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Journal de Théorie des Nombres de Bordeaux, tome 8, n° 2 (1996), p. 315-329

[<http://www.numdam.org/item?id=JTNB_1996__8_2_315_0>](http://www.numdam.org/item?id=JTNB_1996__8_2_315_0)

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Limit Theorem in the space of continuous functions for the Dirichlet polynomial related with the Riemann zeta-funtion

par ANTANAS LAURINČIKAS*

RÉSUMÉ. Dans cet article on prouve un théorème limite dans l'espace des fonctions continues pour le polynôme de Dirichlet

$$\sum_{m \leq T} \frac{d_{\kappa_T}(m)}{m^{\sigma_T + it}},$$

où $d_{\kappa_T}(m)$ sont les coefficients du développement en série de Dirichlet de la fonction $\zeta^{\kappa_T}(s)$ dans le demi-plan $\sigma > 1$, $\kappa_T = (2^{-1} \log l_T)^{-\frac{1}{2}}$, $\sigma_T = \frac{1}{2} + \frac{\log^2 l_T}{l_T}$, $l_T > 0$, $l_T \leq \log T$ et $l_T \rightarrow \infty$ lorsque $T \rightarrow \infty$.

ABSTRACT. A limit theorem in the space of continuous functions for the Dirichlet polynomial

$$\sum_{m \leq T} \frac{d_{\kappa_T}(m)}{m^{\sigma_T + it}},$$

where $d_{\kappa_T}(m)$ denote the coefficients of the Dirichlet series expansion of the function $\zeta^{\kappa_T}(s)$ in the half-plane $\sigma > 1$, $\kappa_T = (2^{-1} \ln l_T)^{-1/2}$, $\sigma_T = \frac{1}{2} + \frac{\ln^2 l_T}{l_T}$ and $l_T > 0$, $l_T \leq \ln T$ and $l_T \rightarrow \infty$ as $T \rightarrow \infty$, is proved.

Let s be a complex variable and $\zeta(s)$, as usual, denote the Riemann zeta-function. To study the distribution of values of the Riemann zeta-function the probabilistic methods can be used, and the obtained results usually are presented as the limit theorems of probability theory. The first theorems of this type were obtained in [1],[2], and they were proved in [3]-[5] using other methods. In modern terminology we can formulate it as follows. Let

Manuscrit reçu le 20 octobre 1993

*This paper has been written during the author's stay at the University Bordeaux I in the framework of the program "S&T Cooperation with Central and Eastern European Countries".

$\mathbf{C}$ be the complex space and let $\mathcal{B}(S)$ denote the class of Borel sets of the space S . Let $\text{meas}\{A\}$ be the Lebesgue measure of the set A and

$$\nu_T^t(\dots) = \frac{1}{T} \text{meas}\{t \in [0, T], \dots\}$$

where in place of dots we write the conditions which are satisfied by t . We define the probability measure

$$P_T(A) = \nu_T^t(\zeta(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbf{C})$$

THEOREM A. *For $\sigma > \frac{1}{2}$ there exists a probability measure P on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ such that P_T converges weakly to P as $T \rightarrow \infty$.*

More general results were obtained in [6]. Let M denote the space of functions meromorphic in the half-plane $\sigma > \frac{1}{2}$, equipped with the topology of uniform convergence on compacta. Define the probability measure

$$Q_T(A) = \nu_T^\tau(\zeta(s + i\tau) \in A), \quad A \in \mathcal{B}(M).$$

THEOREM B. *There exists a probability measure Q on $(M, \mathcal{B}(M))$ such that Q_T converges weakly to Q as $T \rightarrow \infty$.*

Note that the explicit form of the measure Q can be indicated, and, obviously, Theorem A is a corollary of Theorem B.

The situation is more complicated when σ depends on T and tends to $\frac{1}{2}$ as $T \rightarrow \infty$, or $\sigma = \frac{1}{2}$. It turns out that in this case some power norming is necessary. Let $l_T > 0$ and let l_T tend to infinity as $T \rightarrow \infty$, or $l_T = \infty$. We take

$$\tilde{\sigma}_T = \frac{1}{2} + \frac{1}{l_T}, \quad \kappa = \kappa_T = \begin{cases} (2^{-1} \log l_T)^{-1/2}, & l_T \leq \log T, \\ (2^{-1} \log \log T)^{-1/2}, & l_T \geq \log T. \end{cases}$$

The case $l_T = \infty$ corresponds to $\tilde{\sigma}_T = \frac{1}{2}$.

The function

$$w(\tau, k) \stackrel{\text{def}}{=} \int_{\mathbf{C} \setminus \{0\}} |s|^{i\tau} e^{ik \arg s} dP \quad \tau \in \mathbb{R}, k \in \mathbb{Z},$$

is called the characteristic transform of the probability measure P on the space $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ [7]. The lognormal probability measure on $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is defined by the characteristic transform

$$w(\tau, k) = \exp \left\{ -\frac{\tau^2}{2} - \frac{k^2}{2} \right\}.$$

THEOREM C. *The probability measure*

$$\nu_T^t(\zeta^{\kappa_T}(\tilde{\sigma}_T + it) \in A), \quad A \in \mathcal{B}(\mathbf{C}),$$

converges weakly to the lognormal probability measure as $T \rightarrow \infty$.

Here if $\zeta(s) \neq 0$, $a \in \mathbb{R}$, then $\zeta^a(s)$ is understood as $\exp\{a \log \zeta(s)\}$ where $\log \zeta(s)$ is defined by continuous displacement from the point $s = 2$ along the path joining the points 2, $2 + it$ and $\sigma + it$.

When $\tilde{\sigma}_T = \frac{1}{2}$ Theorem C was proved by A.Selberg (unpublished), see also [8], and for different form of l_T , it was obtained in [8]–[10], [5].

Now it arises the problem to obtain some results of the kind of Theorem C in the space of continuous functions.

Let $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere and let $d(s_1, s_2)$ be a metric on $\mathbf{C}_\infty$ given by the formulae

$$d(s_1, s_2) = \frac{2 |s_1 - s_2|}{\sqrt{1 + |s_1|^2} \sqrt{1 + |s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1 + |s|^2}}, \quad d(\infty, \infty) = 0.$$

Here $s, s_1, s_2 \in \mathbf{C}$. This metric is compatible with the topology of $\mathbf{C}_\infty$. Let $C(\mathbb{R}) = C(\mathbb{R}, \mathbf{C}_\infty)$ denote the space of continuous functions $f : \mathbb{R} \rightarrow \mathbf{C}_\infty$ equipped with the topology of uniform convergence on compacta. In this topology, sequence $\{f_n, f_n \in C(\mathbb{R})\}$ converges to the function $f \in C(\mathbb{R})$ if

$$d(f_{n(t)}, f(t)) \rightarrow 0$$

as $n \rightarrow \infty$ uniformly in t on compact subsets of $\mathbb{R}$.

The functional analogue of the probability measure in Theorem C is the measure

$$(1) \quad \nu_T^{\tilde{\tau}}(\zeta^{\kappa_T}(\tilde{\sigma}_T + it + i\tau) \in A), \quad A \in \mathcal{B}(C(\mathbb{R})).$$

Does this measure converge weakly as $T \rightarrow \infty$ to some probability measure on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$? At this moment this question is open and it seems to be very difficult.

In the proof of Theorem C an important role is played by the Dirichlet polynomial

$$S_u(s) = \sum_{m \leq u} \frac{d_\kappa(m)}{m^s}$$

where $d_\kappa(m)$ denote the coefficients of the Dirichlet series expansion of the function $\zeta^\kappa(s)$ in the half-plane $\sigma > 1$ (see [11], [12]). Therefore the aim of this paper is to prove the limit theorem in the space of continuous functions for $S_u(s)$. This theorem will be the first step to study the weak convergence of the probability measure (1).

Now let $l_T \leq \log T$, $\sigma_T = \frac{1}{2} + \frac{\log^2 l_T}{l_T}$, and let

$$P_{T,S_u}(A) = \nu_T^\tau(S_u(\sigma_T + it + i\tau) \in A), \quad A \in \mathcal{B}(C(\mathbb{R})).$$

Moreover we suppose that

$$(2) \quad l_{T+U} - l_T = \frac{BU}{T}$$

for all $U > 0$ as $T \rightarrow \infty$. Here B denotes a number (not always the same) which is bounded by a constant.

THEOREM *There exists a probability measure P on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$ such that P_{T,S_T} converges weakly to P as $T \rightarrow \infty$.*

Proof of the theorem is based on the following probability result. Let S_1 and S_2 be two metric spaces, and let $h : S_1 \rightarrow S_2$ be a measurable function. Then every probability measure P on $(S_1, \mathcal{B}(S_1))$ induces on $(S_2, \mathcal{B}(S_2))$ the unique probability measure Ph^{-1} defined by the equality $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_2)$.

Now let h and h_n be the measurable functions from S_1 into S_2 and

$$E = \{x \in S_1 : h_n(x_n) \not\rightarrow h(x) \text{ for some } x_n \xrightarrow{n \rightarrow \infty} x\}.$$

LEMMA1. *Let P and P_n be the probability measures on $(S_1, \mathcal{B}(S_1))$. Suppose that P_n converges weakly to P as $n \rightarrow \infty$ and that $P(E) = 0$. Then the measure $P_n h_n^{-1}$ converges weakly to Ph^{-1} as $n \rightarrow \infty$.*

Proof. This lemma is Theorem 5.5 from [13].

Let γ denote the unit circle on complex plane, that is $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. We put

$$\Omega = \prod_p \gamma_p$$

where $\gamma_p = \gamma$ for each prime p . With the product topology and point-wise multiplication the infinite-dimensional torus Ω is a compact Abelian topological group. Let P be a probability measure on $(\Omega, \mathcal{B}(\Omega))$.

The Fourier transform $g(\underline{k})$ of the measure P is defined by the formula

$$g(\underline{k}) = \int_{\Omega} \prod_p x_p^{k_p} dP.$$

Here $\underline{k} = (k_2, k_3, \dots)$ where only a finite number of integers k_p are distinct of zero, and $x_p \in \gamma$.

LEMMA 2. Let $\{P_n\}$ be a sequence of probability measures on $(\Omega, \mathcal{B}(\Omega))$ and let $\{g_n(\underline{k})\}$ be a sequence of corresponding Fourier transforms. Suppose that for every vector $\underline{k}$ the limit $g(\underline{k}) = \lim_{n \rightarrow \infty} g_n(\underline{k})$ exists. Then there exists a probability measure P on $(\Omega, \mathcal{B}(\Omega))$ such that P_n converges weakly to P as $n \rightarrow \infty$. Moreover, $g(\underline{k})$ is the Fourier transform of P .

Proof. The lemma is the special case of the continuity theorem for compact Abelian group, see, for example, [14].

Let

$$Q_T(A) = \nu_T((p_1^{i_T}, p_2^{i_T}, \dots) \in A), \quad A \in \mathcal{B}(\Omega).$$

LEMMA 3. The probability measure Q_T converges weakly to the Haar measure m on $(\Omega, \mathcal{B}(\Omega))$ as $T \rightarrow \infty$.

Proof. The Fourier transform $g_T(\underline{k})$ of the measure Q_T is given by

$$\begin{aligned} g_T(\underline{k}) &= \int_{\Omega} \prod_p x_p^{k_p} dQ_T = \frac{1}{T} \int_0^T \prod_{j=1}^{\infty} p_j^{ik_j \tau} d\tau = \\ &= \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ \frac{\exp\{iT \sum_{j=1}^{\infty} k_j \log p_j\} - 1}{iT \sum_{j=1}^{\infty} k_j \log p_j} & \text{if } \underline{k} \neq \underline{0}. \end{cases} \end{aligned}$$

Here $x_p \in \gamma$, $\underline{k} = (k_1, k_2, \dots)$. By definition of the Fourier transform of probability measure on $(\Omega, \mathcal{B}(\Omega))$, only a finite number of k_j are distinct from zero. Since the logarithms of prime numbers are linearly independent over the field of rational numbers, we find that

$$g_T(\underline{k}) \rightarrow \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0} \end{cases}$$

as $T \rightarrow \infty$. In view of Lemma 2, this proves the lemma.

We define the function $h_T : \Omega \rightarrow C(\mathbb{R})$ by the formula

$$(3) \quad h_T(t; e^{i\eta_1}, e^{i\eta_2}, \dots) = \sum_{k \leq T} \frac{d_\kappa(k)}{k^{\sigma_T + it} \prod_{p_j^{\alpha_j} \parallel k} e^{i\alpha_j \eta_j}}.$$

Here $p^\alpha \parallel k$ means that $p^\alpha \mid k$ but $p^{\alpha+1} \nmid k$. Then, clearly,

$$(4) \quad S_T(\sigma_T + it + i\tau) = h_T(t; p_1^{i\tau}, p_2^{i\tau}, \dots).$$

Let, for brevity,

$$h_T(t; e^{i\tau_1}, e^{i\tau_2}, \dots) = S_T(\sigma_T + it + i\tau),$$

and let

$$Z_{nk}(it, \tau) = S_{n+k}(\sigma_{n+k} + it + i\tau) - S_n(\sigma_n + it + i\tau).$$

Let K be a compact subset of $\mathbb{R}$. For every $\epsilon > 0$ we define the set A_{nk}^ϵ by

$$A_{nk}^\epsilon(K) = \{(e^{i\tau_1}, e^{i\tau_2}, \dots) : \sup_{t \in K} |Z_{nk}(it, \tau)| \geq \epsilon\}$$

and we put

$$A_k^\epsilon(K) = \bigcap_{l=1}^{\infty} \bigcup_{n>l} A_{nk}^\epsilon.$$

LEMMA 4. $m(A_k^\epsilon(K)) = 0$ for every $\epsilon > 0$, K , and $k \in \mathbb{N}$.

Proof. By the Chebyshev inequality

$$(5) \quad m(A_{nk}^\epsilon(K)) \leq \frac{1}{\epsilon^2} \int_{\Omega} \sup_{t \in K} |Z_{nk}(it, \tau)|^2 dm.$$

Using the Cauchy formula, we have that

$$Z_{nk}^2(it, \tau) = \frac{1}{2\pi i} \int_L \frac{Z_{nk}^2(z, \tau)}{z - it} dz$$

where L denotes the rectangle, enclosing the set $iK = \{ia, a \in K\}$, with the sides $\sigma = -\frac{1}{l_{n+k}} + it$, $\sigma = \frac{1}{l_{n+k}} + it$, and with two other sides parallel

to the real axis. Moreover, we suppose that the distance of L from the set iK is $\geq \frac{1}{l_{n+k}}$. From this equality it follows that

$$\sup_{t \in K} |Z_{nk}(it, \mathcal{I})|^2 = Bl_{n+k} \int_L |Z_{nk}(z, \mathcal{I})|^2 |dz|.$$

Hence, having in mind the inequality (5), we obtain that

$$\begin{aligned} m(A_{nk}^\epsilon) &= \frac{Bl_{n+k}}{\epsilon^2} \int_L |dz| \int_\Omega |Z_{nk}(z, \mathcal{I})|^2 dm = \\ (6) \quad &= \frac{Bl_{n+k} |L|}{\epsilon^2} \sup_{z \in L} \int_\Omega |Z_{nk}(z, \mathcal{I})|^2 dm \end{aligned}$$

where $|L|$ is the length of L . From the definitions of $Z_{nk}(z, \mathcal{I})$ and $S_n(\sigma_n + z + i\tau)$ we have that, for $z = u + iv$,

$$\begin{aligned} Z_{nk}(z, \mathcal{I}) &= \sum_{l \leq n+k} \frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u+iv} \prod_{p_j^{\alpha_j} \| l} e^{i\alpha_j \tau_j}} - \sum_{l \leq n} \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u+iv} \prod_{p_j^{\alpha_j} \| l} e^{i\alpha_j \tau_j}} = \\ &= \sum_{n < l \leq n+k} \frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u+iv} \prod_{p_j^{\alpha_j} \| l} e^{i\alpha_j \tau_j}} + \\ &+ \sum_{l \leq n} \left(\frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u+iv} \prod_{p_j^{\alpha_j} \| l} e^{i\alpha_j \tau_j}} - \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u+iv} \prod_{p_j^{\alpha_j} \| l} e^{i\alpha_j \tau_j}} \right) \stackrel{\text{def}}{=} V + W. \end{aligned}$$

Since

$$|a + b|^2 \leq 2(|a|^2 + |b|^2),$$

hence we find that

$$(7) \quad |Z_{nk}(z, \tau)|^2 \leq 2(|V|^2 + |W|^2).$$

The properties of the Haar measure m imply the equality

$$\begin{aligned} (8) \quad \int_\Omega |V|^2 dm &= \sum_{n < l \leq n+k} \frac{d_{\kappa_{n+k}}^2(l)}{l^{2(\sigma_{n+k}+u)}} + \sum_{\substack{n < l_1 \leq n+k \\ n < l_2 \leq n+k \\ l_1 \neq l_2}} \frac{d_{\kappa_{n+k}}(l_1) d_{\kappa_{n+k}}(l_2)}{l_1^{\sigma_{n+k}+u+iv} l_2^{\sigma_{n+k}+u-iv}} \times \\ &\times \int_\Omega \frac{\prod_{p_j^{\alpha_j} \| l_2} e^{i\alpha_j \tau_j}}{\prod_{p_j^{\alpha_j} \| l_1} e^{i\alpha_j \tau_j}} dm = \sum_{n < l \leq n+k} \frac{d_{\kappa_{n+k}}^2(l)}{l^{2(\sigma_{n+k}+u)}}. \end{aligned}$$

By a similar manner we find that

$$(9) \quad \int_{\Omega} |W|^2 dm = \sum_{l \leq n} \left(\frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u}} - \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u}} \right)^2.$$

From the definition of the contour L it follows that

$$(10) \quad -\frac{1}{l_{n+k}} \leq u \leq \frac{1}{l_{n+k}}$$

for $z = u + iv \in L$. Then (8) together with (2) and the well-known estimate

$$\sum_{m \leq x} \frac{1}{m} = \log x + \gamma_0 + \frac{B}{x},$$

where γ_0 is the Euler constant, yields

$$(11) \quad \begin{aligned} \int_{\Omega} |V|^2 dm &= Bn \frac{2 \log^2 l_{n+k} - 1}{l_{n+k}} \sum_{n < l \leq n+k} \frac{1}{l} = Be \frac{-\log n \log^2 l_n}{l_n} \left(1 + \frac{Bk}{n} \right) \times \\ &\times \left(\log \frac{n+k}{n} + \frac{B}{n} \right) = \frac{Bk}{n} e^{-c_1 \frac{\log n \log^2 l_n}{l_n}} \end{aligned}$$

for $n \geq n_0$. Here we have used the inequality $0 < d_{\kappa_{n+k}}(l) < 1$, $n \geq n_0$, which follows trivially from the multiplicativity of $d_{\kappa_{n+k}}(m)$ and from the inequality $0 < d_{\kappa_{n+k}}(p^\alpha) < 1$, $n \geq n_0$, implied by the formula [11], [12]

$$(12) \quad d_{\kappa}(p^\alpha) = \frac{\kappa(\kappa+1) \dots (\kappa+\alpha-1)}{\alpha!}.$$

From the assumption on l_T we deduce that, for $n \geq n_0$,

$$(13) \quad \begin{aligned} \sigma_{n+k} &= \sigma_n \left(1 + \frac{Bk}{n} \right), \\ \log l_{n+k} &= \log \left(l_n + \frac{Bk}{n} \right) = \log l_n \left(1 + \frac{Bk}{nl_n} \right) = \\ &= \log l_n + \frac{Bk}{nl_n} = \log l_n \left(1 + \frac{Bk}{nl_n \log l_n} \right). \end{aligned}$$

Thus,

$$\begin{aligned}\kappa_{n+k} &= (2^{-1} \log l_{n+k})^{-\frac{1}{2}} = \kappa_n \left(1 + \frac{Bk}{nl_n \log l_n}\right)^{-\frac{1}{2}} = \\ &= \kappa_n \left(1 + \frac{Bk}{nl_n \log l_n}\right) \stackrel{\text{def}}{=} \kappa_n(1 + r_{nk}).\end{aligned}$$

Consequently, in view of (12), for $n \geq n_0$,

$$\begin{aligned}d_{\kappa_{n+k}}(p^\alpha) &= \frac{\kappa_n(1 + r_{nk})(\kappa_n(1 + r_{nk}) + 1) \dots (\kappa_n(1 + r_{nk}) + \alpha - 1)}{\alpha!} = \\ &= \frac{\kappa_n(1 + r_{nk})(\kappa_n + 1) \left(1 + \frac{\kappa_n r_{nk}}{\kappa_n + 1}\right) \dots (\kappa_n + \alpha - 1) \left(1 + \frac{\kappa_n r_{nk}}{\kappa_n + \alpha - 1}\right)}{\alpha!} = \\ &= d_{\kappa_n}(p^\alpha) \prod_{j=1}^{\alpha} \left(1 + \frac{Br_{nk}}{j}\right) = \\ &= d_{\kappa_n}(p^\alpha)(1 + Br_{nk} \log \alpha).\end{aligned}$$

Hence, for $m \leq n$,

$$\begin{aligned}d_{\kappa_{n+k}}(m) &= \prod_{p^\alpha \parallel m} d_{\kappa_{n+k}}(p^\alpha) = \\ &= d_{\kappa_n}(m) = \prod_{p^\alpha \parallel m} (1 + Br_{nk} \log \alpha) = \\ &= d_{\kappa_n}(m) \exp\{Br_{nk} \sum_{p^\alpha \parallel m} \log \alpha\} = \\ &= d_{\kappa_n}(m) \exp\{Br_{nk} \log \log n \sum_{p^\alpha \parallel m} 1\} = \\ &= d_{\kappa_n}(m) \exp\{Br_{nk} \log \log n \log n\} = \\ (14) \quad &= d_{\kappa_n}(m)(1 + Br_{nk} \log \log n \log n).\end{aligned}$$

Therefore, from (9), (13) and (14) we have that

$$(15) \quad \int_{\Omega} |W|^2 dm = \frac{Bk^2 \log^2 n \log^2 \log n}{n^2 l_n^2 \log^2 l_n} \sum_{l \leq n} \frac{d_{\kappa_n}^2(l)}{l^{2(\sigma_n + u)}}.$$

Repeating the proof of Lemma 3 from [10] and taking into account (10), we see that

$$\sum_{l \leq n} \frac{d_{\kappa_n}^2(l)}{l^{2(\sigma_n+u)}} = B.$$

Consequently, this and (15) give the estimate

$$\int_{\Omega} |W|^2 dm = \frac{Bk^2 \log^2 n \log^2 \log n}{n^2 l_n^2 \log^2 l_n}.$$

From this, (6), (7) and (11) we find that

$$m(A_{nk}^{\epsilon}(K)) = \frac{Bk^2}{\epsilon^2} \left(\frac{1}{n} e^{-c_1 \frac{\log n \log^2 l_n}{l_n}} + \frac{\log^2 n \log^2 \log n}{n^2 l_n^2 \log^2 l_n} \right)$$

for every $\epsilon > 0$ and $k \in \mathbb{N}$. Thus it follows from the definition of the set $A_{nk}^{\epsilon}(K)$ that

$$\begin{aligned} m(A_k^{\epsilon}(K)) &= \lim_{l \rightarrow \infty} m\left(\bigcup_{n>l} A_{nk}^{\epsilon}(K)\right) = \\ &= \lim_{l \rightarrow \infty} \frac{Bk^2}{\epsilon^2} \sum_{n>l} \left(\frac{1}{n} e^{-c_1 \frac{\log n \log^2 l_n}{l_n}} + \frac{\log^2 n \log^2 \log n}{n^2 l_n^2 \log^2 l_n} \right) = 0. \end{aligned}$$

The lemma is proved.

Proof of Theorem. We will deduce the theorem from lemmas 1, 3 and 4. Let

$$(e^{i\tau_1(T)}, e^{i\tau_2(T)}, \dots)$$

converges to

$$(e^{i\tau_1}, e^{i\tau_2}, \dots)$$

as $T \rightarrow \infty$, and let E denote the set $\{(e^{i\tau_1}, e^{i\tau_2}, \dots)\}$ of elements of Ω such that

$$h_T(t; e^{i\tau_1(T)}, e^{i\tau_2(T)}, \dots)$$

does not converge to some function $h(t; e^{i\tau_1}, e^{i\tau_2}, \dots)$ as $T \rightarrow \infty$. In order to prove the theorem we must show that $m(E) = 0$. Since Ω is compact, it is separable. Consequently [13], $E \in \mathcal{B}(\Omega)$.

Let E_1 denote the set $\{(e^{i\tau_1}, e^{i\tau_2}, \dots)\}$ such that

$$h_T(t; e^{i\tau_1}, e^{i\tau_2}, \dots)$$

does not converge to some function $h(t; e^{i\tau_1}, e^{i\tau_2}, \dots)$ as $T \rightarrow \infty$. We will prove that $m(E_1) = 0$. First we consider the sequence $h_n(t; e^{i\tau_1}, e^{i\tau_2}, \dots)$.

Note that there exists a sequence $\{K_j\}$ of compact subsets of $\mathbb{R}$ such that

$$\mathbb{R} = \bigcup_{j=1}^{\infty} K_j,$$

$K_j \subset K_{j+1}$, and if K is compact of $\mathbb{R}$ then $K \subset K_j$ for some j . Let

$$\rho_j(f, g) = \sup_{t \in K_j} d(f(t), g(t))$$

for $f, g \in C(\mathbb{R})$. Then

$$\rho(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f, g)}{1 + \rho_j(f, g)}$$

is a metric in $C(\mathbb{R})$.

Since $C(\mathbb{R})$ is a complete metric space, we have that every fundamental sequence is convergent. Thus it follows from the definition of the fundamental sequence that

$$\begin{aligned} & m((e^{i\tau_1}, e^{i\tau_2}, \dots) : h_n(t; e^{i\tau_1}, e^{i\tau_2}, \dots) \not\rightarrow) = \\ & = m((e^{i\tau_1}, e^{i\tau_2}, \dots) : (e^{i\tau_1}, e^{i\tau_2}, \dots) \in \bigcup_{\epsilon > 0} \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_k^{\epsilon}(K_j)). \end{aligned}$$

Thus, by Lemma 4,

$$(16) \quad m((e^{i\tau_1}, e^{i\tau_2}, \dots) : h_n(t; e^{i\tau_1}, e^{i\tau_2}, \dots) \not\rightarrow) = 0.$$

From the definition of the function h_T , using the estimates of types (13) and (14), we find that

$$h_T(t; e^{i\tau_1}, e^{i\tau_2}, \dots) = \sum_{k \leq [T]} \frac{d_{\kappa_{[T]}}(k)}{k^{\sigma_{[T]} + it} \prod_{p_j^{\alpha_j} \parallel k} e^{i\alpha_j \tau_j}} + \frac{B}{T^{1/4}}$$

uniformly in $t \in \mathbb{R}$ and in $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$. Therefore, in view of (16), $m(E_1) = 0$.

We have shown that there exists a function h such that for almost all $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$

$$(17) \quad \sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T + it} \prod_{p_j^{\alpha_j} \| k} e^{i\alpha_j \tau_j}} \xrightarrow{T \rightarrow \infty} h(t; e^{i\tau_1}, e^{i\tau_2}, \dots)$$

uniformly in t on compact subsets of $\mathbb{R}$. Similarly as above in the case of the variable t it can be proved using the Cauchy formula that for almost all $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$ the relation (17) is valid uniformly in τ_1 on compact subsets of $\mathbb{R}$, uniformly in τ_2 on compact subsets of $\mathbb{R}$, $\dots$. Since the family of sets of m -measure one is closed under countable intersection, hence we have that (17) is true for almost all $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$ uniformly in t on compact subsets of $\mathbb{R}$, the convergence being uniform in τ_j on compact subsets of $\mathbb{R}$, $j = 1, 2, \dots$.

Since, for every $M > 0$,

$$\begin{aligned} m(| \sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T + it} \prod_{p_j^{\alpha_j} \| k} e^{i\alpha_j \tau_j} | \geq M) &\leq \frac{1}{M^2} \int_{\Omega} \left| \sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T + it} \prod_{p_j^{\alpha_j} \| k} e^{i\alpha_j \tau_j} \right|^2 dm = \\ &= \frac{1}{M^2} \sum_{k \leq T} \frac{d_{\kappa_T}^2(k)}{k^{2\sigma_T}} = \frac{B}{M^2} \end{aligned}$$

in view of the estimate

$$\sum_{k \leq T} \frac{d_{\kappa_T}^2(k)}{k^{2\sigma_T}} = B,$$

we have that $h(t; e^{i\tau_1}, e^{i\tau_2}, \dots) \neq \infty$ for almost all $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$.

The relation (17) and the uniform convergence imply that for almost all $(e^{i\tau_1}, e^{i\tau_2}, \dots) \in \Omega$

$$\sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T + it} \prod_{p_j^{\alpha_j} \| k} e^{i\alpha_j \tau_j(T)}} \xrightarrow{T \rightarrow \infty} h(t; e^{i\tau_1}, e^{i\tau_2}, \dots)$$

uniformly in t on compact subsets of $\mathbb{R}$. This yields $m(E) = 0$. The latter equality together with Lemmas 1 and 3 proves the theorem.

Now let $n_T = T^{\frac{\kappa_T}{2}}$.

COROLLARY. *There exists a probability measure P on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$ such that $P_{T, S_{n_T}}$ converges weakly to P as $n \rightarrow \infty$.*

Proof. Let K be a compact subset of $\mathbb{R}$. Denote by $Z_T(it + i\tau)$ the difference

$$S_T(\sigma_T + it + i\tau) - S_{n_T}(\sigma_T + it + i\tau) = \sum_{n_T < k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T + it + i\tau}}.$$

Let $\epsilon_T = (\log l_T)^{-1}$. Then

$$(18) \quad \nu_T^\tau(\sup_{t \in K} |Z_T(it + i\tau)| \geq \epsilon_T) \leq \frac{1}{\epsilon_T^2 T} \int_0^T \sup_{t \in K} |Z_T(it + i\tau)|^2 d\tau.$$

In view of the Cauchy formula

$$\sup_{t \in K} |Z_T(it + i\tau)| = Bl_T \int_L |Z_T(z + i\tau)|^2 |dz|$$

where L is the contour similar to that in the proof of Lemma 4. Hence we find by the Montgomery–Vaughan theorem for trigonometrical polynomials [15], [12] that

$$\begin{aligned} \int_0^T \sup_{t \in K} |Z_T(it + i\tau)|^2 d\tau &= Bl_T \sup_{z \in L} \int_0^T |Z_T(z + i\tau)|^2 d\tau = \\ &= Bl_T T \sup_{z \in L} \sum_{n_T < k \leq T} \frac{d_{\kappa_T}^2(k)}{k^{2\sigma_T + 2u}} = Bl_T T^{1 - \kappa_T} \frac{c_2 \log^2 l_T}{l_T} \sum_{n_T < k \leq T} \frac{d_{\kappa_T}^2(k)}{k} = \\ &= BT^{1 - \kappa_n} \frac{c_2 \log^2 l_T}{l_T} \log T \sum_{k \leq T} \frac{1}{k} = BT^{1 - c_3} \frac{\log^{\frac{3}{2}} l_T}{l_T} \log^2 T. \end{aligned}$$

From this and from (18) we deduce that

$$(19) \quad \nu_T^\tau(\sup_{t \in K} |Z_T(it + i\tau)| \geq \epsilon_T) = o(1)$$

as $T \rightarrow \infty$. Clearly, from the definition of the metric ρ , for $\epsilon > 0$,

$$\nu_T^\tau(\rho(S_T(\sigma_T + it + i\tau), S_{n_T}(\sigma_T + it + i\tau)) \geq \epsilon) \leq$$

$$\begin{aligned}
&\leq \frac{1}{\epsilon T} \int_0^T \rho(S_T(\sigma_T + it + i\tau), S_{n_T}(\sigma_T + it + i\tau)) d\tau \leq \\
&\leq \frac{1}{\epsilon} \sum_{j=1}^{\infty} 2^{-j} \frac{1}{T} \int_0^T \frac{2 \sup_{t \in K_j} |Z_T(it + i\tau)|}{1 + 2 \sup_{t \in K_j} |Z_T(it + i\tau)|} d\tau = \\
&= \frac{1}{\epsilon} \sum_{j=1}^{\infty} 2^{-j} \frac{1}{T} \left(\int_0^T \sup_{t \in K_j} |Z_T(it + i\tau)| \leq \epsilon_T + \int_0^T \sup_{t \in K_j} |Z_T(it + i\tau)| \geq \epsilon_T \right) \times \\
&\quad \times \frac{2 \sup_{t \in K_j} |Z_T(it + i\tau)|}{1 + 2 \sup_{t \in K_j} |Z_T(it + i\tau)|} d\tau.
\end{aligned}
\tag{20}$$

By (19) the second integral in the latter formula is $o(T)$ as $T \rightarrow \infty$, and the first integral trivially is $B_{\epsilon T} T$. Hence and from (20)

$$\nu_T^\tau = (\rho(S_T(\sigma_T + it + i\tau), S_{n_T}(\sigma_T + it + i\tau)) \geq \epsilon) = o(1)$$

as $T \rightarrow \infty$ for every $\epsilon > 0$. Thus, the corollary follows from Theorem and Theorem 4.1 from [13]: Let (S, ρ) be a separable space and X_n and Y_n be S -valued random elements. If $X_n \xrightarrow{\mathcal{D}} X$ and $\rho(X_n, Y_n) \xrightarrow{P} 0$, then $Y_n \xrightarrow{\mathcal{D}} X$.

Acknowledgement. For the warm hospitality during our stay at the University Bordeaux I we express our deep gratitude to Prof. M. MENDÈS FRANCE, Prof. M. BALAZARD and to their colleagues. We also would like to thank Prof. J.-P. ALLOUCHE and the referee for several helpful remarks.

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