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The genus class group I.

par A. Fröhlich

Introduction.

This is something of a variation on a theme of S. Wilson (cf. [3], [4]). Applications to Galois module structure force on us a generalisation of the theory of class groups of orders. Both for additive structure, as in Wilson's work, and more so for multiplicative structure the relevant Galois objects are no longer necessarily locally free over the given order, nor-in the miltiplicative case - even acted on faithfully. Indeed demanding faithful action would be too restricitve. An important concept is now the genus classgroup and our purpose here is to develop a direct "Hom-language" approach to it. This is all the more relevant because of the connection which arises with the new concept of factorisability.

Our approach also throws new light on the foundations of the theory of the locally free classgroup, clarifying the customary dichotomy between the "additive" language of K-theory and the "multiplicative" language of Galois homomorphisms.

Every genus classgroup is isomorphic to a locally free classgroup of some order of endomorphisms. We make heavy use of this for a considerable simplification of proofs. This fact however is almost irrelevant for applications.

The Hom description of the classgroup is naturally forced on us by the prospective applications, relating modules to arithmetic character invariants. As already in the locally free case this Hom language is also the most convenient for the formulation of functorial properties, e.g. on change of genus. We shall come back to this problem in a subsequent paper.

As usual the symbols $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$ stand for the natural numbers, the integers and the rational numbers. R^* is the multiplicative group of invertible elements of a ring R.

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1. Some axiomatics.

We are considering quadruplets consisting of

(i) an additive Abelian semigroup $\mathcal{L}$;

(ii) a surjective homomorphim $r : \mathcal{L} \to \mathbf{N}$ (the positive integers);

(iii) a multiplicative Abelian group $\mathcal{C}$;

(iv) a map $\mathcal{C} \times \mathcal{L} \to \mathcal{L}$, written as $(c, x) \mapsto cx$.

These object are to satisfy the following axioms.

(1.1.a) $1x = x$ for $x \in \mathcal{L}, 1$ the identity of $\mathcal{C}$;

(1.1.b) $(c_1c_2)x = c_1(c_2x)$ for $c_1,\ c_2 \in \mathcal{C},\ x \in \mathcal{L}$;

(1.1.c) $r(cx) = r(x)$ for $c \in \mathcal{C},\ x \in \mathcal{L}$;

(1.1.d) $(c_1x_1) + (c_2x_2) = (c_1c_2)(x_1 + x_2)$ for $c_1,\ c_2 \in \mathcal{C},\ x_1,\ x_2 \in \mathcal{L}$.

Write $\mathcal{K}_0(\mathcal{L})$ for the Grothendieck group of $\mathcal{L}$, *i.e.* the universal object for homomorphims $\mathcal{L} \to$ Abelian groups. The rank map r extends to a homomorphism $\mathcal{K}_0(\mathcal{L}) \to \mathbf{Z}$, whose kernel we denote by $\tilde{\mathcal{K}}_0(\mathcal{L})$.

Write $\overline{\mathcal{L}}$ for the image of $\mathcal{L}$ in $\mathcal{K}_0(\mathcal{L})$, and $x \mapsto \overline{x}$ for the map $\mathcal{L} \to \overline{\mathcal{L}}$.

(1.2). *If $\overline{x} = \overline{y}$ $(x, y \in \mathcal{L})$ and $c \in \mathcal{C}$ then $\overline{cx} = \overline{cy}$.*

Proof : The hypothesis is $x + z = y + z$ for some $z \in \mathcal{L}$. But then by (1.1.a) and (1.1.d), $cx + z = c(x + z) = c(y + z) = cy + z, i.e. \overline{cx} = \overline{cy}$. ■

Using (1.2) we can define $\mathcal{C} \times \overline{\mathcal{L}} \to \overline{\mathcal{L}}$ by $\overline{cx} = c\overline{x}$. We thus get a new quadurplet satisfying our axioms with $\mathcal{L}$ replaced by $\overline{\mathcal{L}}$.

By (1.1.c) $\mathcal{C}$ acts on each of the inverse image sets $r^{-1}(n)$ in $\mathcal{L}$, n any positive integer.

(1.3). *If $\mathcal{C}$ acts fixed point free and transitively on each $r^{-1}(n)$ then $\mathcal{L} = \overline{\mathcal{L}}$, i.e. $\mathcal{L}$ admits the cancellation law.*

Proof : Suppose $\overline{x} = \overline{y}, i.e.\ x + z = y + z$. Then $r(x) = r(y)$, so by transitivity $y = cx$ for some $c \in \mathcal{C}$. Thus $x + z = cx + z = c(x + z)$ by (1.1.d). As action is fixed point free we have $c = 1$, *i.e.* $x = y$. ■

We now define a homomorphism

$$(1.4) \qquad \theta : \mathcal{C} \to \tilde{\mathcal{K}}_0(\mathcal{L})$$

of Abelian groups, which plays a central rôle. By (1.1.c) $c\overline{x} - \overline{x} \in \tilde{\mathcal{K}}_0(\mathcal{L})$. By (1.1.d) $c\overline{x} - \overline{x} = c\overline{y} - \overline{y}$ for all $x, y \in \mathcal{L}$. So the element

$$(1.4.a) \qquad \theta(c) = c\overline{x} - \overline{x}$$

of $\tilde{\mathcal{K}}_0(\mathcal{L})$ does not depend on x at all. Finally applying (1.1.b) and (1.1.d) once more we verify that $\theta(c_1c_2) = \theta(c_1) + \theta(c_2)$. Trivially $\theta(1) = 0$.

(1.5). *The following conditions are equivalent.*

(a) θ is injective;

(b) $\mathcal{C}$ acts fixed point free on $\overline{\mathcal{L}}$;

(c) $\mathcal{C}$ acts faithfully on $\overline{\mathcal{L}}$.

Proof : $(b) \Rightarrow (c)$ trivially. But if $c\overline{x_0} = \overline{x_0}$ for some x_0 then by (1.1.d) $c\overline{x} = \overline{x}$ for all x. Thus $(c) \Rightarrow (b)$. Trivially $(a) \Longleftrightarrow (c)$. ■

We call an element x_0 of $\overline{\mathcal{L}}$ of rank $r(x_0) \doteq 1$ a generator of $\overline{\mathcal{L}}$ if every $x \in \overline{\mathcal{L}}$ is of from $x = cx_0 + mx_0$, $m \in \mathbf{Z}$, $m \geq 0$.

(1.6). *The following conditions are equivalent*

(a) θ is surjective;

(b) $\mathcal{C}$ acts transitively on each set $r^{-1}(n) \in \overline{\mathcal{L}}$ $(n \geq 0)$;

(c) $\overline{\mathcal{L}}$ has a generator;

(d) Each $x_0 \in r^{-1}(1) \subset \overline{\mathcal{L}}$ is a generator of $\overline{\mathcal{L}}$.

Proof : $(b) \Rightarrow (a)$. A typical element of $\tilde{\mathcal{K}}_0(\mathcal{L})$ is of form $\overline{y} - \overline{x}$ where $r(\overline{y}) = r(\overline{x})$. By (b) $\overline{y} = c\overline{x}$, so $\overline{y} - \overline{x} = \theta(c)$.

$(a) \Rightarrow (b)$. Let $\overline{x}, \overline{y} \in r^{-1}(n)$. By (a) $\overline{x} - \overline{y} = \overline{z} - c\overline{z}$ for some z. By (1.1.d) $\overline{z} - c\overline{z} = \overline{x} - c\overline{x}$, whence $\overline{y} = c\overline{x}$.

$(b) \Rightarrow (d)$. Let $r(\overline{y}) = n$. As $r(n\overline{x_0}) = n$ we have, by (b), $\overline{y} = c(n\overline{x_0}) = (n-1)\overline{x_0} + c\overline{x}_0$ by (1.1.d).

$(d) \Rightarrow (c)$ is trivial, and so is $(c) \Rightarrow (b)$.

2. The Grothendieck group and the class group of a genus.

Let F be an algebraic number field (always of finite degree over $\mathbf{Q}$ and contained in a fixed algebraic closure $\mathbf{Q}^c$ of $\mathbf{Q}$). Denote by $\mathcal{O}_F$ or just $\mathcal{O}$ its ring of algebraic integers. Let furthermore B be a finite dimensional semisimple F-algebra and $\mathfrak{B}$ be an $\mathcal{O}$-order spanning B. We consider $\mathfrak{B}$-lattices, *i.e.* finitely generated (right) $\mathfrak{B}$-modules, torsion-free over $\mathcal{O}$. Recall that two such lattices L and M belong to the same genus if for all finite places $\wp$ of F there is an isomorphism of $\mathfrak{B}_\wp$-modules

$$L_\wp \cong M_\wp. \tag{2.1}$$

Here the subscript denotes completion at $\wp$.

Given (2.1) we have for all $\wp$ an isomorphism of $B_\wp$-modules

$$(L \otimes_{\mathcal{O}} F)_\wp \cong (M \otimes_{\mathcal{O}} F)_\wp$$

and hence an isomorphism of B-modules

$$L \otimes_{\mathcal{O}} F \cong M \otimes_{\mathcal{O}} F.$$

Thus given a genus $\mathcal{G}$ we have an associated B-module

$$W = W(\mathcal{G})$$

unique to within isomorphism, such that for all $L \in \mathcal{G}$

$$L \otimes_{\mathcal{O}} F \cong W(\mathcal{G}). \tag{2.2}$$

Fix $\mathcal{G}$ from now on. For $L \in \mathcal{G}$ and $n \in \mathbb{N}$ (*i.e.* $n = 1, 2, \ldots$), write $\mathcal{G}^n$ for the genus of L^n (product of n copies of L). Clearly $\mathcal{G}^n$ does not depend on the choice of L within $\mathcal{G}$. Now let

$$\tilde{\mathcal{G}} = \cup_{n \geq 1} \mathcal{G}^n.$$

Each $M \in \tilde{\mathcal{G}}$ determines a unique $\mathcal{G}^n$, and we put

$$n = r(M) \tag{2.3}$$

the *rank* of M.

$r(M)$ can be described in terms of $W(\mathcal{G}) = W$: there is an isomorphism of B-modules

$$M \otimes_{\mathcal{O}} F \cong W^n. \tag{2.4}$$

We now get a Grothendieck group which by abuse of language we call the Grothendieck group of $\mathcal{G}$ and denote by $\mathcal{K}_0(\mathcal{G})$. It is the Abelian group generated by elements $[M]$ $(M \in \tilde{\mathcal{G}})$, one for each isomorphism class, with relations given by direct sums. The rank map r yields a surjection onto $\mathbf{Z}$; we denote its kernel by $\tilde{\mathcal{K}}_0(\mathcal{G})$. The subsemigroup of $\mathcal{K}_0(\mathcal{G})$ of elements $[M]$ will be denoted by $\mathcal{L}(\mathcal{G})$.

By definition any finite direct sum of modules in $\mathcal{G}$ defines an element of $\tilde{\mathcal{G}}$. The converse is true as well: every module M in $\tilde{\mathcal{G}}$ is a finite direct sum of modules in $\mathcal{G}$. All we have to do is to show that if $r(M) > 1$ then

$$M \cong L \oplus N \quad \text{with } L \in \mathcal{G}. \tag{2.5}$$

Indeed by Krull-Schmidt $N \in \mathcal{G}^{r(M)-1}$. To establish (2.5) let S be a finite, nonempty set of finite places of F, including those for which $\mathfrak{B}_\wp$ is not a maximal order. For each $\wp \in S$ let $L_\wp \subset W_\wp$ be a $\mathfrak{B}_\wp$-lattice in the local isomorphism class associated with $\mathcal{G}$. There is then a splitting surjection $g(\wp) : M_\wp \to L_\wp$ of $\mathfrak{B}_\wp$-lattices, *i.e.* the map

$$\mathrm{Hom}_{\mathfrak{B}_\wp}(L_\wp, M_\wp) \to \mathrm{Hom}_{\mathfrak{B}_\wp}(L_\wp, L_\wp) \tag{2.6}$$

induced by $g(\wp)$ is surjective. By weak approximation there is a surjective homomorphism $f : M \to L$ of $\mathfrak{B}$-modules, with L spanning W, such that for all $\wp \in S$

$$\mathrm{Im} f_\wp = L_\wp \tag{2.7}$$

and that the maps (2.6) induced by $f_\wp$ are still surjective. For any $\wp \notin S$ the Im $f_\wp$ in (2.7) are still the local components of a module in $\mathcal{G}$ and are projective $\mathfrak{B}_\wp$-modules, whence (2.6) is still surjective. Therefore the maps

$$\mathrm{Hom}_{\mathfrak{B}}(L, M)_\wp \to \mathrm{Hom}_{\mathfrak{B}}(L, L)_\wp$$

are surjective for all $\wp$, *i.e.*

$$\mathrm{Hom}_{\mathfrak{B}}(L, M) \to \mathrm{Hom}_{\mathfrak{B}}(L, L)$$

is surjective, *i.e.* $f : M \to L$ splits, which yields (2.5).

Now we are going to prepare the ground for the definition of the classgroup of $\mathcal{G}$. For any F-vector space V we denote by $V^c = V \otimes_F \mathbf{Q}^c$ its extension to the algebraic closure $\mathbf{Q}^c$, assuming always that V^c inherits any additional structure from V. The Grothendieck group $\mathcal{K}_0(B^c)$ carries a symmetric bilinear map into $\mathbf{Z}$, given by

$$< U, U' > = \dim_{\mathbf{Q}^c} \mathrm{Hom}_{B^c}(U, U').$$

Write $(W^c)^\perp$ for the subgroup of $\mathcal{K}_0(B^c)$ generated by the simple B^c-modules orthogonal to W^c. This subgroup is in general non-zero, as we do not assume W to be a faithful B-module. We let $\mathcal{K}_0(B^c, W)$ be the subgroup of $\mathcal{K}_0(B^c)$ generated by the classes of those simple B^c-modules which occur in W^c. Then

$$\mathcal{K}_0(B^c) = \mathcal{K}_0(B^c, W) \oplus (W^c)^\perp. \tag{2.8}$$

We define the idele group $\mathcal{J}(\mathbf{Q}^c)$ in the usual way as the direct limit of the idele groups $\mathcal{J}(E)$ as E runs over the finite extensions of $\mathbf{Q}$ in $\mathbf{Q}^c$. It is a module over the Galois group

$$\Omega_F = \mathrm{Gal}(\mathbf{Q}^c/F),$$

and so are the groups appearing in (2.8). The classgroup $Cl(\mathcal{G})$ of $\mathcal{G}$ will then be defined as a quotient of the Hom group

$$\mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(B^c, W), \mathcal{J}(\mathbf{Q}^c)) = \mathcal{H}(\mathcal{G}). \tag{2.9}$$

For given n and a given B^c-module U (always finite dimensional over $\mathbf{Q}^c$) we consider the $\mathbf{Q}^c$-vector space

$$X_n(U) = \mathrm{Hom}_{B^c}((W^c)^n, U) \tag{2.10}$$

The map $[U] \mapsto [X_n(U)]$ is a homomorphism

$$\mathcal{K}_0(B^c) \to \mathcal{K}_0(\mathbf{Q}^c)$$

of Ω_F-modules, whose kernel contains $(W^c)^\perp$. It thus yields a homomorphism

$$(2.11) \qquad \mathcal{K}_0(B^c, W) \to \mathcal{K}_0(\mathbf{Q}^c)$$

of Ω_F-modules. With $\mathrm{Aut}_B(W^n)$ acting on $X_n(U)$ via its action on $(W^c)^n$ we now obtain a representation $T_{U,n}$ of the group $\mathrm{Aut}_B(W^n)$ by $\mathbf{Q}^c$-linear transformations of $X^n(U)$. Thus for $\beta \in \mathrm{Aut}_B(W^n)$ the determinant $\det(T_{U,n}(\beta))$ is an element of $(\mathbf{Q}^c)^*$. Write then

$$(2.12) \qquad \mathrm{Det}_n(\beta)(U) = \mathrm{Det}(\beta)(U) = \det(T_{U,n}(\beta)).$$

Taking account of what we said above, we now have derived a homomorphism

$$(2.12.\mathrm{a}) \qquad \mathrm{Det}_n = \mathrm{Det} : \mathrm{Aut}_B(W^n) \to \mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(B^c, W), (\mathbf{Q}^c)^*).$$

Going over to ideles we get analogously a homomorphism

$$(2.12.\mathrm{b}) \qquad \mathrm{Det} = \mathrm{Det}_n : \mathcal{J}(\mathrm{Aut}_B(W^n)) \to \mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(B^c, W), \mathcal{J}(\mathbf{Q}^c)).$$

Of course, as usual an element β of $\mathcal{J}(\mathrm{Aut}_B(W^n))$ is given by local components $\beta_\wp$ with the property that if $N \in \mathcal{G}^n$ then

$$\beta_\wp \in \mathrm{Aut}_{B_\wp}(N_\wp)$$

for almost all $\wp$. Those ideles for which this relation holds for all $\wp$ are the ***unit ideles*** of $N_\wp$. These form a subgroup $\mathcal{U}_B(N)$ of $\mathcal{J}(\mathrm{Aut}_B(W^n))$. We shall now show that if also $M \in \mathcal{G}^n$ then

$$(2.13) \qquad \mathrm{Det}_n(\mathcal{U}_B(N)) = \mathrm{Det}_n(\mathcal{U}_B(M)).$$

Indeed we way suppose that N and M both span W^n. For each $\wp$ there is then an automorphism $f_\wp$ of the $B_\wp$-module $W^n_\wp$ with $f_\wp(N_\wp) = M_\wp$, whence $f_\wp \circ (\mathrm{Aut}_B(N_\wp)) \circ f_\wp^{-1} = \mathrm{Aut}_{B_\wp}(M_\wp)$. Here for almost all $\wp$ we may take $f_\wp$ the identity map. This however implies (2.13). We shall now denote the common value of the $\mathrm{Det}_1(\mathcal{U}_B(L))$ for all $L \in \mathcal{G}$ by $\mathrm{Det}(\mathcal{U}(\mathcal{G}))$.

Now we define the classgroup of $\mathcal{G}$:

(2.14)
$$Cl(\mathcal{G}) = \mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(B^c, W), \mathcal{J}(\mathbf{Q}^c))/[\mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(B^c, W), (\mathbf{Q}^c)^*) \,.\, \mathrm{Det}(\mathcal{U}(\mathcal{G}))].$$

Thus $Cl(\mathcal{G})$ is a quotient of the Hom group $\mathcal{H}(\mathcal{G})$ in (2.9) and write

(2.14.a) $$c_{\mathcal{G}} : \mathcal{H}(\mathcal{G}) \to Cl(\mathcal{G})$$

for the quotient map.

We shall lead up to the "action" of $Cl(\mathcal{G})$ on $\mathcal{L}(\mathcal{G})$. Let $M \in \mathcal{G}^n, \beta \in \mathcal{J}(\mathrm{Aut}_B(W^n))$. Assume that M spans W^n. Then, for each $\wp$, $\beta_\wp(M_\wp)$ spans $W_\wp^n$ and there is an isomorphism of $\mathcal{B}_\wp$-lattices

(2.15) $$\beta_\wp(M_\wp) \cong M_\wp.$$

For almost all $\wp$ we have $\beta_\wp(M_\wp) = M_\wp$. Therefore there exists $\beta(M) \in \mathcal{G}^n, \beta(M) \otimes_{\mathcal{O}} F = W^n$ so that $\beta(M)_\wp = \beta_\wp(M_\wp)$, for all $\wp$.

Our construction of the action of idelic automorphisms β seems to depend on a choice of modules within an isomorphism class. This is however superficial. A typical isomorphic copy of the $\mathcal{B}$-module M is of form $\phi(M)$ where $\phi : W^n \cong W'^n$ is an isomorphism of B-modules. Now $\mathcal{J}(\mathrm{Aut}_B(W'^n)) = \phi(\mathcal{J}(\mathrm{Aut}_B(W^n)))\phi^{-1}$, and ϕ will also restrict to an isomorphism
$$\beta(M) \cong (\phi\beta\phi^{-1})(\phi(M)).$$
But, as we shall see, it is really only the elements $\mathrm{Det}(\beta)$ and $\mathrm{Det}(\phi\beta\phi^{-1})$ which count and these coincide. From now on we may suppose that any M in $\mathcal{G}^n$ spans W^n.

Going over to determinants, as above, the operation of β will also respect stable isomorphism. Indeed, let $M, M' \in \mathcal{G}^n$, $\beta, \beta' \in \mathcal{J}(\mathrm{Aut}_B(W^n))$ with $\mathrm{Det}(\beta) = \mathrm{Det}(\beta')$. Suppose moreover that for some $L \in \mathcal{G}^n$ we have $M \oplus L \cong M' \oplus L$. Thus

$$(\beta \oplus 1_L)(M \oplus L) = \beta(M) \oplus L \cong \beta'(M') \oplus L = (\beta' \oplus 1_L)(M \oplus L);$$

and $\mathrm{Det}(\beta \oplus 1_L) = \mathrm{Det}(\beta' \oplus 1_L)$.

THEOREM 1.

(I) Given $c \in Cl(\mathcal{G})$ and given a positive integer n, $\exists\beta \in \mathcal{J}(\mathrm{Aut}_B(W^n))$ with $c = c_{\mathcal{G}}\mathrm{Det}_n(\beta)$. Given moreover a module $M \in \mathcal{G}^n$, the class $[\beta(M)]$ in $\mathcal{K}_0(\mathcal{G})$ will only depend on c, not on β, and on $[M] \in \mathcal{K}_0(\mathcal{G})$. Denote it by $c[M]$.

(II) The quadruplet consisting of

(i) $\mathcal{L}(\mathcal{G}) = \mathcal{L} \subset \mathcal{K}_0(\mathcal{G})$;

(ii) the map r (cf. (2.3));

(iii) the group $C = Cl(\mathcal{G})$;

(iv) the pairing $(c, [M]) \mapsto c[M]$

satisfies axioms (1.1).

Moreover $Cl(\mathcal{G})$ acts fixed point free on $\mathcal{L}(\mathcal{G})$ and transitively on each $r^{-1}(n) \subset \mathcal{L}(\mathcal{G})$.

In the particular case when $\mathcal{G} = \mathcal{G}(\mathcal{B})$ is the genus of the (right) $\mathcal{B}$-lattice $\mathcal{B}$, then $Cl(\mathcal{G})$ and $\mathcal{K}_0(\mathcal{G})$ are the classgroup, and the Grothendieck group, respectively, of locally free $\mathcal{B}$-modules. In this case the assertions of Theorem 1 are - perhaps slightly reformulated - known results (see e.g. [1], [2]). For instance (1.1.b) goes back to the fact that the determinant of the product of two linear transformations is the product of their determinants, and (1.1.d) to the fact that the determinant of a direct sum of transformations is also the product of their determinants.

Our proof of Theorem 1 proceeds by reduction to the locally free case, which we shall take for granted. It must be emphasized however that for applications, e.g. to Galois module structure, a restriction to locally free classgroups would be self-defeating.

§3. The basic isomorphisms.

In this section it will avoid confusion if we are strict and explicit in fixing on which side various rings will act on modules. Thus if X, Y are say right modules we shall often write $\mathrm{Hom}(X_R, Y_R)$ in place of $\mathrm{Hom}_R(X, Y)$.

We still consider the genus $\mathcal{G}$ of right $\mathcal{B}$-lattices as before. Fix a module $L \in \mathcal{G}$ and write

$$(3.1) \qquad \mathcal{A} = \mathrm{End}(L_{\mathcal{B}}), \quad A = \mathrm{End}(W_B), \quad (L =_{\mathcal{A}} L_{\mathcal{B}}).$$

Then $\mathcal{A}$ is an order in the semisimple F-algebra A. We shall write for simplicity

$$Cl(\mathcal{A}) = Cl(\mathcal{G}(\mathcal{A})), \quad \mathcal{K}_0(\mathcal{A}) = \mathcal{K}_0(\mathcal{G}(\mathcal{A})), \quad \text{etc}$$

We define a functor H from right $\mathcal{B}$-modules to right $\mathcal{A}$-modules by

$$(3.2) \qquad H(X_{\mathcal{B}}) = \mathrm{Hom}(L_{\mathcal{B}}, X_{\mathcal{B}}),$$

with $\mathcal{A}$ acting by

$$(fa)(y) = f(ay), \quad a \in \mathcal{A}, \quad y \in L, \quad f \in H(X).$$

Next we have a functor G from right $\mathcal{A}$-modules to right $\mathcal{B}$-modules, where

$$G(Y_{\mathcal{A}}) = Y \otimes_{\mathcal{A}} L \tag{3.3}$$

with $\mathcal{B}$ acting via L. We moreover "extend" H and G to functors of modules over A and B, or A^c and B^c respectively, in the obvious way. Thus e.g.

$$\begin{cases} H(U_{B^c}) = \mathrm{Hom}(W^c_{B^c}, U_{B^c}), \\ G(V_{A^c}) = V \otimes_{A^c} W^c. \end{cases} \tag{3.4}$$

THEOREM 2.

(i) H and G give rise to inverse equivalences between the categories $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}(\mathcal{A})$(= locally free $\mathcal{A}$ – modules). They thus induce inverse isomorphisms

$$\mathcal{L}(\mathcal{G}) \underset{G}{\overset{H}{\rightleftarrows}} \mathcal{L}(\mathcal{A}),$$

$$\mathcal{K}_0(\mathcal{G}) \underset{G}{\overset{H}{\rightleftarrows}} \mathcal{K}_0(\mathcal{A}),$$

preserving ranks;

(ii) For $\beta \in \mathcal{J}(\mathrm{Aut}_B(W^n))$, $M \in \mathcal{G}^n$ we have an isomorphism of $\mathcal{A}$-modules

$$H(\beta(M)) \cong H(\beta)(H(M))$$

and

$$[H(\beta(M))]_{\mathcal{A}} = [H(\beta)(H(M))]_{\mathcal{A}} \quad \textit{in } \mathcal{K}_0(\mathcal{A}).$$

(iii) There are inverse isomorphisms

$$\mathcal{K}_0(B^c, W) \underset{G'}{\overset{H'}{\rightleftarrows}} \mathcal{K}_0(A^c),$$

hence we obtain an isomorphism

$$\tilde{G} : Hom_{\Omega_F}(\mathcal{K}_0(B^c, W), X) \cong Hom_{\Omega_F}(\mathcal{K}_0(A^c), X)$$

natural in the variable Ω_F-module X;

(iv) For $\beta \in \mathcal{J}(Aut_B(W^n))$

$$\tilde{G}(Det(\beta)) = Det(H(\beta));$$

(v) From (iii) and (iv) we obtain an isomorphism

$$\tilde{G} : Cl(\mathcal{G}) \cong Cl(\mathcal{A}).$$

Remark 1. There is an analogue to assertions (i)-(iii) with locally free $\mathcal{A}$-modules replaced by projective $\mathcal{A}$-modules and direct sums of modules in $\mathcal{G}$ by direct summands of such direct sums.

Remark 2. The setup of Theorem 2 also allows us to conclude that the cancellation law holds for direct sums of modules in $\mathcal{G}$ if and only if it holds for locally free $\mathcal{A}$-modules.

Proof of Theorem 2: The rule

$$\text{(3.5)} \qquad \theta(f \otimes_{\mathcal{A}} z) = f(z) \quad \text{for } z \in L, f \in \mathrm{Hom}(L_{\mathcal{B}}, X_{\mathcal{B}})$$

defines a natural homomorphism of right $\mathcal{B}$-modules

$$\text{(3.5.a)} \qquad \theta : GH(X_{\mathcal{B}}) \to X_{\mathcal{B}}.$$

For $X = L$ this is the standard isomorphism $\mathcal{A} \otimes_{\mathcal{A}} L \cong L$. Localising, we conclude that it is an isomorphism for $X \in \mathcal{G}$, hence finally for $X \in \tilde{\mathcal{G}}$.

Next define

$$\text{(3.6)} \qquad \psi(y)z = y \otimes z \quad \text{for } y \in Y_{\mathcal{A}}, z \in L.$$

This is a natural homomorphism

$$\text{(3.6.a)} \qquad \psi : Y_A \to HG(Y_{\mathcal{A}}).$$

It is an isomorphism for $Y = \mathcal{A}$, hence for all projective $\mathcal{A}$-modules, in particular for all locally free $\mathcal{A}$-modules. Theorem 2 (i) is now clear.

The proof of (iii) is a variant of that of (i), using now definitions (3.4). The values of the new functor G are direct sums of those simple B^c-modules which occur in W^c. We thus get a commutative diagram

$$\begin{tikzcd} \mathcal{K}_0(A^c) \arrow[rr, "G"] \arrow[dr, "G'"'] & & \mathcal{K}_0(B^c) \\ & K_0(B^c, W) \arrow[ur] & \end{tikzcd}$$

with injective unlabelled arrow.

Next, H annihilates the orthogonal complement $(W^c)^\perp$ of W^c, and we therefore have a commutative diagram

$$\begin{tikzcd} \mathcal{K}_0(B^c) \arrow[rr, "H"] \arrow[dr] & & \mathcal{K}_0(A^c) \\ & K_0(B^c, W) \arrow[ur, "H'"'] & \end{tikzcd}$$

with surjective unlabelled arrow.

Defining suitable analogues of the natural homomorphisms θ and ψ (cf. (3.5), (3.6)) we then prove that G' and H' are inverse isomorphisms. The existence of the isomophism $\tilde{G}$ is an immediate consequence. Next for $\beta \in \mathrm{Aut}_B(W^n)$ assertion (ii) follows as H is a functor. For $\beta \in \mathcal{J}(\mathrm{Aut}_B(W^n))$ first localise and then proceed locally.

We delay the proof of (iv) and first show how (iii) and (iv) imply (v). With L_B as in (3.1) we have

$$H(\mathrm{Aut}(L_B)) = \mathrm{Aut}(\mathcal{A}_\mathcal{A}).$$

Therefore by (iv)

$$\tilde{G}(\mathrm{Det}(\mathcal{U}\mathcal{G}) = \mathrm{Det}(\mathcal{U}\mathcal{A}).$$

Hence by (iii), with maps being induced by $\tilde{G}$,

$$\begin{aligned} Cl(\mathcal{A}) &= \mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(A^c), \mathcal{J}(\mathbf{Q}^c))/[\mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(A^c), (\mathbf{Q}^c)^*).\mathrm{Det}(\mathcal{U}\mathcal{A})] \\ &\cong \mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(B^c, W), \mathcal{J}(\mathbf{Q}^c))/[\mathrm{Hom}_{\Omega_F}(\mathcal{K}_0(B^c, W), (\mathbf{Q}^c)^*).\mathrm{Det}(\mathcal{U}\mathcal{G})] \\ &= Cl(\mathcal{G}). \end{aligned}$$

There remains the proof of (iv), which we shall give for an element $\beta \in \mathrm{Aut}_B(W^n)$. We identify the ring $M_n(A)$ of n by n matrices over A with $\mathrm{End}(W_B^n)$, and write accordingly

$$\beta = (\beta_{ij})$$

as an n by n matrix, with $\beta_{ij} \in A$. Thus

$$\beta(w_1, \cdots, w_n) = (\sum_j \beta_{1j} w_j, \cdots, \sum_j \beta_{nj} w_j).$$

Next we identify $\mathrm{Hom}_B(W, W^n) = H(W^n)$ with A^n via

$$(a_1, \cdots, a_n)(w) = (a_1 w, \cdots, a_n w), \quad a_i \in A, \ w \in W.$$

Now $H(\beta)$ acts on A^n by the rule

$$[H(\beta)(a_1, \cdots, a_n)](w) = \beta((a_1, \cdots, a_n)w).$$

Thus

$$H(\beta)(a_1, \cdots, a_n) = (\sum_j \beta_{1j} a_j, \cdots, \sum_j \beta_{nj} a_j).$$

We now come to the representation $T_{V,n}$ of $\mathrm{Aut}_A(A^n)$ on $\mathrm{Hom}_{A^c}((A^c)^n, V)$ for any A^c-module V, which underlies the definition of Det_n - compare with (2.12), (2.12a), with B, W^n replaced by A, A^n. We identify

$$\mathrm{Hom}_{A^c}((A^c)^n, V) = V^n$$

via

$$(v_1, \cdots, v_n)(a_1, \cdots, a_n) = \sum_{i=1}^n v_i a_i, \quad v_i \in V, a_i \in A.$$

Then $[(v_1, \cdots, v_n) T_V(H(\beta))](a_1, \cdots, a_n) = (v_1, \cdots, v_n)[H(\beta)(a_1, \cdots, a_n)]$.

Therefore

$$[(v_1, \cdots, v_n) T_V(H(\beta))] = (\sum_{i=1}^n v_i \beta_{i,1}, \cdots, \sum_{i=1}^n v_i \beta_{i,n}). \tag{3.7}$$

Next we come to the representation $T_{V \otimes W^c, n}$ of $\mathrm{Aut}_B(W^n)$ where we abbreviate $V \otimes_{A^c} W^c = V \otimes W^c$. The underlying vector space is $\mathrm{Hom}_{B^c}((W^c)^n, V \otimes W^c)$ with $\mathrm{Aut}_B(W^n)$ acting on W^n. We can identify

$$\mathrm{Hom}_{B^c}((W^c)^n, V \otimes W^c) = V^n$$

via

$$(v_1, \cdots, v_n)(w_1, \cdots, w_n) = \sum_{i=1}^n v_i \otimes w_i, \quad v_i \in V, \ w_i \in W.$$

(This extends the earlier isomorphism ψ of (3.6)). Then

$$[(v_1,\cdots,v_n)T_{V\otimes W,n}(\beta)](w_1,\cdots,w_n)=(v_1,\cdots,v_n)[\beta(w_1,\cdots,w_n)].$$

By comparison with (3.7) we now verify that

$$T_{V,n}(H(\beta))=T_{V\otimes W,n}(\beta).$$

But by definition,

$$\det(T_{V,n}(H(\beta))=\mathrm{Det}(H(\beta))(V)$$

and

$$\det(T_{V\otimes W,n}(\beta))=(\tilde{G}(\mathrm{Det}(\beta)))(V).$$

§4. The proof of Theorem 1.

We put ourselves in the situation of Theorem 2. To establish (i) in Theorem 1 consider $c\in Cl(\mathcal{G})$. Then

$$\begin{aligned}
&\tilde{G}(c)\in Cl(\mathcal{A})\quad \text{(by Theorem 2 (v))}\\
&\tilde{G}(c)=c_{\mathcal{A}}\mathrm{Det}_n(\alpha),\quad \alpha\in\mathcal{J}(\mathrm{Aut}_A(A^n))\quad \text{(by Theorem 1 for } \mathcal{G}(\mathcal{A}))\\
&\qquad =c_{\mathcal{A}}\mathrm{Det}_n(H(\beta))\quad \text{(by Theorem 2 (i))}\\
&\qquad =c_{\mathcal{A}}\tilde{G}\mathrm{Det}_n(\beta)\quad \text{(by Theorem 2 (iv))}\\
&\qquad =\tilde{G}(c_{\mathcal{G}}\mathrm{Det}_n(\beta))\quad \text{(by Theorem 2 (iii, iv))}.
\end{aligned}$$

Therefore, as $\tilde{G}$ is an isomorphism, $c=c_{\mathcal{G}}\mathrm{Det}_n(\beta)$.

Next suppose that $\beta_i\in\mathcal{J}(\mathrm{Aut}_B(W^n))$ (for $i=1,2$), $M_i\in\mathcal{G}^n, c_{\mathcal{G}}\mathrm{Det}(\beta_1)=c_{\mathcal{G}}\mathrm{Det}(\beta_2)$, $M_1\cong M_2$. Then

$$\begin{aligned}
c_{\mathcal{A}}\mathrm{Det}(H(\beta_1))&=c_{\mathcal{A}}\mathrm{Det}(H(\beta_2)).\\
\therefore [H(\beta_1)H(M_1)]&=[H(\beta_2)H(M_2)]\\
\therefore [H(\beta_1M_1)]&=[H(\beta_2M_2)]\\
\therefore [\beta_1M_1]&=[\beta_2M_2].
\end{aligned}$$

We now have established a pairing as postulated in Theorem 1. It remains to prove:

(4.1). *The diagram*

$$\begin{array}{ccc} Cl(\mathcal{G}) \times \mathcal{L}(\mathcal{G}) & \xrightarrow{\tilde{G}\times H} & Cl(\mathcal{A}) \times \mathcal{L}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathcal{G}) & \xrightarrow{H} & \mathcal{L}(\mathcal{A}) \end{array}$$

commutes.

Proof: Let $c = c_{\mathcal{G}}\mathrm{Det}(\beta), \beta \in \mathcal{J}(\mathrm{Aut}_B(W^n)), n = r([M])$. Then

$$\begin{aligned} H([c.M]) &= [H(\beta M)] \\ &= [H(\beta).H(M)] \\ &= (\mathrm{Det}H(\beta))[H(M)] \\ &= \tilde{G}\mathrm{Det}(\beta))[H(M)] \\ &= (\tilde{G}(c)).[H(M)] \quad \text{as required.} \ \blacksquare \end{aligned}$$

By Theorem 2 and (4.1) we conclude that the bijections $H : \mathcal{K}_0(\mathcal{G}) \to \mathcal{K}_0(\mathcal{A}), \tilde{G} : Cl(\mathcal{G}) \to Cl(\mathcal{A})$ preserve all operations. The validity for $\mathcal{G}$ of II in Theorem 1 now follows from that for $\mathcal{G}(\mathcal{A})$.

References

[1] A. Fröhlich, *Locally free modules over arithmetic orders*, Crelle **274/75** (1975), 112-138.

[2] A. Fröhlich, *Classgroups and Hermitian modules*, Progress in Mathematics **48**, Birkhäuser 1984.

[3] S. M. J. Wilson, *Galois module structure of the rings of integers in wildly ramified extensions*, Ann. Inst. Fourier **39** (1989), 529-551.

[4] S. M. J. Wilson, *A projective invariant comparing rings of integers in wildly ramified extensions*, Crelle **412** (1990), 34-47.

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