MARTIN HELM

A generalization of a theorem of Erdös on asymptotic basis of order 2


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ABSTRACT – Let $T$ be a system of disjoint subsets of $\mathbb{N}^*$. In this paper we examine the existence of an increasing sequence of natural numbers, $A$, that is an asymptotic basis of all infinite elements $T_j$ of $T$ simultaneously, satisfying certain conditions on the rate of growth of the number of representations $r_n(A)$; $r_n(A) := |\{(a_i, a_j) : a_i < a_j; a_i, a_j \in A; n = a_i + a_j\}|$, for all sufficiently large $n \in T_j$ and $j \in \mathbb{N}^*$. A theorem of P. Erdős is generalized.

1. Notation

In this paper, $\mathbb{N}^*$ will always denote the set of integers $\{1, 2, \ldots, n, \ldots\}$. An increasing sequence of natural numbers, $A$, is called an asymptotic basis of order 2 of a given set $T$ of natural numbers if every sufficiently large $n \in T$ has at least one representation in the form $n = a_i + a_j; a_i < a_j; a_i, a_j \in A$. Let $r_n(A)$ be the number of such representations of $n \in T$ by elements of $A$.

DEFINITION. A system $T = (T_j)_{j \in \mathbb{N}^*}$ of disjoint subsets of $\mathbb{N}^*$ satisfying $\mathbb{N}^* = \bigcup_{j=1}^{\infty} T_j$ is called a disjoint covering system.

DEFINITION. If for an increasing sequence $A$ of natural numbers there exists a disjoint covering system $T$ such that

\begin{align*}
(1) & \quad \exists j_0 : T_j = \emptyset \ \forall j \geq j_0 \text{ or } |T_j| = \infty \text{ for infinitively many } j \in \mathbb{N}^* \\
\text{and} \\
(2) & \quad A \text{ is an asymptotic basis of order } 2 \text{ of all infinite elements } T_j \text{ of } T,
\end{align*}

then $A$ is called an asymptotic pseudo-basis of $\mathbb{N}^*$.

Remark. Let $A$ be an asymptotic pseudo-basis in regard to a disjoint covering system $\mathcal{T}$. For any infinite element $T_j$ of $\mathcal{T}$ let

$$n_j := \min \{m \in T_j : r_n(A) > 0 \quad \forall n \in T_j, \ n \geq m\}.$$ 

Obviously any asymptotic basis $A$ of order 2 of $\mathbb{N}^*$ is an asymptotic pseudo-basis (e.g. for $\mathcal{T} := \mathbb{N}^*, \emptyset, \emptyset, ...$). But unfortunately the converse in general is not true since for any asymptotic pseudo-bases $A$ of $\mathbb{N}^*$ together with a corresponding disjoint covering system $\mathcal{T}$ the set of all $n_j$ that are defined in the above sense is not necessarily bounded.

2. Introduction

More than fifty years ago S. Sidon [5] asked if there exists an asymptotic basis of order 2 of $\mathbb{N}^*$ that is economic in the sense that for every $\varepsilon > 0$ the assumption $\lim_{n \to \infty} \frac{r_n(A)}{n^{\varepsilon}} = 0$ holds.

In 1953 P. Erdös [1] solved this problem ingeniously. In fact he proved the much sharper:

**THEOREM.** There exists an asymptotic basis $A$ of order 2 of $\mathbb{N}^*$, satisfying:

$$A(n) \sim \alpha n^{\frac{1}{2}} (\log n)^{\frac{3}{2}}, \alpha \in \mathbb{R},$$

with $A(n) := \sum_{\alpha \in A, 1 \leq \alpha \leq n} 1$

and

$$\log n \ll r_n(A) \ll \log n.$$ 

An attractive and still open problem is to decide whether there exists a basis $A$ of $\mathbb{N}^*$ for which there exists $c := \lim_{n \to \infty} \frac{r_n(A)}{\log n}$.

Moreover in [4] I. Rusza asks for a basis for which $r_n(A) \ll \frac{\log n}{\log \alpha n}$ holds.

3. On asymptotic pseudo-bases

In this paper we prove the following:
Theorem. For any \( k \in \mathbb{N}^* \) there exists a disjoint covering system \( T^{(k)} = \{T_1^{(k)}, T_2^{(k)}, \ldots \} \) satisfying:

\[
\forall j \in \mathbb{N}^* : T_j^{(k)} \text{ is an infinite element of } T^{(k)}:
\]

\[
(5) \quad \log_{k-1} n \gg T_j^{(k)}(n) \gg \log_k n \quad (n \to \infty)
\]

(where \( \log_0 n := \text{id}(n) = n \)),

and an asymptotic pseudo-basis \( A \) satisfying:

\[
(6) \quad A(n) \sim 2\alpha(\log_k n)^{\frac{1}{2}} n^{\frac{1}{2}}
\]

and

\[
(7) \quad c_1 \log_k n \leq r_n(A) \leq c_2 \log_k n,
\]

\[
\forall n \in T_j^{(k)} \text{ that are sufficiently large,}
\]

and \( \forall j \in \mathbb{N}^* \) where \( T_j^{(k)} \) is an infinite element of \( T^{(k)} \),

where \( \alpha, c_1 \) and \( c_2 \) are global real constants not depending on \( j \).

Remark. The above theorem generalizes (3,4), which is just the special case \( k = 1 \) (e.g. with \( T := \mathbb{N}^*, \emptyset, \emptyset, \ldots \)).

The proof of the above theorem is based on a slight modification of Erdős\' proof of (3,4). Therefore like the proof of (3,4), it is based on a probabilistic method and not constructive.

3.1 Inductive construction of suitable disjoint covering systems

First of all, for any \( k \in \mathbb{N}^* \), we are going to construct a special disjoint covering system \( T^{(k)} \) satisfying (1) and (5).

The case \( k = 1 \).

For \( k = 1 \) let \( T^{(1)} := \mathbb{N}^*, \emptyset, \emptyset, \ldots \).

Obviously \( T^{(1)} \) is a disjoint covering system and (1) and (5) hold.

The case \( k = 2 \).

For \( k = 2 \) we define \( T^{(2)} \) inductively as follows:

\[
T_1^{(2)} := \{1\},
\]
Now, if $T_1^{(2)}, \ldots, T_r^{(2)}$ are already defined, let:

$$s := \min\{n \in \mathbb{N}^* : n \notin \bigcup_{i=1}^{r} T_i^{(2)}\}$$

and we define

$$T_{r+1}^{(2)} := \{s^j : j \in \mathbb{N}^*\}.$$ 

Now we consider the following equivalence relation on $\mathbb{N}^*$:

$$a \sim b : \iff \exists s, u, v \in \mathbb{N}^* : a = s^u, \ b = s^v.$$ 

$T^{(2)}$ just consists of all equivalence classes concerning the above equivalence relation. Thus $T^{(2)}$ is a disjoint covering system and obviously (1) holds.

For $T_i^{(2)} \in T^{(2)} \setminus \{1\}$ there exists $s \in \mathbb{N}^*$ such that

$$T_i^{(2)} = \{s^j : j \in \mathbb{N}^*, \ s \in \mathbb{N}^* \setminus \{1\}\}.$$ 

For any sufficiently large $m \in \mathbb{N}^*$ there exists $t \in \mathbb{N}^*$ such that

$$s^t \leq m < s^{t+1}.$$ 

Thus $T_i^{(2)}(m) = t$ implies that:

$$T_i^{(2)}(m) \leq \frac{1}{\log s} \log m \leq T_i^{(2)}(m) + 1,$$

and consequently

$$\log m \ll T_i^{(2)}(m) \ll \log m.$$ 

Therefore also (5) holds.

The case $k = 3$.

**Definition.** For $s \in \mathbb{N}^*$ and any non-empty subset $M$ of $\mathbb{N}^*$ we define

$$s^M := \{s^m : m \in M\}.$$ 

We construct $T^{(3)}$ by dividing every element $T_i^{(2)}$ of $T^{(2)}$ except $\{1\}$ into disjoint infinite subsets of $\mathbb{N}^*$.

For any $T_i^{(2)}$ of $T^{(2)}$ there exists $s \in \mathbb{N}^*$:

$$T_i^{(2)} = \{s^j : j \in \mathbb{N}^*\}.$$
Consequently

\[ T_i^{(2)} = \bigcup_{T_j^{(2)} \in T^{(2)}} s_{T_j^{(2)}} \]

and we define \( T^{(3)} \) as the system of all those sets \( s_{T_j^{(2)}} = \{s^{p^j} : j \in \mathbb{N}^*\} \) where \( p \) is a natural constant. Since \( T^{(2)} \) is a disjoint covering system, \( T^{(3)} \) is a disjoint covering system, too; and as (1) holds for \( T^{(2)} \), \( T^{(3)} \) satisfies (1), too.

For any infinite element \( T_i^{(3)} \) for \( T^{(3)} \) and any sufficiently large number \( m \in \mathbb{N}^* \) there exist \( s, p, t \in \mathbb{N}^* \) such that

\[ T_i^{(3)} = \{s^{p^j} : j \in \mathbb{N}^*\}, \]

and

\[ s^{p^t} \leq m < s^{p^{t+1}}. \]

Then \( T_i^{(3)}(m) = t \) implies \( \log_2 m \ll T_i^{(3)}(m) \ll \log_2 m. \)

Consequently \( T^{(3)} \) satisfies also (5).

The general case \( k \geq 4 \).

Let \( T^{(1)}, T^{(2)}, T^{(3)}, \ldots T^{(k)} \) be already constructed by the above procedure. Thus for every infinite element \( T_i^{(k)} \) of \( T^{(k)} \) there exist \( s_1, \ldots, s_{k-1} \in \mathbb{N}^* \) so that

\[ T_i^{(k)} = \{s_1, \ldots, s_{k-1} \in \mathbb{N}^* : j \in \mathbb{N}^*\}, \]

and according to the above procedure \( T^{(k+1)} \) will be constructed out of \( T^{(k)} \) by dividing every infinite \( T_i^{(k)} \) of \( T^{(k)} \) into disjoint subsets

\[ \left( \begin{array}{c} \left( \begin{array}{c} \vdots \end{array} \right) \left( s_1^{(2)} \right) \left( s_{k-1}^{(2)} \right) \vspace{0.2cm} \right. \\
 s_2 \\
 \left. \begin{array}{c} s_1 \\
 \vdots \\
 s_{k-1} \end{array} \right) \right), \quad T_i^{(2)} \in T^{(2)}. \]
It is easy to see that also $T^{(k+1)}$ is a disjoint covering system satisfying (1) and (5).

3.2 Proof of the existence of an asymptotic pseudo-basis $A$ satisfying (6) and (7) in regard to $T^{(k)}$ for any fixed $k \in \mathbb{N}^*$.

This part of the proof of the above theorem uses the probabilistic method of Erdős and Rényi [2]. Since [3] contains an excellent exposition of it, we only give a short survey of those of Erdős' and Rényi's ideas our next steps are based on without proof.

Remark. Since, as we mentioned above, the case $k = 1$ is already solved we restrict ourselves to the case $k \geq 2$.

By the method of Erdős and Rényi ([2] and [3]) for any sequence of real numbers $(\alpha_j)_{j \in \mathbb{N}^*}, 0 \leq \alpha_j \leq 1$, there exists a probability space with probability measure $\mu$ on the space $\Omega$ of all strictly increasing sequences of natural numbers, satisfying:

1. the event $B^{(n)} := \{\omega \in \Omega : n \in \Omega\}$ is measurable, $\mu(B^{(n)}) = \alpha_n$,
2. and the events $B^{(1)}, B^{(2)}, \ldots$ are independent.

We denote by $\rho_n$ the characteristic function of the event $B^{(n)}$.

From now on we consider only those sequences of probabilities $(\alpha_j)_{j \in \mathbb{N}^*}$, satisfying:

1. $0 < \alpha_j < 1$,
2. $\lim_{j \to \infty} \alpha_j = 0$,
3. $\exists j_0 : \alpha_{j+1} < \alpha_j$ $\forall j \geq j_0$,
4. $\sum_{j=1}^{\infty} \alpha_j = \infty$.

Then by a particular variant of the strong law of large numbers, with probability 1,

$$\sum_{j=1}^{n} \alpha_j \sim \omega(n) \ (n \to \infty)$$
LEMMA 1. A sequence of positive real numbers is defined by

\[ \omega(n) := \sum_{j \in \omega; 1 \leq j \leq n} 1. \]

Let

\[ \lambda_n := \sum_{1 \leq j < \frac{n}{2}} \alpha_j \alpha_{n-j}, \quad m_n := \sum_{j=1}^{n} \alpha_j, \]

and

\[ \lambda'_n := \sum_{1 \leq j < \frac{n}{2}} \alpha_j \alpha_{n-j} (1 - \alpha_j \alpha_{n-j})^{-1}. \]

Then we have:

\[ \lambda'_n \sim \lambda_n \quad (n \to \infty), \]

and

\[ \mu(\{ \omega : r_n(\omega) = d \}) \leq \frac{\lambda_n^d}{d!} e^{-\lambda_n}, \quad d \in \mathbb{N}. \]

LEMMA 1. A sequence \((\alpha_j)_{j \in \mathbb{N}}\) of positive real numbers is defined by

\[ \alpha_j := \alpha \frac{(\log_k j)^{c'}}{j^c} \quad \forall j > j_0, \]

where \(j_0, \alpha, k, c\) and \(c'\) are suitably chosen real constants, satisfying

\[ 0 \leq c', \quad 0 < c < 1, \quad 0 < \alpha, \quad 1 \leq k \]

so that \(\log_k(j) > 0, \forall j > j_0\) and (18) and (10 - 13) are compatible. The precise value of \(\alpha_j\) for small \(j\) is unimportant in case that their choice ensures that (18) and (10 - 13) are compatible also for \(\alpha_1, \cdots, \alpha_{j_0}.\) Then as \((n \to \infty)\)

\[ \lambda_n \sim \frac{1}{2} \alpha^2 \frac{(\Gamma(1 - c))^2}{\Gamma(2 - 2c)} (\log_k n)^{2c'} n^{1-2c}, \]

\[ m_n \sim \frac{\alpha}{1-c} (\log_k n)^{c'} n^{1-c}. \]
Remark. The above lemma is a slight generalization of Lemma 11 in [3], p 144. Its proof corresponds essentially to that of the above-mentioned Lemma 11 and is therefore left to the reader.

Now let \( k \) be a fixed natural number. To prove our theorem, corresponding to Erdös' proof of (3,4), we first choose a number \( \alpha \) with \( 0 < \alpha < 1 \), so that

\[
\frac{1}{2} \alpha^2 \pi > 1
\]

holds, and we define the sequence \((\alpha_j)_{j \in \mathbb{N}^*}\) by

\[
\alpha_j = \begin{cases} 
\frac{1}{2} & 1 \leq j \leq j_0, \\
\alpha \left( \frac{\log_k n}{j} \right)^{\frac{1}{2}} & j > j_0,
\end{cases}
\]

where \( j_0 \) is a suitably chosen natural number so that \( \log_k j > 0 \ \forall j > j_0 \) and \((\alpha_j)_{j \in \mathbb{N}^*}\) satisfies (10 - 13).

Therefore by (14) and by Lemma 1 we have with probability 1

\[
\omega(n) \sim 2\alpha \sqrt{\log_k n \sqrt{n}},
\]

\[
\lambda_n \sim \frac{\pi}{2} \alpha^2 \log_k n,
\]

which because of (21) ensures the existence of a number \( \delta > 0 \) such that

\[
e^{-\lambda_n} \ll (\log_{k-1} n)^{-(1+\delta)}.
\]

In view of (17) for any \( n \in \mathbb{N}^*, \ d \in \mathbb{N} : \)

\[
\mu(\{\omega : r_n(\omega) > e^{\lambda'_n}\}) \leq \sum_{d \geq e^{\lambda'_n}} \mu(\{\omega : r_n(\omega) = d\}) \leq \sum_{d \geq e^{\lambda'_n}} \frac{\lambda_n^d}{d!} e^{-\lambda_n}
\]

\[
\leq \left( \frac{e^{\lambda'_n}}{e^{\lambda'_n}} \right) e^{-\lambda_n} = e^{-\lambda_n} \ll \frac{1}{(\log_{k-1} n)^{1+\delta}}.
\]

Let \( T_i^{(k)} \) be an infinite non-empty element of \( T^{(k)} \).
There exists $s_1, \cdots, s_{k-1} \in \mathbb{N}^*$ so that

$$T_i^{(k)} = \{s_1 \cdots (s_{k-1})^j, j \in \mathbb{N}^*\}.$$  

Consequently:

$$\sum_{n \in T_i^{(k)}} \mu(\{\omega : r_n(\omega) > e^{\lambda_n'}\}) \leq \sum_{n \in T_i^{(k)}} e^{-\lambda_n}$$

$$\leq \sum_{j=1}^{\infty} \left( \log_{k-1} \frac{s_1 \cdots (s_{k-1})^j}{j} \right)^{-1(1+\delta)}$$

$$\ll \sum_{j=1}^{\infty} \left( \frac{1}{j} \right)^{1+\delta} < \infty.$$  

Therefore the application of the Borel-Cantelli-Lemma proves the existence of a positive real number $c_2$, such that for any infinite $T_i^{(k)} \in T^{(k)}$

$$\mu(\{\omega : r_n(\omega) \leq c_2 \log_k n, n \in T_i^{(k)}, (n \text{sufficiently large})\}) = 1.$$  

On the other hand for any suitably chosen constant $b < 1$ again in view of (17) we have

$$\mu(\{\omega : r_n(\omega) < b\lambda_n'\}) \leq \sum_{1 \leq d \leq b\lambda_n'} \mu(\{\omega : r_n(\omega) = d\})$$

$$\leq \sum_{1 \leq d \leq b\lambda_n'} \frac{\lambda_n'^d}{d!} e^{-\lambda_n}$$

$$\leq \left( \frac{e\lambda_n'}{b\lambda_n'} \right)^{b\lambda_n'} e^{-\lambda_n}$$

$$= \left( \frac{e}{b} \right)^{b\lambda_n'} e^{-\lambda_n}.$$  

Therefore because of (16) there exists $c_1, 0 < c_1 < 1$ such that

$$\left( \frac{e}{c_1} \right)^{c_1} \lambda_n' e^{-\lambda_n} \ll (\log_{k-1} n)^{-(1+\delta)}.$$
Thus for any fixed infinite $T_i^{(k)} \in T^{(k)}$, with

$$T_i^{(k)} = \{ s_1 \begin{pmatrix} \cdots (s^j_{k-1}) \\ s_2 \end{pmatrix}, j \in \mathbb{N}^* \},$$

we have

$$\sum_{n \in T_i^{(k)}} \mu(\{ \omega : r_n(\omega) < c_1 \lambda_n' \}) \ll \sum_{j=1}^{\infty} \log_{k-1} s_1 \left( \begin{pmatrix} \cdots (s^j_{k-1}) \\ s_2 \end{pmatrix} \right)^{-(1 + \frac{\delta}{2})} \ll \sum_{j=1}^{\infty} \left( \frac{1}{j} \right)^{1 + \frac{\delta}{2}} < \infty.$$

Again we apply the Borel-Cantelli-Lemma to prove the existence of $c_1 > 0$ such that for any infinite $T_i^{(k)} \in T^{(k)}$

$$(28) \quad \mu(\{ \omega : r_n(\omega) \geq c_1 \log_k n, \ n \in T_i^{(k)}, \ (n \text{ sufficiently large}) \}) = 1.$$

We have shown that $\omega$ has each of the desired properties with probability 1 and thus the whole proof is complete.

REFERENCES


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Martin Helm  
Graduate School and University Center  
of the City University of New-York  
Department of Mathematics  
Graduate Center 33 West 42 Street  
New-York 10036-8099 USA  
e-mail: hxm@cunyvms1.gc.cuny.edu