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Mean Square of the Remainder Term in the Dirichlet Divisor Problem

par Yuk-Kam LAU and Kai-Man TSANG

1. Introduction and Main Results

Let $d(n)$ denote the divisor function. In this paper we shall consider a remainder term associated with the mean square of the error term $\Delta(x)$ in the Dirichlet divisor problem, which is defined as

$$\Delta(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1).$$

Here γ is Euler's constant. The upper bound $\Delta(x) \ll x^{1/2}$ was first obtained by Dirichlet in 1838. This was gradually sharpened by many authors in the ensuing one and a half century. Iwaniec and Mozzochi [5] proved in 1988 that $\Delta(x) \ll x^{7/22+\varepsilon}$ for any $\varepsilon > 0$, by employing intricate techniques for the estimation of certain exponential sums. Such methods, however, do not seem capable of proving the conjectured best bound: $\Delta(x) \ll x^{1/4+\varepsilon}$.

Besides this problem, there are plenty of papers written on other interesting properties of $\Delta(x)$. For instance, Tong [9] showed that $\Delta(x)$ changes sign at least once in every interval of the form $[X, X + c_0\sqrt{X}]$ where c_0 is a certain positive constant. Recently Heath-Brown and Tsang [2] showed that this is essentially best possible: – the length of the intervals cannot be reduced to $o(\sqrt{X} \log^{-5} X)$. In contrast to this erratic behaviour, $\Delta(x)$, when considered in the mean, has very nice asymptotic formula. A classical result of Tong [10] says that

$$(1.1) \quad \int_2^X \Delta(x)^2 dx = \left((6\pi^2)^{-1} \sum_{m=1}^{\infty} d(m)^2 m^{-3/2} \right) X^{3/2} + F(X)$$

with $F(X) \ll X \log^5 X$. The order of the remainder term $F(X)$ has significant connection with that of $\Delta(x)$. Indeed, Ivić's argument in Theorem

3.8 of [4] shows that $\Delta(x) \ll (U \log x)^{1/3}$ for any upper bound U of $F(X)$. Thus from the result $\Delta(x) = \Omega(x^{1/4})$ we infer that

$$(1.2) \quad F(X) = \Omega(X^{3/4}/\log X) .$$

Ivić conjectured that $F(X) \ll X^{3/4+\varepsilon}$ is true for any $\varepsilon > 0$. This is a very strong bound since it implies $\Delta(x) \ll x^{1/4+\varepsilon}$. There are not many results on $F(X)$ in the literature. Tong's bound was slightly improved to $F(X) \ll X \log^4 X$ by Preissmann [7] in 1988. However, the gap between this and the Ω -result (1.2) is still very wide.

In this paper we shall prove the following.

THEOREM 1. *We have*

$$F(X) = \Omega_-(X \log^2 X) .$$

THEOREM 2. *For $X \geq 2$ we have*

$$\int_2^X F(x) dx = -(8\pi^2)^{-1} X^2 \log^2 X + c_1 X^2 \log X + \mathcal{O}(X^2)$$

for a certain constant c_1 .

Theorem 1, which is a direct consequence of Theorem 2, disproves the above conjecture of Ivić. Unfortunately we are still unable to obtain a comparable Ω_+ -result for $F(x)$. In fact we believe that there is an asymptotic formula for $F(x)$ of the form

$$(1.3) \quad F(x) = -(4\pi^2)^{-1} x \log^2 x + c_2 x \log x + \mathcal{O}(x)$$

with a certain constant c_2 . In a forthcoming paper, the second author [11] proves that

$$\int_X^{2X} (F(x + \sqrt{X}) - F(x))^2 dx \asymp X^3 .$$

Using Preissmann's bound we see easily that

$$\begin{aligned} \int_X^{2X} (F(x + \sqrt{X}) - F(x)) dx &= \int_{2X}^{2X+\sqrt{X}} - \int_X^{X+\sqrt{X}} F(x) dx \\ &\ll X^{3/2} \log^4 X . \end{aligned}$$

These two results together shows that $F(x + \sqrt{X}) - F(x)$ changes signs in $[X, 2X]$ and

$$F(x + \sqrt{X}) - F(x) = \Omega(X) .$$

Consequently, if (1.3) is true the $\mathcal{O}$ -term on the right hand side is oscillatory and cannot be reduced.

One of the key ingredients in our argument is an asymptotic formula for the sum

$$\sum_{m \leq x} d(m)d(m+h) .$$

Such a sum has been investigated by several authors in connection with other problems in analytic number theory. In our proof we use a result of Heath-Brown [1] which is quite sufficient for our purpose. (see (2.12)-(2.15) below)

2. Notations and some Preparation

Throughout the paper, ε denotes an arbitrary small positive number which need not be the same at each occurrence. The symbols $c_0, c_1, c_2, \dots$ etc. denote certain constants. We shall also use the well-known inequality $d(n) \ll n^\varepsilon$ from time to time without explicit reference. The constants implicit in the symbols $\mathcal{O}$ and $\ll$ depend at most on ε .

A useful formula for studying problems concerning $\Delta(x)$ was obtained by Voronoi [12] at the beginning of this century. The formula expresses $\Delta(x)$ as an infinite series involving the Bessel functions. In practice, the following truncated form of the formula

$$\begin{aligned} \Delta(x) &= (\pi\sqrt{2})^{-1} x^{1/4} \sum_{n \leq N} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) \\ &\quad + \mathcal{O}(x^\varepsilon + x^{1/2+\varepsilon} N^{-1/2}) \end{aligned}$$

for $1 \leq N \ll x$ is quite sufficient. However, for our present problem, the above $\mathcal{O}$ -term is far too large and we shall use instead the following approximation to $\Delta(x)$ given by Meurman [6, Lemma 3].

LEMMA 1. *For $x \geq 1$ and $M \gg x$, let*

$$\delta_M(x) = (\pi\sqrt{2})^{-1} x^{1/4} \sum_{n \leq M} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) .$$

Then $\Delta(x) = \delta_M(x) + R(x)$ where $R(x) \ll x^{-1/4}$ if $\|x\| \gg x^{5/2}M^{-1/2}$ and $R(x) \ll x^\varepsilon$ otherwise.

Using this we obtain

LEMMA 2. Let $x \geq 2$ and $x^7 \ll M \ll x^{100}$. Then

$$\int_2^x \Delta(u)^2 du = \int_2^x \delta_M(u)^2 du + \mathcal{O}(x) .$$

Proof. Firstly,

$$\int_2^x \Delta(u)^2 du = \int_2^x \delta_M(u)^2 du + 2 \int_2^x \delta_M(u)R(u)du + \int_2^x R(u)^2 du .$$

Next, by Lemma 1, we have

$$(2.1) \quad \int_2^x R(u)^2 du \ll \sum_{n=2}^{[x]+1} n^\varepsilon n^{5/2} M^{-1/2} + \int_2^x (u^{-1/4})^2 du \ll \sqrt{x} .$$

Moreover, following the argument of [3, Theorem 13.5] we show that

$$\int_2^x \delta_M(u)^2 du \asymp x^{3/2}$$

for $M \ll x^{100}$. Thus, by Cauchy-Schwarz's inequality and (2.1) we have

$$\int_2^x \delta_M(u)R(u)du \ll x$$

and hence our lemma.

Square out $\delta_M(u)$ and then integrate term by term, we get

$$\begin{aligned} & \int_2^x \delta_M(u)^2 du \\ &= (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u} \cos(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}) du \\ &+ (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u} \sin(4\pi(\sqrt{n} + \sqrt{m})\sqrt{u}) du. \end{aligned}$$

In the first double sum the diagonal terms yield a total contribution of

$$\begin{aligned} & (4\pi^2)^{-1} \sum_{m \leq M} d(m)^2 m^{-3/2} \frac{2}{3} (x^{3/2} - 2^{3/2}) \\ &= (6\pi^2)^{-1} \sum_{m=1}^{\infty} d(m)^2 m^{-3/2} x^{3/2} + \mathcal{O}(x^{3/2} M^{\varepsilon-1/2} + 1) . \end{aligned}$$

Here the main term is the same as that in (1.1). Hence by Lemma 2, we can write

$$(2.2) \quad F(x) = S_1(x) + S_2(x) + \mathcal{O}(x) ,$$

where for any $y \geq 2$,

$$(2.3) \quad \begin{aligned} S_1(y) &= (2\pi^2)^{-1} \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4} \times \\ &\quad \int_2^y \sqrt{u} \cos(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}) du , \end{aligned}$$

and

$$(2.4) \quad \begin{aligned} S_2(y) &= (4\pi^2)^{-1} \sum_{m, n \leq M} d(m)d(n)(mn)^{-3/4} \times \\ &\quad \int_2^y \sqrt{u} \sin(4\pi(\sqrt{n} + \sqrt{m})\sqrt{u}) du . \end{aligned}$$

From now on, we let X to be a sufficiently large number, $M = X^7$ and $L = \log X$. For any $\nu \geq 0$, let

$$(2.5) \quad g(\nu) = \nu^{-3/2} J_{3/2}(\nu) - 4\nu^{-5/2} J_{5/2}(\nu) ,$$

where J_k denotes the Bessel function of order k . It is well-known that [13, §§3.3, 3.4]

$$J_k(z) \ll \min(|z|^k, |z|^{-1/2})$$

for any real z . Hence,

$$(2.6) \quad g(\nu) \ll \min(1, \nu^{-2})$$

for any $\nu \geq 0$.

LEMMA 3. *We have*

$$(2.7) \quad \int_0^X F(x)dx = \sqrt{2\pi}^{-3/2} X^{5/2} \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4} g(\theta_{m,n}) + \mathcal{O}(X^2)$$

where $\theta_{m,n} = 4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}$.

Proof. By [8, Lemma 4.2], for any real α and y , $y \geq 2$ we have

$$(2.8) \quad \int_2^y \sqrt{u} e^{i\alpha\sqrt{u}} du \ll y|\alpha|^{-1}.$$

We first obtain some preliminary bounds for $S_1(y)$ and $S_2(y)$. According to (2.3) and on applying (2.8), we have

$$(2.9) \quad S_1(y) \ll y \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4} (\sqrt{n} - \sqrt{m})^{-1}$$

$$(2.9) \quad S_1(y) \ll y \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4} (\sqrt{n} - \sqrt{m})^{-1}$$

$$\begin{aligned} &\ll yM^\varepsilon \left\{ \sum_{m < n \leq 2m \leq M} (mn)^{-3/4} (\sqrt{n} + \sqrt{m})(n - m)^{-1} \right. \\ &\quad \left. + \sum_{2m < n \leq M} m^{-3/4} n^{-5/4} \right\} \\ &\ll yM^\varepsilon \left\{ \sum_{m \leq M/2} m^{-1} \sum_{m < n \leq M} (n - m)^{-1} + \log M \right\} \ll yM^\varepsilon. \end{aligned}$$

Similarly,

$$(2.10) \quad S_2(y) \ll y \sum_{m, n \leq M} d(m)d(n)(mn)^{-3/4} (\sqrt{n} + \sqrt{m})^{-1} \ll yM^\varepsilon.$$

Next, for $x \in [\sqrt{X}, X]$ we have $x^7 \ll M \ll x^{14}$ so that, by (2.2)

$$\begin{aligned} (2.11) \quad &\int_{\sqrt{X}}^X F(x)dx = \int_{\sqrt{X}}^X S_1(x)dx + \int_{\sqrt{X}}^X S_2(x)dx + \mathcal{O}(X^2) \\ &= \int_2^X S_1(x)dx + \int_2^X S_2(x)dx + \mathcal{O}(XM^\varepsilon + X^2), \end{aligned}$$

since, by (2.9) and (2.10), $\int_2^{\sqrt{X}} S_i(x)dx \ll XM^\epsilon$. The main term on the right hand side of (2.7) arises from $\int_2^X S_1(x)dx$. Indeed, by (2.3),

$$\begin{aligned} \int_2^X S_1(x)dx &= (2\pi^2)^{-1} \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4} \times \\ &\quad \int_2^X \int_2^x \sqrt{u} \cos(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u})dudx . \end{aligned}$$

Write $\theta = 4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}$ for short. Then the above double integral is equal to

$$\begin{aligned} &\int_2^X (X - u)\sqrt{u} \cos(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u})du \\ &= 2X^{5/2} \int_{\sqrt{2/X}}^1 (1 - v^2)v^2 \cos(\theta v)dv \\ &= 2X^{5/2} \left\{ \int_0^1 (1 - v^2) \cos(\theta v)dv - \int_0^1 (1 - v^2)^2 \cos(\theta v)dv \right. \\ &\quad \left. - \int_0^{\sqrt{2/X}} (1 - v^2)v^2 \cos(\theta v)dv \right\} . \end{aligned}$$

By the well-known integral representation

$$J_{k+\frac{1}{2}}(z) = \frac{2}{\sqrt{\pi}} \left(\frac{z}{2}\right)^{k+\frac{1}{2}} \frac{1}{k!} \int_0^1 (1 - v^2)^k \cos(zv)dv , \quad k = 0, 1, 2, \dots$$

for the Bessel functions [13, §3.3], the first two integrals on the right hand side is equal to

$$\sqrt{2\pi}(\theta^{-3/2} J_{3/2}(\theta) - 4\theta^{-5/2} J_{5/2}(\theta)) = \sqrt{2\pi}g(\theta) ,$$

by (2.5). Moreover, using integration by parts we find that

$$\int_0^{\sqrt{2/X}} (1 - v^2)v^2 \cos(\theta v)dv \ll X^{-1}\theta^{-1} .$$

Hence

$$\int_2^X (X - u)\sqrt{u} \cos(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u})du = 2\sqrt{2\pi}X^{5/2}g(\theta) + \mathcal{O}(X^{3/2}\theta^{-1}),$$

and then

$$\begin{aligned} \int_2^X S_1(x)dx &= \sqrt{2}\pi^{-3/2}X^{5/2} \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4}g(\theta) \\ &\quad + \mathcal{O}\left(X \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4}(\sqrt{n} - \sqrt{m})^{-1}\right). \end{aligned}$$

The sum inside the $\mathcal{O}$ -term can be treated by the argument in (2.9), and we then find that the $\mathcal{O}$ -term is bounded by XM^ε , which is smaller than that on the right hand side of (2.7).

In view of (2.11) and (2.7), it remains to bound the two integrals $\int_0^{\sqrt{X}} F(x)dx$ and $\int_2^X S_2(x)dx$ by X^2 . By Preissmann's bound, we have

$$\int_0^{\sqrt{X}} F(x)dx \ll X \log^4 X$$

which is acceptable. Next, by (2.4),

$$\begin{aligned} \int_2^X S_2(x)dx &= (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \times \\ &\quad \int_2^X (X-u)\sqrt{u} \sin(4\pi(\sqrt{n} + \sqrt{m})\sqrt{u})du \\ &= (2\pi^2)^{-1}X^{5/2} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \times \\ &\quad \int_{\sqrt{2/X}}^1 (1-v^2)v^2 \sin(4\pi(\sqrt{n} + \sqrt{m})\sqrt{X}v)dv. \end{aligned}$$

The inner integral, on applying integration by parts twice, is found to be

$$\ll X^{-3/2}(\sqrt{n} + \sqrt{m})^{-1} + X^{-1}(\sqrt{n} + \sqrt{m})^{-2}.$$

Thus,

$$\begin{aligned} \int_2^X S_2(x)dx &\ll X \sum_{m \leq n \leq M} d(m)d(n)(mn)^{-3/4}n^{-1/2} + \\ &\quad X^{3/2} \sum_{m \leq n \leq M} d(m)d(n)(mn)^{-3/4}n^{-1} \\ &\ll XM^\varepsilon + X^{3/2} \ll X^{3/2}. \end{aligned}$$

This completes the proof of Lemma 3.

For any $y > 0$, let

$$(2.12) \quad \psi_h(y) = \sum_{m \leq y} d(m)d(m+h) .$$

In his work on the fourth power moment of the Riemann zeta-function on the critical line, Heath-Brown [1] proved that

$$(2.13) \quad \psi_h(y) = I_h(y) + E_h(y),$$

where the main term $I_h(y)$ is of the form

$$(2.14) \quad I_h(y) = y \sum_{i=0}^2 \log^i y \sum_{d|h} d^{-1} (\alpha_{i0} + \alpha_{i1} \log d + \alpha_{i2} \log^2 d)$$

for certain constants α_{ij} , and the remainder $E_h(y)$ satisfies

$$(2.15) \quad E_h(y) \ll y^{5/6+\varepsilon}$$

uniformly for $1 \leq h \leq y^{5/6}$. In particular $\alpha_{20} = 6\pi^{-2}$, $\alpha_{21} = \alpha_{22} = 0$. We note that $I_h(y)$ is roughly of order $y \log^2 y$. In our proof of Theorem 2 in §3 we shall need $I'_h(y)$, the derivative of $I_h(y)$. By (2.14)

$$(2.16) \quad I'_h(y) = a_2(h) \log^2 y + a_1(h) \log y + a_0(h)$$

where

$$(2.17) \quad \begin{aligned} a_2(h) &= 6\pi^{-2} \sum_{d|h} d^{-1} , \\ a_1(h) &= \sum_{d|h} d^{-1} (12\pi^{-2} + \alpha_{10} + \alpha_{11} \log d + \alpha_{12} \log^2 d) , \\ a_0(h) &= \sum_{d|h} d^{-1} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) \log^j d . \end{aligned}$$

For any $y > 0$, $Q > 3$ let

$$(2.18) \quad \xi(y, Q) = \sum_{h \leq y} h^{-1} (4a_2(h) \log^2 Qh + 2a_1(h) \log Qh + a_0(h)) .$$

LEMMA 4. *We have*

$$\begin{aligned} \xi(y, Q) = & \frac{4}{3} \log^3 Q y + c_3 \log^2 Q y - \frac{4}{3} \log^3 Q + c_4 \log^2 Q + c_5 \log Q \\ & + c_6 \log y + c_7 + \mathcal{O}(y^{-1} \log^3 y \log^2 Q y) . \end{aligned}$$

Proof. In the argument below we use the symbol c to denote a certain constant which may not be the same at each occurrence.

Firstly, for $j = 0, 1, 2$ there are constants $\beta_0, \beta_1, \beta_2$ such that

$$(2.19) \quad \sum_{h \leq y} a_j(h) = \beta_j y + B_j(y)$$

with $B_j(y) \ll \log^3 y$. (Note $B_j(1^-) = -\beta_j$). Indeed, by (2.17),

$$\begin{aligned} \sum_{h \leq y} a_0(h) &= \sum_{d \leq y} d^{-1} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) (\log^j d) (y d^{-1} + \mathcal{O}(1)) \\ &= y \sum_{d \leq y} d^{-2} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) \log^j d + \mathcal{O}\left(\sum_{d \leq y} d^{-1} \log^2 d\right) \\ &= y \sum_{d=1}^{\infty} d^{-2} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) \log^j d + \mathcal{O}\left(y \sum_{d > y} d^{-2} \log^2 d\right) + \\ &\quad + \mathcal{O}(\log^3 y) \\ &= \beta_0 y + \mathcal{O}(\log^3 y) \end{aligned}$$

with

$$\beta_0 = \sum_{d=1}^{\infty} d^{-2} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) \log^j d .$$

Similar argument establishes (2.19) for $j = 1$ and 2 . Further we find that $\beta_2 = 1$.

Next by Riemann Stieltjes integration and (2.19), we have

$$\begin{aligned}
& \sum_{h \leq y} a_2(h) h^{-1} \log^2 Q h \\
&= \int_1^y t^{-1} \log^2 Q t dt + [t^{-1} \log^2 Q t B_2(t)]_1^y \\
&\quad - \int_1^y B_2(t) t^{-2} (2 \log Q t - \log^2 Q t) dt \\
&= \frac{1}{3} (\log^3 Q y - \log^3 Q) + \log^2 Q + \mathcal{O}(y^{-1} \log^3 y \log^2 Q y) \\
&\quad - \int_1^y B_2(t) t^{-2} (-\log^2 Q + 2(1 - \log t) \log Q + 2 \log t - \log^2 t) dt \\
&= \frac{1}{3} (\log^3 Q y - \log^3 Q) + \log^2 Q + c \log^2 Q + c \log Q + c \\
&\quad + \mathcal{O}\left(\int_y^\infty (\log^3 t) t^{-2} (\log^2 Q + \log^2 t) dt\right) + \mathcal{O}(y^{-1} \log^3 y \log^2 Q y) \\
&= \frac{1}{3} (\log^3 Q y - \log^3 Q) + c \log^2 Q + c \log Q + c + \mathcal{O}(y^{-1} \log^3 y \log^2 Q y) .
\end{aligned}$$

In the same way, we find that

$$\begin{aligned}
\sum_{h \leq y} a_1(h) h^{-1} \log Q h &= \frac{1}{2} \beta_1 (\log^2 Q y - \log^2 Q) + c \log Q + c + \\
&\quad + \mathcal{O}(y^{-1} \log^3 y \log Q y)
\end{aligned}$$

and

$$\sum_{h \leq y} a_0(h) h^{-1} = \beta_0 \log y + c + \mathcal{O}(y^{-1} \log^3 y) .$$

Collecting all these in (2.18) our lemma follows.

Lastly we evaluate some integrals involving the function $g(\nu)$.

LEMMA 5. *We have*

$$\begin{aligned}
\int_0^\infty g(\nu) d\nu &= 0 , \\
\int_0^\infty g(\nu) \log \nu d\nu &= -\sqrt{\pi} 2^{-7/2} .
\end{aligned}$$

Proof. It is known that [13, §13.24]

$$\int_0^\infty J_k(\nu) \nu^{s-k-1} d\nu = \Gamma\left(\frac{s}{2}\right) 2^{s-k-1} / \Gamma\left(k - \frac{s}{2} + 1\right)$$

for $0 < \operatorname{Re} s < \operatorname{Re} k + 1/2$. Hence

$$\begin{aligned} \int_0^\infty (\nu^{-k} J_k(\nu) - (2k+1) \nu^{-k-1} J_{k+1}(\nu)) \nu^s d\nu \\ = -s 2^{s-k-1} \Gamma\left(\frac{s+1}{2}\right) / \Gamma\left(k - \frac{s}{2} + \frac{3}{2}\right) \end{aligned}$$

for $-1 < \operatorname{Re} s < \operatorname{Re} k - 1/2$. Setting $k = 3/2$ and in view of (2.5) we have

$$(2.20) \quad \int_0^\infty g(\nu) \nu^s d\nu = -s 2^{s-5/2} \Gamma\left(\frac{s+1}{2}\right) / \Gamma\left(3 - \frac{s}{2}\right)$$

for $-1 < \operatorname{Re} s < 1$. On putting $s = 0$ we get $\int_0^\infty g(\nu) d\nu = 0$. The remaining integral is equal to

$$\frac{d}{ds} \left(\int_0^\infty g(\nu) \nu^s d\nu \right) \Big|_{s=0}$$

which can be evaluated by differentiating the right hand side of (2.20).

3. Proof of Theorem 2

We shall now complete the proof of Theorem 2 by evaluating the double sum

$$(3.1) \quad T = \sum_{m < n \leq M} u_{m,n}$$

in Lemma 3, where

$$u_{m,n} = d(m)d(n)(mn)^{-3/4} g(4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}) .$$

In view of Lemma 3, we can allow errors of order up to $X^{-1/2}$ in the course of our analysis.

First of all, we consider those terms $u_{m,n}$ for which $m < n/2$. In this case $\sqrt{n} - \sqrt{m} \asymp \sqrt{n}$ so that, by (2.6)

$$g(4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}) \ll (nX)^{-1}.$$

The contribution to T from these $u_{m,n}$ is therefore

$$\ll X^{-1} \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4}n^{-1} \ll X^{-1},$$

which is acceptable.

For the remaining terms $u_{m,n}$ in T , we write $n = m + h$ with $1 \leq h \leq m$. Then

$$T = \sum_{h \leq M/2} \sum_{h \leq m \leq M-h} u_{m,m+h} + \mathcal{O}(X^{-1}).$$

For $h \leq m$, we have $4\pi(\sqrt{m+h} - \sqrt{m})\sqrt{X} \asymp 2\pi h\sqrt{X/m}$ so that, by (2.6) again

$$(3.2) \quad g(4\pi(\sqrt{m+h} - \sqrt{m})\sqrt{X}) \ll mh^{-2}X^{-1}$$

and each term $u_{m,m+h}$ satisfies

$$u_{m,m+h} \ll M^\varepsilon m^{-3/2}mh^{-2}X^{-1}.$$

Thus, the contribution to T from those $u_{m,m+h}$ with $h > \sqrt{M}$ is $\ll X^{-1}M^\varepsilon$ and the error caused by extending the upper limit for the summation on m to M is $\mathcal{O}(X^{-1}M^{-1/2+\varepsilon})$. Hence we have

$$T = \sum_{h \leq \sqrt{M}} \sum_{h \leq m \leq M} u_{m,m+h} + \mathcal{O}(X^{-1}M^\varepsilon).$$

For simplicity let

$$D_h = h^2XL^{-8}.$$

Then we can further write

$$(3.3) \quad \begin{aligned} T &= \sum_{h \leq \sqrt{M}} \sum_{h \leq m \leq \min(D_h, M)} + \sum_{h \leq X^3L^4} \sum_{D_h < m \leq M} + \mathcal{O}(X^{-1}M^\varepsilon) \\ &= \sum_1 + \sum_2 + \mathcal{O}(X^{-1}M^\varepsilon), \end{aligned}$$

say. Using the same bound (3.2), each term $u_{m,m+h}$ in $\sum_1$ is

$$\begin{aligned} &\ll (d(m)^2 + d(m+h)^2)(m(m+h))^{-3/4}mh^{-2}X^{-1} \\ &\ll d(m)^2m^{-1/2}h^{-2}X^{-1} + d(m+h)^2(m+h)^{-1/2}h^{-2}X^{-1}, \end{aligned}$$

since $m+h \asymp m$ for $1 \leq h \leq m$. An application of the well-known estimate

$$\sum_{m \leq y} d(m)^2m^{-1/2} \ll \sqrt{y} \log^3 y \quad \text{for } y > 1,$$

then yields

$$\sum_1 \ll X^{-1} \sum_{h \leq \sqrt{M}} h^{-2} \sqrt{\min(D_h, M)} \log^3 M \ll X^{-1/2}.$$

Putting this into (3.3), we have

$$\begin{aligned} (3.4) \quad T &= \sum_{h \leq X^3 L^4} \sum_{D_h < m \leq M} d(m)d(m+h)(m(m+h))^{-3/4} g(\theta_{m,m+h}) \\ &\quad + \mathcal{O}(X^{-1/2}) \end{aligned}$$

with $\theta_{m,m+h} = 4\pi(\sqrt{m+h} - \sqrt{m})\sqrt{X}$.

Next, we transform the above inner sum over m into an integral. By (2.12), (2.13) and Riemann Stieltjes integration we have

$$\begin{aligned} (3.5) \quad \sum_{D_h < m \leq M} &= \int_{D_h}^M (y(y+h))^{-3/4} g(\theta_{y,y+h}) d\psi_h(y) \\ &= \int_{D_h}^M (y(y+h))^{-3/4} g(\theta_{y,y+h}) I'_h(y) dy \\ &\quad + \left[(y(y+h))^{-3/4} g(\theta_{y,y+h}) E_h(y) \right]_{D_h}^M \\ &\quad - \int_{D_h}^M E_h(y) \frac{d}{dy} \{ (y(y+h))^{-3/4} g(\theta_{y,y+h}) \} dy \\ &= W_1(h) + W_2(h) + W_3(h), \end{aligned}$$

say. We bound $W_2(h)$ by using (2.15) and the trivial estimate $g(\nu) \ll 1$. Whence

$$W_2(h) \ll M^{-3/2} M^{5/6+\varepsilon} + D_h^{-3/2} D_h^{5/6+\varepsilon} \ll D_h^{-2/3+\varepsilon} \ll h^{-4/3} X^{-2/3+\varepsilon}.$$

For $W_3(h)$, by [13, §3.2] we have

$$g'(\nu) = -\nu^{-3/2} J_{5/2}(\nu) + 4\nu^{-5/2} J_{7/2}(\nu) \ll \nu \quad \text{for } \nu \geq 0 ,$$

since $J_k(\nu) \ll \nu^k$. Hence, by $g(\nu) \ll 1$ and (2.15) we have

$$\begin{aligned} W_3(h) &\ll \int_{D_h}^M y^{5/6+\varepsilon} \left\{ y^{-5/2} + y^{-3/2} |\theta_{y,y+h}| \left| \frac{d}{dy} \theta_{y,y+h} \right| \right\} dy \\ &\ll \int_{D_h}^M \{ y^{-5/3+\varepsilon} + y^{-2/3+\varepsilon} h y^{-1/2} X^{1/2} h y^{-3/2} X^{1/2} \} dy \\ &\ll h^{-4/3} X^{-2/3+\varepsilon} . \end{aligned}$$

In view of (3.4) and (3.5), the contribution to T from $W_2(h)$ and $W_3(h)$ is therefore

$$\ll \sum_{h \leq X^3 L^4} h^{-4/3} X^{-2/3+\varepsilon} \ll X^{-2/3+\varepsilon} ,$$

which is again acceptable. Thus,

$$(3.6) \quad T = \sum_{h \leq X^3 L^4} \int_{D_h}^M (y(y+h))^{-3/4} g(\theta_{y,y+h}) I'_h(y) dy + \mathcal{O}(X^{-1/2}) .$$

To evaluate the inner integral, we begin by making the change of variable

$$\omega = \theta_{y,y+h} = 4\pi(\sqrt{y+h} - \sqrt{y})\sqrt{X} .$$

Then

$$y = 4\pi^2 X \omega^{-2} h^2 - \frac{1}{2} h + (64\pi^2 X)^{-1} \omega^2 = 4\pi^2 X \omega^{-2} h^2 (1 + \mathcal{O}(\omega^2 X^{-1} h^{-1})) ,$$

so that

$$\begin{aligned} (y(y+h))^{-3/4} &= (4\pi^2 X \omega^{-2} h^2 - (64\pi^2 X)^{-1} \omega^2)^{-3/2} \\ &= (2\pi h)^{-3} X^{-3/2} \omega^3 (1 + \mathcal{O}(\omega^4 X^{-2} h^{-2})) \end{aligned}$$

and

$$\frac{dy}{d\omega} = -8\pi^2 X \omega^{-3} h^2 (1 + \mathcal{O}(\omega^4 X^{-2} h^{-2})) .$$

Moreover, by (2.16)

$$I'_h(y) = 4a_2(h) \log^2(2\pi\sqrt{X}\omega^{-1}h) + 2a_1(h) \log(2\pi\sqrt{X}\omega^{-1}h) + a_0(h) \\ + \mathcal{O}(\omega^2 X^{-1} h^{-1} (|a_2(h)|L + |a_1(h)|)) .$$

Set

$$(3.7) \quad u_1 = 4\pi(\sqrt{M+h} - \sqrt{M})\sqrt{X} = 2\pi h X^{-3} + \mathcal{O}(h^2 X^{-10})$$

and

$$(3.8) \quad u_2 = 4\pi(\sqrt{D_h+h} - \sqrt{D_h})\sqrt{X} = 2\pi L^4 + \mathcal{O}(h^{-1} X^{-1} L^{12}) .$$

Then with the help of all these estimates we find that

$$\int_{D_h}^M (y(y+h))^{-3/4} g(\theta_{y,y+h}) I'_h(y) dy \\ = \frac{1}{\pi\sqrt{X}} \int_{u_1}^{u_2} g(\omega) h^{-1} \{4a_2(h) \log^2(2\pi\sqrt{X}\omega^{-1}h) \\ + 2a_1(h) \log(2\pi\sqrt{X}\omega^{-1}h) + a_0(h)\} d\omega + \mathcal{O}(h^{-2} X^{-3/2+\varepsilon}) .$$

In obtaining the above $\mathcal{O}$ -term, we have used $g(\omega) \ll 1$, (3.8) and the observation that $a_j(h) \ll \log^3 h \ll L^3$. The integration limits u_1 and u_2 can be replaced by $2\pi h X^{-3}$ and $2\pi L^4$ respectively, since the error thus caused is

$$\ll X^{-1/2} h^{-1} L^5 (h^2 X^{-10} + h^{-1} X^{-1} L^{12}) \ll h X^{-21/2} L^5 + h^{-2} X^{-3/2} L^{17} \\ \ll h^{-2} X^{-3/2+\varepsilon} ,$$

by (3.7) and (3.8). Collecting these into (3.6), we get

$$T = \frac{1}{\pi\sqrt{X}} \sum_{h \leq X^3 L^4} \int_{2\pi h X^{-3}}^{2\pi L^4} g(\omega) h^{-1} \{4a_2(h) \log^2(2\pi\sqrt{X}\omega^{-1}h) \\ + 2a_1(h) \log(2\pi\sqrt{X}\omega^{-1}h) + a_0(h)\} d\omega + \mathcal{O}(X^{-1/2}) .$$

Next we interchange the summation and integration. In view of (2.18) we have

$$(3.9) \quad T = \frac{1}{\pi\sqrt{X}} \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega) \xi((2\pi)^{-1} \omega X^3, 2\pi\sqrt{X}\omega^{-1}) d\omega + \mathcal{O}(X^{-1/2}) .$$

By Lemma 4, and after some simplifications, we have

$$\xi((2\pi)^{-1}\omega X^3, 2\pi\sqrt{X}\omega^{-1}) = \log \omega \log^2 X + (c_8 \log \omega + c_9 \log^2 \omega) \log X + \\ + c_{10} \log \omega + c_{11} \log^2 \omega + c_{12} \log^3 \omega + \Phi(X) + \mathcal{O}(\omega^{-1} X^{-3} L^5),$$

where $\Phi(X) = c_{13} \log^3 X + c_{14} \log^2 X + c_{15} \log X + c_{16}$ and $c_8, c_9, \dots, c_{16}$ are certain constants. Finally inserting this into (3.9) we get

$$(3.10) \quad T = \frac{1}{\pi\sqrt{X}} \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega) \{ \log \omega \log^2 X + (c_8 \log \omega + c_9 \log^2 \omega) \log X \\ + c_{10} \log \omega + c_{11} \log^2 \omega + c_{12} \log^3 \omega + \Phi(X) \} d\omega + \mathcal{O}(X^{-1/2}).$$

It remains to evaluate the integrals

$$K_j = \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega) \log^j \omega d\omega$$

for $j = 0, 1, 2, 3$. Writing

$$K_j = \int_0^\infty g(\omega) \log^j \omega d\omega - \int_0^{2\pi X^{-3}} g(\omega) \log^j \omega d\omega - \int_{2\pi L^4}^\infty g(\omega) \log^j \omega d\omega,$$

we see, by (2.6), that the last two integrals are bounded by $X^{-3} L^j$ and L^{-4+j} respectively. Hence, by Lemma 5 we have

$$K_0 \ll L^{-4}, \quad K_1 = -\sqrt{\pi} 2^{-7/2} + \mathcal{O}(L^{-3})$$

and by (2.6),

$$K_2, K_3 = \text{constant} + \mathcal{O}(L^{-1}).$$

When these are inserted into (3.10) we obtain

$$T = -2^{-3} (2\pi X)^{-1/2} \log^2 X + c_{17} X^{-1/2} \log X + \mathcal{O}(X^{-1/2}),$$

and Theorem 2 now follows from (3.1) and Lemma 3.

REFERENCES

- [1] D. R. HEATH-BROWN, *The fourth power moment of the Riemann zeta-function*, Proc. London Math. Soc. **38** (3) (1979), 385–422.
- [2] D. R. HEATH-BROWN and K.M. TSANG, *Sign changes of $E(T)$, $\Delta(x)$ and $P(x)$* , J. Number Theory **49** (1994), 73–83.
- [3] A. IVIĆ, *The Riemann zeta-function*, Wiley, New York-Brisbane-Singapore 1985.
- [4] A. IVIĆ, *Lectures on mean values of the Riemann zeta-function*, Springer Verlag, Berlin-Heidelberg-New York-Tokyo, 1991.
- [5] H. IWANIEC and C.J. MOZZOCHI, *On the divisor and circle problems*, J. Number Theory **29** (1988), 60–93.
- [6] T. MEURMAN, *On the mean square of the Riemann zeta-function*, Quart. J. Math. (Oxford) (2)**38**, (1987), 337–343.
- [7] E. PREISSMANN, *Sur la moyenne quadratique du terme de reste du problème du cercle*, C.R. Acad. Sci. Paris Sér. I **306** (1988), 151–154.
- [8] E. C. TITCHMARSH, *The theory of the Riemann zeta-function*, (2nd ed. revised by D.R. HEATH-BROWN), Clarendon Press, Oxford, 1986.
- [9] K. C. TONG, *On divisor problems I*, Acta Math. Sinica **5**, no.3 (1955), 313–324.
- [10] K. C. TONG, *On divisor problems III*, Acta Math. Sinica **6**, no.4 (1956), 515–541.
- [11] K. TSANG, *A mean value theorem in the dirichlet divisor problem*, preprint.
- [12] G. F. VORONOI, *Sur une fonction transcendante et ses applications à la sommation de quelques séries*, Ann. École Normale **21**(3) (1904), 207–268 ; 459–534.
- [13] G. N. WATSON, *A treatise on the theory of Bessel functions*, 2nd edition, Cambridge University Press, 1962.

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