $r \leq m, r_0 \leq r_1$: Γ' has only one side with lengths of the projections to the axis : $\ell = p^{m-r_0+1} = e$, $h = 1$ if $p > 2$ and $\ell = 2e$, $h = 2$ if $p = 2$ (see fig. 2). Therefore $\varepsilon := (\ell, h) = 1$ or 2 according to $p > 2$ or $p = 2$. In the latter case the associated polynomial is congruent (mod 2) to $Y^2 + Y + 1$, which is irreducible. By Theorem 6, (2.2.2) and (2.2.8) are proved. Since $F(X)$ is Γ -regular we have :

$$I = I_{r-1} + e(r-1) \quad \text{if } p > 2,$$

$$I = I_{r-2} + e(2r-3) \quad \text{if } p = 2,$$

hence, $I = J$ in both cases, as desired.

Figure 2

Case $r \leq m, r_0 > r_1$: If $p > 2$, Γ' has two sides S, S' with projections to the axis $\ell = e - 1, h = 1$ and $\ell' = 1, h' = r_0 - r_1$ respectively (see fig. 3). If $p = 2$, Γ' contains the side S_{r-1} and two more sides with the same dimensions of S and S' above, except for the case $r_1 = m, r_0 = m + 1$ in which besides S_{m-1} there is only one side with projections to the axis $\ell = h = 2$ and associated polynomial congruent (mod 2) to $Y^2 + Y + 1$, which is irreducible (see fig. 3). By Theorem 6, (2.2.3) and (2.2.9) are proved. Since $F(X)$ is Γ -regular in any case, we have :

$$\begin{aligned} I &= I_{m-1} + 2m - 1 && \text{if } p = 2, r_1 = m \text{ and } r_0 = m + 1, \\ I &= I_{r-1} + e(r - 1) + 1 && \text{otherwise,} \end{aligned}$$

hence $I = J + 1$ in both cases, as desired.

Figure 3

This ends the discussion of the case $r \leq m$.

Assume from now on that $r = m + 1$. If we study Γ' in this case as above, we are led to many p -irregular cases. For this reason, instead of the polynomial $f(X - B)$ we seek an opportune substitute providing a much more regular situation.

Since $r_1 = v_p(n(-B)^{n-1} + A) > m$, we have :

$$v_p(A) = m \quad \text{and} \quad A_p \equiv -1 \pmod{p}.$$

Thus, from $r_0 = v_p((-B)^{n-1} + A - 1) > m$, we get :

$$(3.2.2.) \quad (-B)^{n-1} \equiv 1 + p^m \pmod{p^{m+1}}.$$

Let $\beta = -nB/(n-1)A$. Since $v_p(\beta) = 0$, β is a p -adic integer and it is clear that Theorem 6 is also applicable to the polynomial :

$$G(X) := f(X + \beta) = \sum_{i=0}^{n-2} \binom{n}{i} \beta^i X^{n-i} + f'(\beta)X + f(\beta).$$

Computation leads to :

$$f(\beta) = (-1)^{\frac{n(n+1)}{2}} \frac{BD}{(n-1)^n A^n}, f'(\beta) = (-1)^{\frac{n(n+1)}{2}-1} \frac{D}{(n-1)^{n-1} A^{n-1}},$$

hence, $s_0 := v_p(f(\beta)) = v_p(D) - nm$ and $s_1 := v_p(f'(\beta)) = s_0 + m$. It is easy to check that :

$$A_p^n \equiv (-1)^n \pmod{p^{m+1}} \text{ and } (n-1)^{n-1} \equiv (-1)^{n-1}(1+n) \pmod{p^{m+1}},$$

hence, by (3.2.2) :

$$\frac{(-1)^{\frac{n(n-1)}{2}} D}{n^n} = B^{n-1} + (-1)^{n-1} (n-1)^{n-1} A_p^n \equiv 0 \pmod{p^{m+1}},$$

so that $s_0 = v_p(D/n^n) > m$. Thus, Newton's polygon Γ_β of $G(X)$ with respect to p can be also expressed as :

$$\Gamma_\beta = S_1 \cup \dots \cup S_{m-1} \cup \Gamma'_\beta,$$

and we need only to study Γ'_β in order to find the prime-ideal decomposition of the respective ideals $\mathfrak{b}$ of Theorems 3 and 4. Again, we have to distinguish several cases :

Case $r = m + 1, p > 3$ or $p = 3$ and $s_0 > m + 2$: Γ'_β contains S_m and one more side of dimensions $\ell = 2, h = s_0 - m$ (see fig. 4). For this latter side, $\varepsilon = (\ell, h) = 1$ or 2 according to $s_0 - m$ odd or even. In the latter case the associated polynomial is :

$$\begin{aligned} \frac{n-1}{2} \beta^{n-2} Y^2 + \frac{f(\beta)}{p^{s_0}} \\ \equiv \frac{B^{n-2}}{2} Y^2 + (-1)^{\frac{n(n+1)}{2}} BD_p \pmod{p}, \end{aligned}$$

and its discriminant is congruent to $(-1)^{n(n-1)/2} 2D_p$. Since $v_p(D) \equiv s_0 - m \pmod{2}$, (2.2.6) is proved by Theorem 6. Moreover, since we are in a regular case we have :

$$I = I_m + 2m - 1 + \frac{s_0 - m + \varepsilon}{2} = J + \left[\frac{s_0 - m}{2} \right] + 1,$$

as desired.

Figure 4

Case $r = m + 1$, $p = 3$ and $s_0 \leq m + 2$: Γ'_β has only one side with $\ell = 3$ and $h = 2$ or 3 according to $s_0 = m + 1$ or $m + 2$ (see fig. 4). In the latter case $\varepsilon = 3$ and the associated polynomials is

$$\begin{aligned} & \frac{(n-1)(n-2)}{2} \beta^{n-3} Y^3 + \frac{n-1}{2} \beta^{n-2} Y^2 + \frac{f(\beta)}{3^{s_0}} \\ & \equiv B^{n-3} Y^3 - B^{n-2} + (-1)^{m-1} B D_3 \pmod{3}. \end{aligned}$$

Since $(-1)^{n(n+1)/2} = (-1)^{m-1}$ in this case, multiplying by B^2 we get the polynomial $\Phi(Y) = Y^3 - B Y^2 + (-1)^{m-1} B D_3$, which is irreducible $\pmod{3}$ if $D_3 \equiv (-1)^{m-1} \pmod{3}$ and factorizes :

$$\phi(Y) \equiv (Y + B)(Y^2 + B Y - 1) \pmod{3},$$

if $D_3 \equiv (-1)^m \pmod{3}$. By Theorem 6, (2.2.5) is proved. Since we are in a regular case we have :

$$I = I_{m-1} + 3m - 2 = J \quad \text{if } s_0 = m + 1,$$

$$I = I_{m-1} + 3m = J + 2 \quad \text{if } s_0 = m + 2.$$

Case $r = m + 1, p = 2$: Γ'_β has only one side with $\ell = 2$ and $h = s_0 - m + 1$ (see fig.5), hence $\varepsilon = 1$ or 2 according to $s_0 - m + 1$ odd or even, or equivalently according to $v_2(D) - m$ even or odd. In the latter case, the associated polynomial is congruent $\pmod{2}$ to $Y^2 + 1$, hence, it is an irregular case. In the former case Theorem 6 proves (2.2.10) and :

$$I = I_{m-1} + 2m - 2 + \frac{s_0 - m}{2} = J + u - 1,$$

as desired.

Figure 5

Finally, in order to deal with the case $v_2(D) - m$ odd it is necessary to change again Newton's polygon. Let $2u = s_0 - m + 1$ and $\delta = (2^u - B)/(n-1)A_2$. Computation leads to :

$$(3.2.3) \quad (n-1)^n A_2^n f(\delta) = \sum_{i=0}^{n-2} \binom{n}{i} 2^{(n-i)u} (-B)^i + (B - 2^{u+m}) D_0,$$

where $D_0 = D/n^n = B^{n-1} - (n-1)^{n-1}A_2^n$. Since $v_2(D_0) = s_0 = 2u + m - 1 > m, u > 0$ and there are exactly two summands in (3.2.3) with v_2 minimum and equal to $2u + m - 1$, hence, $v_2(f(\delta)) \geq 2u + m$. From the relation :

$$nf(X) - Xf'(X) = (n-1)AX + nB,$$

and being $v_2((n-1)A\delta + nB) = u + m$, we conclude that $v_2(f'(\delta)) = u + m$. Thus Newton's polygon Γ_δ with respect to p of the polynomial $f(X + \delta)$ is again expressible as : $\Gamma_\delta = S_1 \cup \dots \cup S_{n-1} \cup \Gamma'_\delta$. We have now three possibilities (see fig.5) :

- a) $v_2(f(\delta)) = 2u + m$. Γ'_δ has only one side with $\ell = 2, h = 2u + 1$ hence $\varepsilon = (\ell, h) = 1$ and $\mathfrak{a} = \mathfrak{p}^2, N(\mathfrak{p}) = 2$. Moreover $I = I_{m-1} + 2(m-1) + u = J + u - 1$.
- b) $v_2(f(\delta)) = 2u + m + 1$. Γ'_δ has only one side with associated polynomial congruent (mod 2) to $Y^2 + Y + 1$, which is irreducible, hence $\mathfrak{a} = \mathfrak{p}, N(\mathfrak{p}) = 4$. Moreover $I = I_{m-1} + 2(m-1) + u + 1 = J + u$.
- c) $v_2(f(\delta)) > 2u + m + 1$. Γ'_δ has two sides and $\mathfrak{a} = \mathfrak{p}.\mathfrak{p}', N(\mathfrak{p}) = N(\mathfrak{p}') = 2, I = J + u$ like in case b).

Taking congruence (mod 2^{2u+m+2}) of (3.2.3) we shall be able to decide in which case falls our polynomial. All summands of (3.2.3) vanish (mod 2^{2u+m+2}) except for the following :

$$\binom{n}{4} 2^{4u} (-B)^{n-4} + \binom{n}{3} 2^{3u} (-B)^{n-3} + \binom{n}{2} 2^{2u} (-B)^{n-2} + BD_0.$$

Dividing by 2^{2u+m+1} and taking congruence (mod 8) we obtain :

$$(3.2.4) \quad 2^{2u+m+1} - 2^{2u-1} + 2^{u+1} + 2^m - 1 + BD_2 \pmod{8}$$

From (3.2.2) we get $B \equiv -1 + 2^m \pmod{2^{m+1}}$, hence (3.2.4) is equal to :

$$2^{2u+m-2} - 2^{2u-1} + 2^{u+1} - 1 - D_2 \pmod{8}$$

which is equal to $-1 - D_2 \pmod{8}$ if $u > 1$ and to $2^m + 1 - D_2$ if $u = 1$. Therefore cases a) b) and c) are equivalent to the following respective conditions :

$$\begin{aligned} a) &\Leftrightarrow \begin{cases} D_2 \equiv 1 \pmod{4} & \text{if } u > 1 \\ D_2 \equiv -1 \pmod{4} & \text{if } u = 1 \end{cases} \\ b) &\Leftrightarrow \begin{cases} D_2 \equiv 3 \pmod{8} & \text{if } u > 1 \\ D_2 \equiv 5 + n \pmod{8} & \text{if } u = 1 \end{cases} \\ c) &\Leftrightarrow \begin{cases} D_2 \equiv -1 \pmod{8} & \text{if } u > 1 \\ D_2 \equiv 1 + n \pmod{8} & \text{if } u = 1 \end{cases} \end{aligned}$$

This ends the proof of (2.2.10) and (2.2.11).

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