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Limit theorem in the space of continuous functions for the Dirichlet polynomial related with the Riemann zeta-function


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Limit Theorem in the space of continuous functions for the Dirichlet polynomial related with the Riemann zeta-function

par Antanas Laurinčikas*

Résumé. Dans cet article on prouve un théorème limite dans l'espace des fonctions continues pour le polynôme de Dirichlet

$$\sum_{m \leq T} \frac{d_{\kappa_T}(m)}{m^{\sigma_T+i\epsilon}},$$

où $d_{\kappa_T}(m)$ sont les coefficients du développement en série de Dirichlet de la fonction $\zeta^{\kappa_T}(s)$ dans le demi-plan $\sigma > 1$, $\kappa_T = (2^{-1} \log l_T)^{-\frac{1}{2}}$, $\sigma_T = \frac{1}{2} + \log^2 l_T$, $l_T > 0$, $l_T \leq \log T$ et $l_T \rightarrow \infty$ lorsque $T \rightarrow \infty$.

Abstract. A limit theorem in the space of continuous functions for the Dirichlet polynomial

$$\sum_{m \leq T} \frac{d_{\kappa_T}(m)}{m^{\sigma_T+i\epsilon}},$$

where $d_{\kappa_T}(m)$ denote the coefficients of the Dirichlet series expansion of the function $\zeta^{\kappa_T}(s)$ in the half-plane $\sigma > 1$, $\kappa_T = (2^{-1} \ln l_T)^{-1/2}$, $\sigma_T = \frac{1}{2} + \ln^2 l_T$ and $l_T > 0$, $l_T \leq \ln T$ and $l_T \rightarrow \infty$ as $T \rightarrow \infty$, is proved.

Let $s$ be a complex variable and $\zeta(s)$, as usual, denote the Riemann zeta-function. To study the distribution of values of the Riemann zeta-function the probabilistic methods can be used, and the obtained results usually are presented as the limit theorems of probability theory. The first theorems of this type were obtained in [1],[2], and they were proved in [3]--[5] using other methods. In modern terminology we can formulate it as follows. Let

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C be the complex space and let $B(S)$ denote the class of Borel sets of the space $S$. Let $\text{meas}\{A\}$ be the Lebesgue measure of the set $A$ and

$$\nu_T^{\mu}(\ldots) = \frac{1}{T}\text{meas}\{t \in [0, T], \ldots\}$$

where in place of dots we write the conditions which are satisfied by $t$. We define the probability measure

$$P_T(A) = \nu_T^{\mu}(\zeta(\sigma + it) \in A), \quad A \in B(C)$$

**Theorem A.** For $\sigma > \frac{1}{2}$ there exists a probability measure $P$ on $(C, B(C))$ such that $P_T$ converges weakly to $P$ as $T \to \infty$.

More general results were obtained in [6]. Let $M$ denote the space of functions meromorphic in the half-plane $\sigma > \frac{1}{2}$, equipped with the topology of uniform convergence on compacta. Define the probability measure

$$Q_T(A) = \nu_T^{\mu}(\zeta(s + i\tau) \in A), \quad A \in B(M).$$

**Theorem B.** There exists a probability measure $Q$ on $(M, B(M))$ such that $Q_T$ converges weakly to $Q$ as $T \to \infty$.

Note that the explicit form of the measure $Q$ can be indicated, and, obviously, Theorem A is a corollary of Theorem B.

The situation is more complicated when $\sigma$ depends on $T$ and tends to $\frac{1}{2}$ as $T \to \infty$, or $\sigma = \frac{1}{2}$. It turns out that in this case some power norming is necessary. Let $l_T > 0$ and let $l_T$ tend to infinity as $T \to \infty$, or $l_T = \infty$. We take

$$\tilde{\sigma}_T = \frac{1}{2} + \frac{1}{l_T}, \quad \kappa = \kappa_T = \begin{cases} (2^{-1} \log l_T)^{-1/2}, & l_T \leq \log T, \\ (2^{-1} \log \log T)^{-1/2}, & l_T \geq \log T. \end{cases}$$

The case $l_T = \infty$ corresponds to $\tilde{\sigma}_T = \frac{1}{2}$.

The function

$$w(\tau, k) \overset{\text{def}}{=} \int_{C \setminus \{0\}} s^{|\tau r|} e^{ikr} \text{d}P \quad \tau \in \mathbb{R}, k \in \mathbb{Z},$$

is called the characteristic transform of the probability measure $P$ on the space $(C, B(C))$ [7]. The lognormal probability measure on $(C, B(C))$ is defined by the characteristic transform

$$w(\tau, k) = \exp \left\{ -\frac{\tau^2}{2} - \frac{k^2}{2} \right\}.$$
THEOREM C. The probability measure

\[ \nu^T_T(\zeta^{\kappa T}(\sigma_T + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \]

converges weakly to the lognormal probability measure as \( T \to \infty \).

Here if \( \zeta(s) \neq 0, a \in \mathbb{R} \), then \( \zeta^a(s) \) is understood as \( \exp\{a \log \zeta(s)\} \) where \( \log \zeta(s) \) is defined by continuous displacement from the point \( s = 2 \) along the path joining the points \( 2, 2 + it \) and \( \sigma + it \).

When \( \sigma_T = \frac{1}{2} \) Theorem C was proved by A. Selberg (unpublished), see also [8], and for different form of \( l_T \), it was obtained in [8]–[10], [5].

Now it arises the problem to obtain some results of the kind of Theorem C in the space of continuous functions.

Let \( \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \) be the Riemann sphere and let \( d(s_1, s_2) \) be a metric on \( \mathbb{C}_\infty \) given by the formulae

\[
d(s_1, s_2) = \frac{2 |s_1 - s_2|}{\sqrt{1 + |s_1|^2} \sqrt{1 + |s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1 + |s|^2}}, \quad d(\infty, \infty) = 0.
\]

Here \( s, s_1, s_2 \in \mathbb{C} \). This metric is compatible with the topology of \( \mathbb{C}_\infty \). Let \( C(\mathbb{R}) = C(\mathbb{R}, \mathbb{C}_\infty) \) denote the space of continuous functions \( f : \mathbb{R} \to \mathbb{C}_\infty \) equipped with the topology of uniform convergence on compacta. In this topology, sequence \( \{f_n, f_n \in C(\mathbb{R})\} \) converges to the function \( f \in C(\mathbb{R}) \) if

\[
d(f_n(t), f(t)) \to 0
\]
as \( n \to \infty \) uniformly in \( t \) on compact subsets of \( \mathbb{R} \).

The functional analogue of the probability measure in Theorem C is the measure

(1) \[ \nu^T_T(\zeta^{\kappa T}(\sigma_T + it + i\tau) \in A), \quad A \in \mathcal{B}(C(\mathbb{R})). \]

Does this measure converge weakly as \( T \to \infty \) to some probability measure on \( (C(\mathbb{R}), \mathcal{B}(C(\mathbb{R}))) \)? At this moment this question is open and it seems to be very difficult.

In the proof of Theorem C an important role is played by the Dirichlet polynomial

\[
S_u(s) = \sum_{m \leq u} \frac{d_K(m)}{m^s}
\]
where $d_\kappa(m)$ denote the coefficients of the Dirichlet series expansion of the function $\zeta^n(s)$ in the half-plane $\sigma > 1$ (see [11], [12]). Therefore the aim of this paper is to prove the limit theorem in the space of continuous functions for $S_u(s)$. This theorem will be the first step to study the weak convergence of the probability measure (1).

Now let $l_T \leq \log T$, $\sigma_T = \frac{1}{2} + \frac{\log^2 l_T}{4l_T}$, and let

$$P_{T,S_u}(A) = \nu_T^r(S_u(\sigma_T + it + i\tau) \in A), \quad A \in \mathcal{B}(C(\mathbb{R})).$$

Moreover we suppose that

$$l_{T+U} - l_T = \frac{BU}{T}$$

for all $U > 0$ as $T \to \infty$. Here $B$ denotes a number (not always the same) which is bounded by a constant.

**THEOREM** There exists a probability measure $P$ on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$ such that $P_{T,S_T}$ converges weakly to $P$ as $T \to \infty$.

Proof of the theorem is based on the following probability result. Let $S_1$ and $S_2$ be two metric spaces, and let $h : S_1 \to S_2$ be a measurable function. Then every probability measure $P$ on $(S_1, \mathcal{B}(S_1))$ induces on $(S_2, \mathcal{B}(S_2))$ the unique probability measure $P h^{-1}$ defined by the equality $P h^{-1}(A) = P(h^{-1}A), \quad A \in \mathcal{B}(S_2)$.

Now let $h$ and $h_n$ be the measurable functions from $S_1$ into $S_2$ and

$$E = \{x \in S_1 : h_n(x_n) \not\to h(x) \quad \text{for some} \quad x_n \to x\}.$$

**LEMMA 1.** Let $P$ and $P_n$ be the probability measures on $(S_1,\mathcal{B}(S_1))$. Suppose that $P_n$ converges weakly to $P$ as $n \to \infty$ and that $P(E) = 0$. Then the measure $P_n h_n^{-1}$ converges weakly to $P h^{-1}$ as $n \to \infty$.

**Proof.** This lemma is Theorem 5.5 from [13].

Let $\gamma$ denote the unit circle on complex plane, that is $\gamma = \{s \in \mathbb{C} : |s| = 1\}$. We put

$$\Omega = \prod_p \gamma_p$$

where $\gamma_p = \gamma$ for each prime $p$. With the product topology and pointwise multiplication the infinite-dimensional torus $\Omega$ is a compact Abelian topological group. Let $P$ be a probability measure on $(\Omega, \mathcal{B}(\Omega))$. 

The Fourier transform \( g(\mathbf{k}) \) of the measure \( P \) is defined by the formula

\[
g(\mathbf{k}) = \int_{\Omega} \prod_{p} x_p^{k_p} \, dP.
\]

Here \( \mathbf{k} = (k_2, k_3, \ldots) \) where only a finite number of integers \( k_p \) are distinct from zero, and \( x_p \in \gamma \).

**Lemma 2.** Let \( \{P_n\} \) be a sequence of probability measures on \((\Omega, B(\Omega))\) and let \( \{g_n(\mathbf{k})\} \) be a sequence of corresponding Fourier transforms. Suppose that for every vector \( \mathbf{k} \) the limit \( g(\mathbf{k}) = \lim_{n \to \infty} g_n(\mathbf{k}) \) exists. Then there exists a probability measure \( P \) on \((\Omega, B(\Omega))\) such that \( P_n \) converges weakly to \( P \) as \( n \to \infty \). Moreover, \( g(\mathbf{k}) \) is the Fourier transform of \( P \).

**Proof.** The lemma is the special case of the continuity theorem for compact Abelian group, see, for example, [14].

Let

\[
Q_T(A) = \nu_T((p_1^\tau, p_2^\tau, \ldots) \in A), \quad A \in B(\Omega).
\]

**Lemma 3.** The probability measure \( Q_T \) converges weakly to the Haar measure \( m \) on \((\Omega, B(\Omega))\) as \( T \to \infty \).

**Proof.** The Fourier transform \( g_T(\mathbf{k}) \) of the measure \( Q_T \) is given by

\[
g_T(\mathbf{k}) = \int_{\Omega} \prod_{p} x_p^{k_p} \, dQ_T = \frac{1}{T} \int_{0}^{T} \prod_{j=1}^{\infty} p_j^{i k_j \tau} \, d\tau =
\begin{cases}
1 & \text{if } \mathbf{k} = \mathbf{0}, \\
\exp\left( i T \sum_{j=1}^{\infty} k_j \log p_j \right) - 1 & \text{if } \mathbf{k} \neq \mathbf{0}
\end{cases}
\]

Here \( x_p \in \gamma, \mathbf{k} = (k_1, k_2, \ldots) \). By definition of the Fourier transform of probability measure on \((\Omega, B(\Omega))\), only a finite number of \( k_j \) are distinct from zero. Since the logarithms of prime numbers are linearly independent over the field of rational numbers, we find that

\[
g_T(\mathbf{k}) \to \begin{cases} 
1 & \text{if } \mathbf{k} = \mathbf{0}, \\
0 & \text{if } \mathbf{k} \neq \mathbf{0}
\end{cases}
\]

as \( T \to \infty \). In view of Lemma 2, this proves the lemma.
We define the function $h_T : S_2 \to C(\mathbb{R})$ by the formula

$$h_T(t; e^{i\eta_1}, e^{i\eta_2}, ...) = \sum_{k \leq T} \frac{d_\kappa(k)}{k^{\sigma_T + it} \prod_{p_j \parallel k} e^{i\tau_j \eta_j}}.$$

Here $p^\alpha \parallel k$ means that $p^\alpha \mid k$ but $p^{\alpha + 1} \not\mid k$. Then, clearly,

$$S_T(\sigma_T + it + i\tau) = h_T(t; p_1^{i\tau}, p_2^{i\tau}, ...).$$

Let, for brevity,

$$h_T(t; e^{i\tau_1}, e^{i\tau_2}, ...) = S_T(\sigma_T + it + i\tau),$$

and let

$$Z_{nk}(it, \tau) = S_{n+k}(\sigma_{n+k} + it + i\tau) - S_n(\sigma_n + it + i\tau).$$

Let $K$ be a compact subset of $\mathbb{R}$. For every $\epsilon > 0$ we define the set $A_{nk}^\epsilon$ by

$$A_{nk}^\epsilon(K) = \{(e^{i\tau_1}, e^{i\tau_2}, ...) : \sup_{t \in K} |Z_{nk}(it, \tau)| \geq \epsilon\}$$

and we put

$$A_k^\epsilon(K) = \bigcap_{n \geq 1} \bigcup_{n > k} A_{nk}^\epsilon.$$

**Lemma 4.** $m(A_k^\epsilon(K)) = 0$ for every $\epsilon > 0, K$, and $k \in \mathbb{N}$.

**Proof.** By the Chebyshev inequality

$$m(A_{nk}^\epsilon(K)) \leq \frac{1}{\epsilon^2} \int_{\Omega} \sup_{t \in K} |Z_{nk}(it, \tau)|^2 dm.$$

Using the Cauchy formula, we have that

$$Z_{nk}^2(it, \tau) = \frac{1}{2\pi i} \int_L \frac{Z_{nk}^2(z, \tau)}{z - it} dz$$

where $L$ denotes the restangle, enclosing the set $iK = \{ia, a \in K\}$, with the sides $\sigma = -\frac{1}{n+k} + it$, $\sigma = \frac{1}{n+k} + it$, and with two other sides parallel
to the real axis. Moreover, we suppose that the distance of $L$ from the set $iK$ is $\geq \frac{1}{l_{n+k}}$. From this equality it follows that

$$\sup_{t \in K} |Z_{nk}(it, \ell)|^2 = Bl_{n+k} \int_{L} |Z_{nk}(z, \ell)|^2 \, dz.$$ 

Hence, having in mind the inequality (5), we obtain that

$$m(A_{nk}) = \frac{Bl_{n+k}}{\epsilon^2} \int_{L} \left( \int_{\Omega} |Z_{nk}(z, \ell)|^2 \, dm \right) \frac{1}{\epsilon^2} \sup_{z \in L} \int_{\Omega} |Z_{nk}(z, \ell)|^2 \, dm =$$

$$(6) \quad = \frac{Bl_{n+k}}{\epsilon^2} \frac{|L|}{\epsilon^2} \sup_{z \in L} \int_{\Omega} |Z_{nk}(z, \ell)|^2 \, dm$$

where $|L|$ is the length of $L$. From the definitions of $Z_{nk}(z, \ell)$ and $S_n(\sigma_n + z + i\tau)$ we have that, for $z = u + iv$,

$$Z_{nk}(z, \ell) = \sum_{l \leq n+k} \frac{d_{\kappa_n+l}(l)}{l^{\sigma_n+u+iv}} \prod_{\sigma_j} e^{i\alpha_j \tau_j} - \sum_{l \leq n} \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u+iv}} \prod_{\sigma_j} e^{i\alpha_j \tau_j} =$$

$$= \sum_{n<l \leq n+k} \frac{d_{\kappa_n+l}(l)}{l^{\sigma_n+u+iv}} \prod_{\sigma_j} e^{i\alpha_j \tau_j} + \sum_{l \leq n} \frac{d_{\kappa_n+l}(l)}{l^{\sigma_n+u+iv}} \prod_{\sigma_j} e^{i\alpha_j \tau_j} - \frac{d_{\kappa_n}(l)}{l^{\sigma_n+u+iv}} \prod_{\sigma_j} e^{i\alpha_j \tau_j} \overset{\text{def}}{=} V + W.$$

Since

$$|a + b|^2 \leq 2(|a|^2 + |b|^2),$$

hence we find that

$$(7) \quad |Z_{nk}(z, \tau)|^2 \leq 2(|V|^2 + |W|^2).$$

The properties of the Haar measure $m$ imply the equality

$$(8) \quad \int_\Omega |V|^2 \, dm = \sum_{n<l \leq n+k} \frac{d_{\kappa_n+l}(l)}{l^{2(\sigma_n+u+u)}} + \sum_{n<l_1 \leq n+k} \sum_{n<l_2 \leq n+k} \frac{d_{\kappa_n+l_1}(l_1)d_{\kappa_n+l_2}(l_2)}{l_1^{2(\sigma_n+u+iv)}l_2^{\sigma_n+u+iv}} \times$$

$$\times \int_\Omega \prod_{\sigma_j} \frac{e^{i\alpha_j \tau_j}}{l_1^{\sigma_j}} \prod_{\sigma_j} \frac{e^{i\alpha_j \tau_j}}{l_2^{\sigma_j}} \, dm = \sum_{n<l \leq n+k} \frac{d_{\kappa_n+l}(l)}{l^{2(\sigma_n+u+u)}}.$$
By a similar manner we find that

\[ \int \Omega | W |^2 \, dm = \sum_{l \leq n} \left( \frac{d_{\kappa_{n+k}}(l)}{l^{\sigma_{n+k}+u}} - \frac{d_{\kappa_{n}}(l)}{l^{\sigma_{n}+u}} \right)^2. \]

From the definition of the contour \( L \) it follows that

\[ \frac{1}{l_{n+k}} \leq u \leq \frac{1}{l_{n+k}} \]

for \( z = u + iv \in L \). Then (8) together with (2) and the well-known estimate

\[ \sum_{m \leq x} \frac{1}{m} = \log x + \gamma_0 + \frac{B}{x}, \]

where \( \gamma_0 \) is the Euler constant, yields

\[ \int \Omega | V |^2 \, dm = Bn \frac{-2 \log^2 l_{n+k} - 1}{l_{n+k}} \sum_{n < l \leq n+k} \frac{1}{l} = B \epsilon \frac{-\log n \log^2 l_n}{l_n} \left( 1 + \frac{Bk}{n} \right) \times \]

\[ \times \left( \log \frac{n+k}{n} + \frac{B}{n} \right) = \frac{Bk}{n} e^{-c_1 \log n \log^2 l_n / l_n} \]

for \( n \geq n_0 \). Here we have used the inequality \( 0 < d_{\kappa_{n+k}}(l) < 1 \), \( n \geq n_0 \), which follows trivially from the multiplicativity of \( d_{\kappa_{n+k}}(m) \) and from the inequality \( 0 < d_{\kappa_{n+k}}(p^\alpha) < 1 \), \( n \geq n_0 \), implied by the formula [11], [12]

\[ d_{\kappa}(p^\alpha) = \frac{\kappa(\kappa + 1) \cdots (\kappa + \alpha - 1)}{\alpha!}. \]

From the assumption on \( l_T \) we deduce that, for \( n \geq n_0 \),

\[ \sigma_{n+k} = \sigma_n(1 + \frac{Bk}{n}), \]

\[ \log l_{n+k} = \log \left( l_n + \frac{Bk}{n} \right) = \log l_n \left( 1 + \frac{Bk}{nl_n} \right) = \]

\[ = \log l_n + \frac{Bk}{nl_n} = \log l_n \left( 1 + \frac{Bk}{nl_n \log l_n} \right). \]
Thus,
\[ \kappa_{n+k} = (2^{-1} \log l_{n+k})^{-\frac{1}{2}} = \kappa_n \left( 1 + \frac{Bk}{n l_n \log l_n} \right)^{-\frac{1}{2}} = \]
\[ = \kappa_n \left( 1 + \frac{Bk}{n l_n \log l_n} \right)^{\text{def}} = \kappa_n (1 + r_{nk}). \]
Consequently, in view of (12), for \( n \geq n_0 \),
\[ d_{\kappa_{n+k}}(p^\alpha) = \frac{\kappa_n (1 + r_{nk})(\kappa_n (1 + r_{nk}) + 1) \cdots (\kappa_n (1 + r_{nk}) + \alpha - 1)}{\alpha!} = \]
\[ = \frac{\kappa_n (1 + r_{nk})(\kappa_n + 1) \left( 1 + \frac{\kappa_n r_{nk}}{\kappa_n + 1} \right) \cdots (\kappa_n + \alpha - 1) \left( 1 + \frac{\kappa_n r_{nk}}{\kappa_n + \alpha - 1} \right)}{\alpha!} = \]
\[ = d_{\kappa_n}(p^\alpha) \prod_{j=1}^{\alpha} \left( 1 + \frac{Br_{nk}}{j} \right) = \]
\[ = d_{\kappa_n}(p^\alpha)(1 + Br_{nk} \log \alpha). \]
Hence, for \( m \leq n \),
\[ d_{\kappa_{n+k}}(m) = \prod_{p^\alpha \parallel m} d_{\kappa_{n+k}}(p^\alpha) = \]
\[ = d_{\kappa_n}(m) = \prod_{p^\alpha \parallel m} (1 + Br_{nk} \log \alpha) = \]
\[ = d_{\kappa_n}(m) \exp\{Br_{nk} \sum_{p^\alpha \parallel m} \log \alpha\} = \]
\[ = d_{\kappa_n}(m) \exp\{Br_{nk} \log \log n \sum_{p^\alpha \parallel m} 1\} = \]
\[ = d_{\kappa_n}(m) \exp\{Br_{nk} \log \log n \log n\} = \]
\[ = d_{\kappa_n}(m)(1 + Br_{nk} \log \log n \log n). \]
Therefore, from (9), (13) and (14) we have that
\[ \int_{\Omega} |W|^2 \, dm = \frac{Bk^2 \log^2 n \log^2 \log n}{n^2 l_n^2 \log^2 l_n} \sum_{l \leq n} \frac{d_{\kappa_n}^2(l)}{l^2(\sigma_n^2 + u)}. \]
Repeating the proof of Lemma 3 from [10] and taking into account (10), we see that
\[ \sum_{l \leq n} \frac{d_{n}^{2}(l)}{l^{2}(\sigma_{n}+u)} = B. \]
Consequently, this and (15) give the estimate
\[ \int_{\Omega} |W|^2 \, dm = \frac{Bk^{2} \log^{2} n \log^{2} \log n}{n^{2}l_{n}^{2} \log^{2} l_{n}}. \]
From this, (6), (7) and (11) we find that
\[ m(A_{nk}^{\varepsilon}(K)) = \frac{Bk^{2}}{\varepsilon^{2}} \left( \frac{1}{n}e^{-c_{1} \frac{\log n \log^{2} l_{n}}{l_{n}}} + \frac{\log^{2} n \log^{2} \log n}{n^{2}l_{n}^{2} \log^{2} l_{n}} \right) \]
for every \( \varepsilon > 0 \) and \( k \in \mathbb{N} \). Thus it follows from the definition of the set \( A_{nk}^{\varepsilon}(K) \) that
\[ m(A_{k}^{\varepsilon}(K)) = \lim_{l \to \infty} m(\bigcup_{n > l} A_{nk}^{\varepsilon}(K)) = \]
\[ = \lim_{l \to \infty} \frac{Bk^{2}}{\varepsilon^{2}} \sum_{n > l} \left( \frac{1}{n}e^{-c_{1} \frac{\log n \log^{2} l_{n}}{l_{n}}} + \frac{\log^{2} n \log^{2} \log n}{n^{2}l_{n}^{2} \log^{2} l_{n}} \right) = 0. \]
The lemma is proved.

Proof of Theorem. We will deduce the theorem from lemmas 1, 3 and 4. Let
\[ (e^{i\tau_{1}(T)}, e^{i\tau_{2}(T)}, ...) \]
converges to
\[ (e^{i\tau_{1}}, e^{i\tau_{2}}, ...) \]
as \( T \to \infty \), and let \( E \) denote the set \( \{ (e^{i\tau_{1}}, e^{i\tau_{2}}, ...) \} \) of elements of \( \Omega \) such that
\[ h_{T}(t; e^{i\tau_{1}(T)}, e^{i\tau_{2}(T)}, ...) \]
does not converge to some function \( h(t; e^{i\tau_{1}}, e^{i\tau_{2}}, ...) \) as \( T \to \infty \). In order to prove the theorem we must show that \( m(E) = 0 \). Since \( \Omega \) is compact, it is separable. Consequently [13], \( E \in B(\Omega) \).

Let \( E_{1} \) denote the set \( \{ (e^{i\tau_{1}}, e^{i\tau_{2}}, ...) \} \) such that
\[ h_{T}(t; e^{i\tau_{1}}, e^{i\tau_{2}}, ...) \]
does not converge to some function \( h(t; e^{it_1}, e^{it_2}, \ldots) \) as \( T \to \infty \). We will prove that \( m(E_1) = 0 \). First we consider the sequence \( h_n(t; e^{it_1}, e^{it_2}, \ldots) \).

Note that there exists a sequence \( \{K_j\} \) of compact subsets of \( \mathbb{R} \) such that
\[
\mathbb{R} = \bigcup_{j=1}^{\infty} K_j,
\]
\( K_j \subset K_{j+1} \), and if \( K \) is as compact of \( \mathbb{R} \) then \( K \subset K_j \) for some \( j \). Let
\[
\rho_j(f, g) = \sup_{t \in K_j} d(f(t), g(t))
\]
for \( f, g \in C(\mathbb{R}) \). Then
\[
\rho(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\rho_j(f, g)}{1 + \rho_j(f, g)}
\]
is a metric in \( C(\mathbb{R}) \).

Since \( C(\mathbb{R}) \) is a complete metric space, we have that every fundamental sequence is convergent. Thus it follows from the definition of the fundamental sequence that
\[
m(\{e^{it_1}, e^{it_2}, \ldots\} : h_n(t; e^{it_1}, e^{it_2}, \ldots) \not\to) =
\]
\[= m(\{e^{it_1}, e^{it_2}, \ldots\} : \{e^{it_1}, e^{it_2}, \ldots\} \in \bigcup_{\varepsilon>0} \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_{\epsilon}^k(K_j)).
\]
Thus, by Lemma 4,
\[
(16) \quad m(\{e^{it_1}, e^{it_2}, \ldots\} : h_n(t; e^{it_1}, e^{it_2}, \ldots) \not\to) = 0.
\]
From the definition of the function \( h_T \), using the estimates of types (13) and (14), we find that
\[
h_T(t; e^{it_1}, e^{it_2}, \ldots) = \sum_{k \leq [T]} \frac{d_{\kappa[T]}(k)}{k^{\sigma[T]+it} \prod_{\alpha_j \neq \alpha_k} e^{it\alpha_j}} + \frac{B}{T^{1/4}}
\]
uniformly in \( t \in \mathbb{R} \) and in \( \{e^{it_1}, e^{it_2}, \ldots\} \in \Omega \). Therefore, in view of (16), \( m(E_1) = 0 \).
We have shown that there exists a function $h$ such that for almost all $(e^{i\tau_1}, e^{i\tau_2}, \ldots) \in \Omega$

(17) \[
\sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it}} \prod_{p_j \mid k} e^{ia_j \tau_j} \to h(t; e^{i\tau_1}, e^{i\tau_2}, \ldots)
\]

uniformly in $t$ on compact subsets of $\mathbb{R}$. Similarly as above in the case of the variable $t$ it can be proved using the Cauchy formula that for almost all $(e^{i\tau_1}, e^{i\tau_2}, \ldots) \in \Omega$ the relation (17) is valid uniformly in $\tau_1$ on compact subsets of $\mathbb{R}$, uniformly in $\tau_2$ on compact subsets of $\mathbb{R}$, $\ldots$. Since the family of sets of $m$–measure one is closed under countable intersection, hence we have that (17) is true for almost all $(e^{i\tau_1}, e^{i\tau_2}, \ldots) \in \Omega$ uniformly in $t$ on compact subsets of $\mathbb{R}$, the convergence being uniform in $\tau_j$ on compact subsets of $\mathbb{R}$, $j = 1, 2, \ldots$. 

Since, for every $M > 0$,

\[
m(\left| \sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it}} \prod_{p_j \mid k} e^{ia_j \tau_j} \right| \geq M) \leq \frac{1}{M^2} \int_{\Omega} \left| \sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it}} \prod_{p_j \mid k} e^{ia_j \tau_j} \right|^2 dm = \frac{1}{M^2} \sum_{k \leq T} \frac{d_{\kappa_T}^2(k)}{k^{2\sigma_T}} = \frac{B}{M^2}
\]

in view of the estimate

\[
\sum_{k \leq T} \frac{d_{\kappa_T}^2(k)}{k^{2\sigma_T}} = B,
\]

we have that $h(t; e^{i\tau_1}, e^{i\tau_2}, \ldots) \neq \infty$ for almost all $(e^{i\tau_1}, e^{i\tau_2}, \ldots) \in \Omega$.

The relation (17) and the uniform convergence imply that for almost all $(e^{i\tau_1}, e^{i\tau_2}, \ldots) \in \Omega$

\[
\sum_{k \leq T} \frac{d_{\kappa_T}(k)}{k^{\sigma_T+it}} \prod_{p_j \mid k} e^{ia_j \tau_j(T)} \to h(t; e^{i\tau_1}, e^{i\tau_2}, \ldots)
\]

uniformly in $t$ on compact subsets of $\mathbb{R}$. This yields $m(E) = 0$. The latter equality together with Lemmas 1 and 3 proves the theorem.

Now let $n_T = T^{\frac{\sigma_T}{2}}$. 


COROLLARY. There exists a probability measure $P$ on $(C(\mathbb{R}), B(C(\mathbb{R})))$ such that $P_{T, s_{nT}}$ converges weakly to $P$ as $n \to \infty$.

Proof. Let $K$ be a compact subset of $\mathbb{R}$. Denote by $Z_T(it+i\tau)$ the difference

$$S_T(\sigma_T + it + i\tau) - S_{nT}(\sigma_T + it + i\tau) = \sum_{nT < k \leq T} \frac{d_{\kappa \tau}(k)}{k^{\sigma_T + it + i\tau}}.$$

Let $\varepsilon_T = (\log l_T)^{-1}$. Then

$$(18) \quad \nu_T^T(\sup_{t \in K} |Z_T(it + i\tau)| \geq \varepsilon_T) \leq \frac{1}{\varepsilon_T^2 T} \int_0^T \sup_{t \in K} |Z_T(it + i\tau)|^2 d\tau.$$

In view of the Cauchy formula

$$\sup_{t \in K} |Z_T(it + i\tau)| = B l_T \int_L |Z_T(z + i\tau)|^2 |dz|$$

where $L$ is the contour similar to that in the proof of Lemma 4. Hence we find by the Montgomery–Vaughan theorem for trigonometrical polynomials [15], [12] that

$$\int_0^T \sup_{t \in K} |Z_T(it + i\tau)|^2 d\tau = B l_T \sup_{z \in L} \int_0^T |Z_T(z + i\tau)|^2 d\tau =$$

$$= B l_T \sup_{z \in L} \sum_{nT < k \leq T} \frac{d_{\kappa \tau}(k)}{k^{2\sigma_T + 2u}} = B l_T \frac{1 - \kappa_T c_2 \log^2 l_T}{l_T} \sum_{nT < k \leq T} \frac{d_{\kappa \tau}(k)}{k} =$$

$$= B T \frac{1 - \kappa_T c_2 \log^2 l_T}{l_T} \log T \sum_{k \leq T} \frac{1}{k} = B T \frac{1 - c_3 \log^3 l_T}{l_T} \log^2 T.$$

From this and from (18) we deduce that

$$(19) \quad \nu_T^T(\sup_{t \in K} |Z_T(it + i\tau)| \geq \varepsilon_T) = o(1)$$

as $T \to \infty$. Clearly, from the definition of the metric $\rho$, for $\varepsilon > 0$,

$$\nu_T^T(\rho(S_T(\sigma_T + it + i\tau), S_{nT}(\sigma_T + it + i\tau)) \geq \varepsilon) \leq$$

By (19) the second integral in the latter formula is $o(T)$ as $T \to \infty$, and
the first integral trivially is $B \varepsilon T$. Hence and from (20)
as $T \to \infty$ for every $\varepsilon > 0$. Thus, the corollary follows from Theorem
and Theorem 4.1 from [13]: Let $(S, \rho)$ be a separable space and $X_n$ and
$Y_n$ be $S$-valued random elements. If $X_n \xrightarrow{D} X$ and $\rho(X_n, Y_n) \xrightarrow{P} 0$, then
$Y_n \xrightarrow{D} X$.

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