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Irrationality of quick convergent series

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There are many papers concerning the irrationality of infinite series. Erdős [4] proved that if the sequence \(\{a_n\}_{n=1}^{\infty}\) of positive integers converges quickly to infinity, then the series \(\sum_{n=1}^{\infty} 1/a_n\) is an irrational number.

The author [7] defined the irrational sequences and proved criterion for them. Another result is due to Erdős and Strauss [5]. They proved that if \(\{a_n\}_{k=1}^{\infty}\) is a sequence of positive integers with \(\limsup_{n \to \infty} a_1 \cdots a_n/a_{n+1} < \infty\) and \(\limsup_{n \to \infty} a_n^2/a_{n+1} \leq 1\), then the number \(\sum_{n=1}^{\infty} 1/a_n\) is rational if and only if \(a_{n+1} = a_n^2 - a_n + 1\) holds for every \(n > n_0\). Sándor [8] proved that if \(\{a_n\}_{n=1}^{\infty}\) and \(\{b_n\}_{n=1}^{\infty}\) are two sequences of positive integers such that \(\limsup_{n \to \infty} a_n/(a_1 \cdots a_{n-1}b_n) = \infty\) and \(\liminf_{n \to \infty} a_n b_{n-1}/(a_{n-1} b_n) > 1\), then the number \(\sum_{n=1}^{\infty} b_n/a_n\) is irrational.

Finally Badea [1] proved that if \(\{a_n\}_{n=1}^{\infty}\) and \(\{b_n\}_{n=1}^{\infty}\) are two sequences of positive integers such that \(b_{n+1} > (b_n^2 - b_n)a_{n+1}/a_n + 1\), then the sum \(\sum_{n=1}^{\infty} a_n/b_n\) is an irrational number. Later he generalized his result ([2]).

In this paper we will generalize Badea’s result in another way and prove the following theorem.
THEOREM. Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be two sequences of positive integers. If there is a natural number \( m \) such that the following three inequalities hold for every \( n > n_0 \)

(1) \( b_n > m + 1 \)

\[
(2) \quad b_n \sum_{k=1}^{m} (-1)^k \binom{m}{k} \left( \prod_{j=n-m}^{n-k-1} b_j \right) \sum_{j=m-k}^{m-1} \frac{a_{n-m-j}}{b_{n-m-j}} >
\]

\[
> -a_n \sum_{k=0}^{m} (-1)^k \binom{m}{k} \prod_{j=n-m}^{n-1-k} b_j + \sum_{i=1}^{m} \sum_{k=0}^{m} (-1)^{i+k+1} \text{sgn}(k+1-i) \binom{m}{i} \times
\]

\[
\times \left( \binom{m}{k} \prod_{i=1}^{n-i} b_s / \prod_{i=1}^{n-1-k} b_s \right)^{j= \max (m-i, m-k-1)} \sum_{j= \min (m-i, m-k-1)+1}^{m} \frac{a_{n-m-j}}{b_{n-m-j}}
\]

and

(3)

\[
(3) \quad b_n \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} \left( \prod_{j=n-m-1}^{n-k-1} b_j \right) \sum_{j=m+1-k}^{m} \frac{a_{n-m-j-1}}{b_{n-m-j-1}} <
\]

\[
< -a_n \sum_{k=0}^{m+1} (-1)^k \binom{m+1}{k} \prod_{j=n-m-1}^{n-1-k} b_j + \sum_{i=1}^{m+1} \sum_{k=0}^{m+1} (-1)^{i+k+1} \text{sgn}(k+1-i) \times
\]

\[
\times \left( \binom{m+1}{i} \binom{m+1}{k} \prod_{i=1}^{n-i} b_s / \prod_{i=1}^{n-1-k} b_s \right)^{j= \max (m+1-i, m-k)} \sum_{j= \min (m+1-i, m-k)+1}^{m+1} \frac{a_{n-m-j-1}}{b_{n-m-j-1}}
\]

then the number \( A = \sum_{n=1}^{\infty} \frac{a_n}{b_n} \) is irrational.

Proof: For the sake of simplicity we will suppose that (1) – (3) hold for every \( n \). (If not, we define \( a'_n = a_{n+m+n_0} \), \( b'_n = b_{n+m+n_0} \) for every \( n = \)
1, 2, ⋯ and these two sequences \( \{ a'_n \}_{n=1}^{\infty} \) and \( \{ b'_n \}_{n=1}^{\infty} \) satisfy then our above requirements.)

Let us denote

(4) \[ B_n = B_{n,0} = \prod_{i=1}^{n} b_i \]

(5) \[ A_n = A_{n,0} = B_n \sum_{i=1}^{n} a_i/b_i \]

\[ B_{n,i} = B_{n,i-1} - B_{n-1,i-1} \quad i = 1, \cdots, m + 1 \]

\[ A_{n,i} = A_{n,i-1} - A_{n-1,i-1} \quad i = 1, \cdots, m + 1 \]

One can prove by induction that

(6) \[ B_{n,i} = \sum_{j=0}^{i} \binom{i}{j} B_{n-j} (-1)^j \]

and

(7) \[ A_{n,i} = \sum_{j=0}^{i} \binom{i}{j} A_{n-j} (-1)^j \]

hold for \( i = 0, 1, \cdots, m + 1 \). (1) and (4) yield

(8) \[ \binom{i}{j} B_{n-j} - \binom{i}{j+1} B_{n-j-1} = \binom{i}{j} B_{n-j-1} (b_{n-j} - \frac{i-j}{j+1}) > 0 \]

and

(9) \[ \binom{i}{j} A_{n-j} - \binom{i}{j+1} A_{n-j-1} = \binom{i}{j} (A_{n-j} - \frac{i-j}{j+1} A_{n-j-1}) = \]

\[ = \binom{i}{j} B_{n-j-1} \left( a_{n-j} + (b_{n-j} - \frac{i-j}{j+1} \sum_{k=1}^{n-j-1} a_k/b_k) \right) > 0 \]
for every natural number \( n \). Then (4) - (9) imply that \( B_{n,i} > 0 \) and \( A_{n,i} > 0 \) for every positive integer \( n \) and \( i = 0, 1, \ldots, m - 1 \).

First we will prove that

\[
A_{n,m}/B_{n,m} < A_{n+1,m}/B_{n+1,m} < \cdots
\]

and secondly

\[
A_{n,m+1}/B_{n,m+1} > A_{n+1,m+1}/B_{n+1,m+1}
\]

(11) implies that there is a number \( c \geq 0 \) such that

\[
c = \lim_{n \to \infty} A_{n,m+1}/B_{n,m+1}.
\]

Using the famous theorem of Stolz (see e.g. [6]), we obtain

\[
A = \lim_{n \to \infty} A_n/B_n = \cdots = \lim_{n \to \infty} A_{n,m+1}/B_{n,m+1} = c.
\]

On the other hand (10), (11), (12) and Brun’s Theorem (see e.g. [3]) imply the irrationality of the number \( A \).

Now we will prove (10) and (11). Using (4) and (5) we have

\[
\frac{A_{n,m}}{B_{n,m}} - \frac{A_{n-1,m}}{B_{n-1,m}} = \frac{\sum_{i=0}^{m} \binom{m}{i} A_{n-i}(-1)^i}{\sum_{i=0}^{m} \binom{m}{i} B_{n-i}(-1)^i} - \frac{\sum_{i=0}^{m} \binom{m}{i} A_{n-1-i}(-1)^i}{\sum_{i=0}^{m} \binom{m}{i} B_{n-1-i}(-1)^i}
\]

\[
= \frac{\sum_{i=0}^{m} (-1)^i \binom{m}{i} B_{n-i}(A_{n-m}/B_{n-m} + \sum_{j=1}^{m-i} a_{m+j}/b_{n-m+j})}{\sum_{i=0}^{m} \binom{m}{i} B_{n-i}(-1)^i} - \frac{\sum_{i=0}^{m} (-1)^i \binom{m}{i} B_{n-i-1}(A_{n-m-1}/B_{n-m-1} + \sum_{j=1}^{m-i} a_{n-m-j-1}/b_{n-m+j-1})}{\sum_{i=0}^{m} (-1)^i \binom{m}{i} B_{n-i-1}}
\]
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\[
\begin{align*}
\sum_{i=0}^{m} (-1)^i \binom{m}{i} B_{n-i} \sum_{j=0}^{m-i} a_{n-m+j}/b_{n-m+j} &= \\
\sum_{i=0}^{m} (-1)^i \binom{m}{i} B_{n-i} - \\
\sum_{i=0}^{m} (-1)^i \binom{m}{i} B_{n-i-1} \sum_{j=1}^{m-i} a_{n-m+j-1}/b_{n-m+j-1} &= \\
\sum_{i=0}^{m} (-1)^i \binom{m}{i} B_{n-i-1} = \\
\sum_{i=0}^{m} \sum_{k=0}^{m} (-1)^{i+k} \binom{m}{i} \binom{m}{k} B_{n-i} B_{n-k-1} \left( \sum_{j=0}^{m-i} a_{n-m+j}/b_{n-m+j} - \\
- \sum_{s=1}^{m-k} a_{n-m-1+s}/b_{n-m-1+s} \right) / (B_{n,m} B_{n-1,m}) &= \\
= \left( B_n \sum_{k=0}^{m} (-1)^k \binom{m}{k} B_{n-1-k} \sum_{j=m-k}^{m} a_{n-m+j}/b_{n-m+j} - \\
- \sum_{i=1}^{m} \sum_{k=0}^{m} (-1)^{i+k+1} \binom{m}{i} \binom{m}{k} B_{n-i} B_{n-k-1} \left( \sum_{j=0}^{m-i} a_{n-m+j}/b_{n-m+j} - \\
- \sum_{s=1}^{m-k} a_{n-m+s-1}/b_{n-m+s-1} \right) / (B_{n,m} B_{n-1,m}) &= \\
= \left( B_n \sum_{k=1}^{m} (-1)^k \binom{m}{k} B_{n-1-k} \sum_{j=m-k}^{m-1} a_{n-m+j}/b_{n-m+j} + \\
+ B_n a_n \sum_{k=0}^{m} (-1)^k \binom{m}{k} B_{n-k-1} - \sum_{i=1}^{m} \sum_{k=0}^{m} (-1)^{i+k+1} \binom{m}{i} \binom{m}{k} B_{n-i} B_{n-k-1} - \\
\times \max(m-i, m-k-1) \sum_{j=\min(m-i, m-k-1)+1}^{\max(m-i, m-k-1)} \text{sgn}(k+1-i) a_{n-m+j}/b_{n-m+j} \right) / (B_{n,m} B_{n-1,m}).
\end{align*}
\]
(13) and (2) yield (10). Similarly (13) (if we substitute \( m + 1 \) instead of \( m \)) and (3) yield (11).

**Remark:** If we put \( m = 0 \) in the main theorem, then we receive \( b_n > 1, \ 0 > -a_n \) and \(-b_na_{n-1}/b_{n-1} < -a_n(b_{n-1} - 1) - a_{n-1}/b_{n-1}. \) Thus \( b_n > (b_{n-1}^2 - b_{n-1})a_n/a_{n-1} + 1 \) and this is the famous theorem due to Badea (see e.g. [1]).

Consequence 1: Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be two sequences of positive integers. If

\[
(14) \quad b_n > 2
\]

\[
(15) \quad b_n < (b_{n-1}^2 - b_{n-1})a_n/a_{n-1} + 1
\]

\[
(16) \quad b_n(-b_{n-1}a_{n-2} + 2b_{n-2}^2a_{n-1} - b_{n-2}a_{n-1}) >
\]

\[
> a_n b_{n-1} b_{n-2}(b_{n-1} b_{n-2} - 2b_{n-2} + 1) + 3a_{n-1} b_{n-2}^2 - 2b_{n-1} a_{n-2}
\]

\[
- 2b_{n-2} a_{n-1} + a_{n-2}
\]

hold for every \( n > n_0 \), then the number \( A = \sum_{n=1}^{\infty} a_n/b_n \) is irrational.

**Proof:** Let us put \( m = 1 \) in the main theorem. Then (14) is (1), (15) is (2) and (16) is (3).

Consequence 2: Let \( \{b_n\}_{n=1}^{\infty} \) be a sequence of positive integers such that \( b_1 > 2 \) and

\[
(17) \quad kb_{n-1}^2 - (3k - 1)b_{n-1} < b_n < kb_{n-1}^2 - kb_{n-1}
\]

hold for every \( n > n_0 \) where \( k \) is a positive integer. Then the number \( A = \sum_{n=1}^{\infty} k^n/b_n \) is irrational.

**Proof:** Let us put \( a_n = k^n \) in consequence 1. Then (17) immediately implies (15) and

\[
b_n > kb_{n-1}^2 - (3k - 1)b_{n-1} = kb_{n-1}(b_{n-1} - 3) + b_{n-1}.
\]

This and \( b_1 > 2 \) imply that the sequence \( \{b_n\}_{n=1}^{\infty} \) is increasing. Thus (15) is fulfilled too. Condition (16) can be rewritten in the following way
Let us define the sequence \( \{s_n\}_{n=1}^{\infty} \) of nonnegative integers such that
\[
(17) \implies 0 < s_n < (2k - 1)b_n - 1.
\]
Substituting (19) for (18) we obtain the equivalent inequality (21) with (18):
\[
(21) \quad (kb_{n-1}^2 - kb_{n-1} - s_n)(kb_{n-2}^2 + s_{n-1}) >
\]
\[
> k^2b_{n-1}b_{n-2}(b_{n-1}b_{n-2} - 2b_{n-2} + 1) + 3kb_{n-2}^2 - 2b_{n-1} - 2kb_{n-2} + 1.
\]
Carrying out the equivalent calculations step by step, we receive
\[
k^2b_{n-1}s_{n-1} - s_nb_{n-2}^2 - s_ns_{n-1} - kb_{n-1}s_{n-1} + k^2b_{n-1}b_{n-2}^2 - k^2b_{n-1}b_{n-2}
\]
\[
- 3kb_{n-2}^2 + 2b_{n-1} + 2kb_{n-2} - 1 > 0.
\]
Using (20) and the fact that \( \{b_n\}_{n=1}^{\infty} (b_1 > 2) \) is an increasing sequence, it is enough to prove that
\[
(22) \quad kb_{n-1} - (k - 1)kb_{n-1}b_{n-2} - Kb_{n-1}b_{n-2} > 0,
\]
where \( K \) is a suitable constant. (22) is equivalent with
\[
k^2b_{n-1} + k^2b_{n-2} > 0.
\]
(17) implies that
\[
(23) \quad -(3k - 1)b_{n-2} < b_{n-1} - kb_{n-2}.
\]
Because of (23), it is enough to prove that
\[
(24) \quad kb_{n-1}b_{n-2}^2 - K_1b_{n-1}b_{n-2} > 0,
\]
where \( K_1 \) is a suitable constant too. But (24) is true for every \( n > n_0 \). Thus (18) is right and the number \( A \) is irrational.

Examples: The numbers \( \sum_{n=1}^{\infty} 2^n/b_n \) and \( \sum_{n=1}^{\infty} 3^n/a_n \) where \( a_1 > 2, b_1 > 2, b_n = 2b_{n-1}^2 - 2b_{n-1} - 1 \) and \( a_n = 3a_{n-1}^2 - 3a_{n-1} - 4 \) are irrational.
REFERENCES


