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On the discrepancy of Markov-normal sequences

par M.B. LEVIN

RÉSUMÉ. On construit une suite normale de Markov dont la discrepancy est $O(N^{-1/2} \log^2 N)$, améliorant en cela un résultat donnant l'estimation $O(e^{-c(\log N)^{1/2}})$.

ABSTRACT. We construct a Markov normal sequence with a discrepancy of $O(N^{-1/2} \log^2 N)$. The estimation of the discrepancy was previously known to be $O(e^{-c(\log N)^{1/2}})$.

A number $\alpha \in (0, 1)$ is said to be *normal* to the base q , if in a q -ary expansion of α ,

$$\alpha = .d_1 d_2 \cdots = \sum_{i=1}^{\infty} d_i / q^i, \quad d_i \in \{0, 1, \dots, q-1\}$$

each fixed finite block of digits of length k appears with an asymptotic frequency of q^{-k} along the sequence $(d_i)_{i \geq 1}$. Normal numbers were introduced by Borel (1909). Borel proved that almost every number (in the sense of Lebesgue measure) is normal to the base q . But only in 1935 did Champernowne give the explicit construction of such a number, namely

$$\theta = .1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ \dots$$

obtained by successively concatenating all the natural numbers.

Let $P = (p_{i,j})_{0 \leq i,j \leq q-1}$ be an irreducible Markov transition matrix, $(p_i)_{0 \leq i \leq q-1}$ the stationary probability vector of P and $\bar{\mu}$ its probability measure.

A number α (sequence $(d_i)_{i \geq 1}$) is said to be *Markov-normal* if in a q -ary expansion of α each fixed finite block of digits $b_0 b_1 \dots b_k$ appears with an asymptotic frequency of $p_{b_0} p_{b_0 b_1} \dots p_{b_{k-1} b_k}$.

According to the individual ergodic theorem $\bar{\mu}$ -almost all sequences (numbers) are normals.

Markov normal numbers were introduced by Postnikov and Piatecki-Shapiro [1]. They also obtained, by generalizing Champernowne's method, the explicit construction of these numbers. Another Champernowne construction of Markov normal numbers was obtained in Smorodinsky-Weiss

[2] and in Bertrand-Mathis [3]. In [4] Chentsov gave the construction of Markov normal numbers using *completely uniformly distributed sequences* (for the definition, see [5]) and the standard method of modelling Markov chains. In [6] Shahov proposed using a *normal periodic systems of digits* (for the definition, see [5]) to construct Markov normal numbers. In [7] he obtained the estimate of discrepancy of the sequence $\{\alpha q^n\}_{n=1}^N$ to be $O(e^{-c(\log N)^{1/2}})$. In this article we construct a Markov normal sequence with the discrepancy of sequence $\{\alpha q^n\}_{n=1}^N$ equal to $O(N^{-1/2} \log^2 N)$.

Let $(x_n)_{n \geq 1}$ be a sequence of real numbers, μ - measure on $[0, 1)$. The quantity

$$(1) \quad D(\mu, N) = \sup_{\gamma \in [0, 1)} \left| \frac{1}{N} \# \{n \in [1, N] \mid 0 \leq \{x_n\} < \gamma\} - \mu[0, \gamma) \right|$$

is called the *discrepancy* of $(x_n)_{n=1}^N$.

The sequence $(\{x_n\})_{n \geq 1}$ is said to be μ -distributed in $[0, 1)$ if $D(\mu, N) \rightarrow 0$.

Let the measure μ be such that

$$(2) \quad \mu([\gamma_n, \gamma_n + 1/q^n)) = p_{c_1} p_{c_1 c_2} \dots p_{c_{n-1} c_n}, \quad \gamma_n = .c_1 \dots c_n, \quad n = 1, 2, \dots,$$

where $c_k \in \{0, 1, \dots, q-1\}$, $k = 1, 2, \dots$.

It is known that if and only if α is Markov normal number, the sequence $\{\alpha q^n\}_{n=1}^\infty$ is μ -distributed.

The discrepancy $D(\mu, N)$ satisfies $D(\mu, N) = O(N^{-1/2} (\log \log N)^{1/2})$ for almost all α .

The following facts are known from the theory of finite Markov chains [8, 9]:

Let a Markov chain have d cyclic class $C_1, \dots, C_d$. We enumerate the states $e_1, \dots, e_q$ of the Markov chain in such a way, that if $e_i \in C_m$, $e_j \in C_n$ and $i > j$, then $m \geq n$. Here matrix P has d^2 blocks $(\bar{P}_{i,j})_{0 \leq i, j \leq d-1}$, where $\bar{P}_{i,j} = 0$ except for $\bar{P}_{1,2}, \bar{P}_{2,3}, \bar{P}_{d-1,d}, \bar{P}_{d,1}$. Matrix P^d has a block-diagonal structure. Let $P_1, \dots, P_d$ be the block diagonal of matrix P^d . There exists a number k_0 such that all the elements of matrices $P_i^{k_0}$ ($i = 1, \dots, d$) are greater than zero [9, ch. 4]. Let θ be the minimal element of these matrices, and $p_{ij}^{(k)}$ the ij element of matrix P^k , $k = 1, 2, \dots$.

It is evident that

$$(3) \quad \theta = \min_{i,j} p_{ij}^{(dk_0)},$$

where we choose minimum values for i, j so that e_i, e_j are included in the same cyclic class.

Let $f(j)$ be the number of cyclic class states e_j ($e_j \in C_{f(j)}, j = 0, \dots, q-1$).

According to [9, ch.4] we obtain

$$(4) \quad |p_{ij}^{(kd+f(j)-f(i))} - dp_j| \leq (1-2\theta)^{-1+k/k_0},$$

$$p_{ij}^{(kd+f(j)-f(i)+l)} = 0, \quad l = 1, 2, \dots, d-1, \quad k = 1, 2, \dots$$

Let

$$(5) \quad p = \max_{0 \leq i, j \leq q-1} (p_i, p_{ij}), \quad A_n = [p^{-n}], \quad n = 1, 2, \dots$$

We have, from the irreducibility of matrix P , that

$$(6) \quad p < 1 \quad \text{and} \quad A_n \rightarrow \infty.$$

We use matrices $P_n = (p_{ij}(n))_{0 \leq i, j \leq q-1}$ with the rational elements

$$(7) \quad p_{ij}(n) = v_{ij}(n)/A_n,$$

and we choose $v_{ij}(n)$ as follows:

Let i be fixed and p_{ij_0} be greater than zero. Then we denote

$$v_{ij}(n) = [A_n p_{ij}], \quad \text{if } j \neq j_0, \quad \text{and} \quad v_{ij_0}(n) = A_n - \sum_{j \neq j_0} v_{ij}(n).$$

It is evident that

$$(8) \quad \sum_{j=0}^{q-1} p_{ij}(n) = 1, \quad |v_{ij}(n) - A_n p_{ij}| \leq q, \quad i, j = 0, \dots, q-1, \quad n = 1, 2, \dots$$

If k_1 is sufficiently large, then using (3) and (6)-(8), we obtain

$$(9) \quad \min_{ij} p_{ij}^{(dk_0)}(n) \geq \theta/2, \quad n > k_1,$$

where we choose minimum values for i, j so that e_i, e_j belong to the same cyclic class.

It is evident that P_n ($n > k_1$) is an irreducible matrix with a d -cyclic class.

Applying (3), (4) and (9) we obtain

$$(10) \quad |p_{ij}^{(kd+f(j)-f(i))}(n) - dp_j(n)| \leq (1-\theta)^{-1+k/k_0}, \quad k = 1, 2, \dots$$

$$p_{ij}^{(kd+f(j)-f(i)+l)}(n) = 0, \quad l = 1, 2, \dots, d-1, \quad i, j = 0, \dots, q-1,$$

where $n > k_1$, and $(p_j(n))_{j < q}$ is the stationary probability vector of P_n .

According to [7, 10] there exist integers $v_0(n), \dots, v_{q-1}(n), L_n > 0$, such that

$$p_j(n) = v_j(n)/L_n \quad v_0(n) + \dots + v_{q-1}(n) = L_n$$

$$(11) \sum_{j=0}^{q-1} v_j(n) p_{ji}(n) = v_i(n), \quad |p_i - v_i(n)/L_n| < B/A_n, \quad (i = 0, \dots, q-1).$$

If k_1 is sufficiently large, then applying (5)-(8) and (11), we obtain

$$(12) \quad \max_{i,j} (p_i(n), p_{ij}(n)) \leq (p+1)/2 < 1, \quad n > k_1,$$

$$(13) \quad \min_{0 \leq i \leq q-1} p_i(n) \geq \bar{p} = 1/2 \min_{0 \leq i \leq q-1} p_i > 0.$$

Let the measure μ_n on $[0, 1)$ be such that

$$(14) \quad \mu_n([\gamma_r, \gamma_r + 1/q^r)) = p_{c_1}(n) p_{c_1 c_2}(n) \dots p_{c_{r-1} c_r}(n),$$

$$\gamma_r = .c_1 \dots c_r, \quad n, r = 1, 2, \dots$$

where $c_r \in \{0, 1, \dots, q-1\}$, $r = 1, 2, \dots$.

LEMMA 1. Let $\gamma = .c_1 \dots c_n \dots$. Then

$$(15) \quad \mu[0, \gamma) = \mu_n[0, \gamma_n) + O(np^n),$$

$$(16) \quad \mu[0, \gamma) = \mu_n[0, \gamma_n + 1/q^n) + O(np^n),$$

where the O -constant depends only on P .

Proof. It follows from (2), (5) and (6) that

$$(17) \quad \mu[0, \gamma) = \mu[0, \gamma_n) + \sum_{r \geq n+1} \sum_{b=0}^{c_r-1} p_{c_1} p_{c_1 c_2} \dots p_{c_{r-1} b} = \mu[0, \gamma_n) + O(p^n).$$

We apply (2), (14) and obtain

$$(18) \quad \mu[0, \gamma_n) = \mu_n[0, \gamma_n) + \sum_{r=1}^n \sum_{b=0}^{c_r-1} \sigma_r(b),$$

$$\sigma_r(b) = p_{c_1} p_{c_1 c_2} \dots p_{c_{r-1} b} - p_{c_1}(n) p_{c_1 c_2}(n) \dots p_{c_{r-1} b}(n).$$

If $p_{c_1} p_{c_1 c_2} \dots p_{c_{r-1} b} = 0$, then $p_{c_i c_j} = 0$ and according to (5), (7), (8), (11) we have $p_{c_i c_j}(n) = O(p^n)$ and

$$(19) \quad \sigma_r(b) = O(p^n).$$

Let $p_{c_1} p_{c_1 c_2} \dots p_{c_{r-1} b} \neq 0$. Then

$$(20) \quad \sigma_r(b) = p_{c_1} p_{c_1 c_2} \dots p_{c_{r-1} b} \Delta_r,$$

where

$$\Delta_r = 1 - \left(1 + \frac{a_{c_1}(n) - L_n p_{c_1}}{L_n p_{c_1}}\right) \prod_{k=1}^{r-1} \left(1 + \frac{a_{c_k v_k}(n) - A_n p_{c_k v_k}}{A_n p_{c_k v_k}}\right),$$

and $v_k = c_{k+1}$ or b .

On the basis of (5), (7), (8) and (11) we deduce that

$$|\Delta_r| \leq (1 + \frac{B}{p'A_n})(1 + \frac{q}{p'A_n})^{r-1} - 1 \leq (1 + \epsilon p^n)^r - 1, \quad p' = \min_{i,j} p_{ij},$$

where $|\epsilon| < 2qB/p'$.

It is easy to compute that

$$\Delta_r = O(rp^n), \quad r \leq n.$$

Hence and from (17) - (20) we obtain

$$\mu[0, \gamma] - \mu_n[0, \gamma_n] = O(np^n + np^n \sum_{r=1}^n \sum_{b=0}^{c_r-1} p_{c_1} p_{c_1 c_2} \dots p_{c_{r-1} b}) = O(np^n)$$

and formula (15) is proved. Statement (16) is proved analogously. ■

We obtain the Markov normal number $\alpha = .d_1 d_2 \dots$ by concatenating blocks $\alpha'_n = (a_1, \dots, a_{A_{2n}})$, where $a_i \in \{0, 1, \dots, q-1\}$, $i = 1, 2, \dots$

$$(21) \quad \alpha = .\alpha'_1 \dots \alpha'_n \dots,$$

We choose the numbers a_i as follows:

Let

$$(22) \quad \begin{aligned} \Omega_n &= \{\omega_n = (b_0, \dots, b_{A_{2n}+n}) \mid b_0 \in \{0, \dots, L_n - 1\}, b_1, b_2, \dots \in \{0, \dots, A_n - 1\}\} \\ S_0 &= [0, v_0(n)), S_j = [v_0(n) + \dots + v_{j-1}(n), v_0(n) + \dots + v_j(n)), \\ S_{i,0} &= [0, v_{i,0}(n)), \\ S_{i,j} &= [v_{i,0}(n) + \dots + v_{i,j-1}(n), v_{i,0}(n) + \dots + v_{i,j}(n)) \\ &\quad (i = 0, \dots, q-1, j = 1, \dots, q-1). \end{aligned}$$

We set $a_0 = i$, if $b_0 \in S_i$, $i = 0, \dots, q-1$. If we choose the numbers $a_0, \dots, a_{k-1}$, then we set

$$(23) \quad a_k = i, \text{ if } b_k \in S_{a_{k-1}, i}, \quad i = 0, \dots, q-1.$$

Let

$$(24) \quad \alpha_n = \alpha_n(\omega_n) = .a_1, \dots, a_{A_{2n}+n}, \quad n = 1, 2, \dots$$

$$(25) \quad R_{[\beta, \gamma]}(\mu_n, \alpha, M) = \#\{n \in [1, M] \mid \beta \leq \{\alpha q^n\} < \gamma\} - M\mu_n[\beta, \gamma],$$

$$(26) \quad E_n(\omega_n) = \max_{1 \leq M \leq A_{2n}} \max_{\gamma_n} |R_{[0, \gamma_n]}(\mu_n, \alpha_n(\omega_n), M)|,$$

$$(27) \quad E_n = \min_{\omega_n \in \Omega_n} E_n(\omega_n).$$

We choose ω_n (and consequently $\alpha_n(\omega_n)$) such that

$$(28) \quad E_n(\omega_n) = E_n.$$

LEMMA 2.

$$E_n = O(p^{-n}n^2).$$

Proof. (To follow later.)

Let

$$(29) \quad n_1 = 0, \dots, n_{k+1} = n_k + A_{2k}, \quad k = 1, 2, \dots.$$

Every natural N can be represented uniquely in the following form with integers k

$$(30) \quad N = n_k + M_1, \quad 0 \leq M_1 < A_{2k}, \quad k = 1, 2, \dots.$$

Let

$$T_\gamma(\alpha, Q, M) = \#\{n \in (Q, Q + M] \mid \{\alpha q^n\} < \gamma\},$$

$$(31) \quad R_\gamma(\mu, \alpha, Q, M) = T_\gamma(\alpha, Q, M) - M\mu[0, \gamma).$$

For $Q = 0$ we use the symbols $T_\gamma(\alpha, M)$ and $R_\gamma(\mu, \alpha, M)$.

THEOREM 1. *Let the number α be defined by (21), (23), (24) and (27). Then α is Markov-normal and the following estimate is true:*

$$(32) \quad D(\mu, N) = O(N^{-1/2} \log^2 N),$$

where the O -constant depends only on P .

Proof. Using (29), (30) and (31), we obtain

$$(33) \quad R_\gamma(\mu, \alpha, N) = \sum_{r=1}^{k-1} R_\gamma(\mu, \alpha, n_r, A_{2r}) + R_\gamma(\mu, \alpha, n_k, M_1).$$

According to (21), (24) and (25) we have

$$(34) \quad R_\gamma(\mu, \alpha, n_r, M) = R_\gamma(\mu, \alpha_r, M), \quad M < A_{2r} - 2r.$$

It follows from (31) that

$$R_\gamma(\mu, \alpha_r, M) = T_\gamma(\alpha_r, M) - M\mu[0, \gamma),$$

and

$$T_{\gamma_r}(\alpha_r, M) \leq T_\gamma(\alpha_r, M) \leq T_{\gamma_{r+1/q^r}}(\alpha_r, M).$$

It is evident that

$$|R_\gamma(\mu, \alpha_r, M)| \leq |T_{\gamma_r}(\alpha_r, M) - M\mu[0, \gamma]| + |T_{\gamma_r+1/q^r}(\alpha_r, M) - M\mu[0, \gamma]|.$$

We apply (31) and obtain

$$|R_\gamma(\mu, \alpha_r, M)| \leq |R_{\gamma_r}(\mu, \alpha_r, M)| + |R_{\gamma_r+1/q^r}(\mu, \alpha_r, M)| + M(|\mu[0, \gamma] - \mu[0, \gamma_r]| + |\mu[0, \gamma] - \mu[0, \gamma_r + 1/q^r]|).$$

On the basis of (26)-(28), Lemma 1 and Lemma 2 we deduce that

$$R_\gamma(\mu, \alpha_r, M) = O(p^{-r}r^2).$$

According to (34) we have for $M < A_{2r} - r$

$$(35) \quad R_\gamma(\mu, \alpha, n_r, M) = O(p^{-r}r^2).$$

It follows from (31) that

$$R_\gamma(\mu, \alpha_r, M) = R_\gamma(\mu, \alpha_r, M - 2r) + O(r),$$

It is evident from this that statement (35) is valid both for $M < A_{2r} - 2r$ as well as for $M \in [A_{2r} - 2r, A_{2r}]$.

Substituting (35) into (33) and bearing in mind (30) we deduce

$$R_\gamma(\mu, \alpha, N) = \sum_{r=1}^{k-1} O(p^{-r}r^2) + O(p^{-k}k^2) = O(p^{-k}k^2).$$

Using (29), (30) and (5) we obtain

$$R_\gamma(\mu, \alpha, N) = O(N^{1/2} \log^2 N).$$

Hence and from (1), (31) the statement of the theorem follows. ■

We denote

$$(36) \quad \delta(a) = \begin{cases} 1, & \text{if } a = 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(37) \quad \delta(a) = \frac{1}{N} \sum_{m=1}^N e^{2\pi i \frac{ma}{N}}, \quad 0 \leq a \leq N-1.$$

LEMMA 3. Let $1 \leq M \leq A_{2n}$ and

$$G_M = \sum_{x=1}^M g_x.$$

Then

$$(38) \quad |G_M| \leq \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1} \left| \sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{mx}{A_{2n}}} \right|.$$

Proof. According to (36) we have

$$G_M = \sum_{y=1}^M \sum_{x=1}^{A_{2n}} g_x \delta(x-y).$$

Using (37), we obtain

$$(39) \quad |G_M| = \left| \sum_{m=0}^{A_{2n}-1} \frac{1}{A_{2n}} \sum_{y=1}^M \sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{m(x-y)}{A_{2n}}} \right| \leq \\ \leq \sum_{m=0}^{A_{2n}-1} \frac{1}{A_{2n}} \left| \sum_{y=1}^M e^{2\pi i \frac{-my}{A_{2n}}} \right| \left| \sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{mx}{A_{2n}}} \right|.$$

Let $0 < N_2 - N_1 < A_{2n}$. It is known [5, p. 1] that

$$(40) \quad \frac{1}{A_{2n}} \left| \sum_{y=N_1}^{N_2} e^{2\pi i \frac{-my}{A_{2n}}} \right| \leq \min(1, \frac{1}{A_{2n} |\sin \frac{\pi m}{A_{2n}}|}) \leq \frac{1}{m+1}.$$

From (39) and (40) we give the assertion of the lemma. ■

LEMMA 4. Let $0 \leq u_1 \leq u_2 < A_{2n}$, $m \geq 0$, $i, j = 0, \dots, q-1$, $n > k_1$. Then

$$S = \sum_{x=u_1}^{u_2} e^{2\pi i \frac{mx}{A_{2n}}} (p_{ij}^{(x)}(n)/p_j(n) - 1) = O(1),$$

where the constant in symbol O depends only on P .

Proof. Let $N_1 = [u_1/d]$, $N_2 = [u_2/d]$. We change the variable $x = dy + z$ and obtain according to (13)

$$S = \epsilon \frac{2d}{\bar{p}} + \sum_{y=N_1}^{N_2} \sum_{z=1}^d e^{2\pi i \frac{m(dy+z)}{A_{2n}}} (p_{ij}^{(dy+z)}(n)/p_j(n) - 1), \quad \text{where } |\epsilon| < 1$$

Let

$$\sigma_y = \sum_{z=1}^d e^{2\pi i \frac{mz}{A_{2n}}} (p_{ij}^{(dy+z)}(n)/p_j(n) - 1).$$

It follows that

$$(41) \quad S = \epsilon \frac{2d}{\bar{p}} + \sum_{y=N_1}^{N_2} e^{2\pi i \frac{myd}{A_{2n}}} \sigma_y.$$

Applying (13), (10), we obtain

$$\sigma_y = de^{2\pi i \frac{mz_1}{A_{2n}}} + \epsilon_1 \frac{d}{p_j(n)} (1 - \theta)^{-1+y/k_0} - \sum_{z=1}^d e^{2\pi i \frac{mz}{A_{2n}}},$$

where $|\epsilon_1| < 1$, $z_1 = f(j) - f(i)$.

Substituting this formula into (41), we obtain according to (13), that

$$(42) \quad S = S_1 S_2 + \epsilon_1 \sum_{y=N_1}^{N_2} \frac{d}{p} (1 - \theta)^{-1+y/k_0}, \quad |\epsilon_1| \leq 1,$$

where

$$(43) \quad S_1 = \sum_{y=N_1}^{N_2} e^{2\pi i \frac{myd}{A_{2n}}} \quad S_2 = \sum_{z=1}^d (e^{2\pi i \frac{mz_1}{A_{2n}}} - e^{2\pi i \frac{mz}{A_{2n}}}).$$

It is known that

$$(44) \quad |e^{2\pi i \frac{m(z_1-z)}{A_{2n}}} - 1| = 2|\sin \pi m(z_1 - z)/A_{2n}| \leq 2\pi md/A_{2n}.$$

Using (40) we get

$$S_1 \leq A_{2n}/(md + 1).$$

Hence and from (42-44) the assertion of the lemma follows. ■

We consider further that a_i , $i = 1, 2, \dots$ is the sign of the number $\alpha_n(\omega_n)$.

It follows from (25), that

$$(45) \quad R_{[0, \gamma_n]}(\mu_n, \alpha_n, M) = \sum_{r=1}^n \sum_{b=0}^{c_r-1} R_{[\gamma_{r-1}+b/q^r, \gamma_{r-1}+(b+1)/q^r]}(\mu_n, \alpha_n, M),$$

and

$$R_{[\gamma_{r-1}+b/q^r, \gamma_{r-1}+(b+1)/q^r]}(\mu_n, \alpha_n, M) = \sum_{x=1}^M \delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - b) - M \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r].$$

Hence and from (45) we get

$$(46) \quad R_{[0, \gamma_n]}(\mu_n, \alpha_n, M) = \sum_{r=1}^n \sum_{b=0}^{c_r-1} \sum_{x=1}^M (\delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - b) - \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r]).$$

LEMMA 5. Let $n > k_1$,

$$(47) \quad B(r, c) = \sum_{x, y=1}^{A_{2n}} e^{2\pi i \frac{m(x-y)}{A_{2n}}} (\mu_n^2[\gamma_r, \gamma_r + \frac{1}{q^r}] + \sigma_1(x, y) - \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}](\sigma_2(x) + \sigma_2(y))),$$

where

$$(48) \quad \sigma_1(x, y) = \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - c_r) \delta(a_{y+1} - c_1) \dots \delta(a_{y+r} - c_r),$$

$$(49) \quad \sigma_2(x) = \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - c_r).$$

Then

$$(50) \quad E_n \leq \sum_{m=0}^{A_{2n}-1} \frac{(nq)^{1/2}}{m+1} \left(\sum_{r=1}^n \sum_{c_1=0}^{q-1} \dots \sum_{c_r=0}^{q-1} B(r, c) \right)^{1/2}.$$

Proof. It follows from (46) and Lemma 3 that

$$|R_{[0, \gamma_n)}(\mu_n, \alpha_n, M)| \leq \sum_{r=1}^n \sum_{b=0}^{c_r-1} \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1} \left| \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}} \right.$$

$$\left. (\delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - b) - \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r]) \right|.$$

Changing the order of summation and applying the Cauchy inequality

$$(51) \quad \left| \frac{1}{N} \sum_{n=1}^N g_n \right| \leq \left(\frac{1}{N} \sum_{n=1}^N |g_n|^2 \right)^{1/2},$$

we obtain that

$$\begin{aligned} & |R_{[0, \gamma_n)}(\mu_n, \alpha_n, M)| \leq \\ & \leq \sum_{m=0}^{A_{2n}-1} \frac{(qn)^{1/2}}{m+1} \left(\sum_{r=1}^n \sum_{b=0}^{c_r-1} \left| \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}} (\delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - b) - \right. \right. \\ & \quad \left. \left. - \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r]) \right|^2 \right)^{1/2}. \end{aligned}$$

We change the variable b to c_r and assume, on the right-hand side, the summation on c_i , $i = 1, \dots, r-1$.

It is evident that

$$(52) \quad |R_{[0, \gamma_n)}(\mu_n, \alpha_n, M)| \leq \sum_{m=0}^{A_{2n}-1} \frac{(qn)^{1/2}}{m+1} \left(\sum_{r=1}^n \sum_{c_1=0}^{q-1} \dots \sum_{c_r=0}^{q-1} \right.$$

$$\left| \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}} (\delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - c_r) - \mu_n[\gamma_r, \gamma_r + 1/q^r]) \right|^2 \Big)^{1/2}.$$

We denote by $S(\omega_n)$ the right-hand side of formula (52).

It is evident that $S(\omega_n)$ does not depend on M and γ_n .

Applying (26), we obtain

$$E_n(\omega_n) \leq S(\omega_n)$$

and

$$E_n \leq \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} E_n(\omega_n) \leq \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} S(\omega_n).$$

Changing the order of summation and using (51), we obtain

$$E_n \leq \sum_{m=0}^{A_{2n}-1} \frac{(qn)^{1/2}}{m+1} \left(\sum_{r=1}^n \sum_{c_1=0}^{q-1} \dots \sum_{c_r=0}^{q-1} \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \left| \sum_{x=1}^{A_{2n}} e^{2\pi i \frac{mx}{A_{2n}}} (\delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - c_r) - \mu_n[\gamma_r, \gamma_r + 1/q^r]) \right|^2 \right)^{1/2}.$$

Hence and from (47)-(49) we deduce formula (50). ■

LEMMA 6. Let $n > k_1$. Then

$$\sigma_2(x) = \mu_n[\gamma_r, \gamma_r + 1/q^r].$$

Proof. Applying (49) and (22), we get

$$\sigma_2(x) = \frac{1}{L_n A_n^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \dots \sum_{b_{x+r}=0}^{A_n-1} \delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - c_r).$$

According (23), we obtain

$$(53) \quad a_{x+i} = c_i \quad \text{if and only if} \quad b_{x+i} \in S_{c_{i-1}c_i} \quad i = 2, 3, \dots$$

It follows that

$$\begin{aligned} \sigma_2(x) &= \frac{1}{L_n A_n^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \dots \sum_{b_{x+1}=0}^{A_n-1} \delta(a_{x+1} - c_1) \sum_{b_{x+2} \in S_{c_1 c_2}} \dots \sum_{b_{x+r} \in S_{c_{r-1} c_r}} 1 = \\ &= \frac{1}{L_n A_n^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \dots \sum_{b_{x+1}=0}^{A_n-1} \delta(a_{x+1} - c_1) v_{c_1 c_2}(n) \dots v_{c_{r-1} c_r}(n). \end{aligned}$$

Using (7) we get

$$(54) \quad \sigma(x) = \sigma p_{c_1 c_2}(n) \dots p_{c_{r-1} c_r}(n),$$

where

$$(55) \quad \sigma = \frac{1}{L_n A_n^{x+1}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \dots \sum_{b_{x+1}=0}^{A_n-1} \delta(a_{x+1} - c_1).$$

It is obvious that

$$(56) \quad \sum_{d_0, \dots, d_x=0}^{q-1} \prod_{i=0}^x \delta(a_i - d_i) = 1.$$

Hence and from (55) we obtain, changing the order of summation

$$(57) \quad \sigma = \sum_{d_0, \dots, d_x=0}^{q-1} \frac{1}{L_n A_n^{x+1}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \dots \sum_{b_{x+1}=0}^{A_n-1} \prod_{i=0}^x \delta(a_i - d_i) \delta(a_{x+1} - c_1).$$

According to (53), (36) and (22), we have

$$\begin{aligned} \sigma &= \sum_{d_0, \dots, d_x=0}^{q-1} \frac{1}{L_n A_n^{x+1}} \sum_{b_0 \in S_{d_0}} \sum_{b_1 \in S_{d_0 d_1}} \dots \sum_{b_{x+1} \in S_{d_x c_1}} 1 = \\ &= \sum_{d_0, \dots, d_x=0}^{q-1} \frac{1}{L_n A_n^{x+1}} v_{d_0}(n) v_{d_0 d_1}(n) v_{d_x c_1}(n). \end{aligned}$$

Applying (7) and (11), we obtain

$$\sigma = p_{c_1}(n).$$

On the basis of (54) and (14) the lemma is proved. ■

LEMMA 7. Let $n > k_1$, $|y - x| > r$. Then

$$\sigma_1(x, y) = \mu_n^2[\gamma_r, \gamma_r + 1/q^r] p_{c_r c_1}^{(|y-x|-r)}(n) / p_{c_1}(n).$$

Proof. Let $y > x$.

Applying (48) and (22), we obtain $\sigma_1(x, y) =$

$$\frac{1}{L_n A_n^{y+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \dots \sum_{b_{y+r}=0}^{A_n-1} \delta(a_{x+1} - c_1) \dots \delta(a_{x+r} - c_r) \delta(a_{y+1} - c_1) \dots \delta(a_{y+r} - c_r).$$

As in the proof of Lemma 6, we get

$$(58) \quad \sigma_1(x, y) = p_{c_1}(n) (p_{c_1 c_2}(n) \dots p_{c_{r-1} c_r}(n))^2 \sigma,$$

where

$$(59) \quad \sigma = \frac{1}{A_n^{y-x-r}} \sum_{b_{x+r+1}=0}^{A_n-1} \dots \sum_{b_{y+1}=0}^{A_n-1} \delta(a_{x+r} - c_r) \delta(a_{y+1} - c_1).$$

As in (56), we have

$$\sum_{d_1, \dots, d_{y-x-r}=0}^{q-1} \prod_{i=1}^{y-x-r} \delta(a_{x+r+i} - d_i) = 1.$$

Hence and from (59), changing the order of summation, we obtain

$$\begin{aligned} \sigma = & \sum_{d_1, \dots, d_{y-x-r}=0}^{q-1} \frac{1}{A_n^{y-x-r}} \sum_{b_{x+r+1}=0}^{A_n-1} \dots \sum_{b_{y+1}=0}^{A_n-1} \delta(a_{x+r} - c_r) \times \\ & \times \prod_{i=1}^{y-x-r} \delta(a_{x+r+i} - d_i) \delta(a_{y+1} - c_1). \end{aligned}$$

Using (53), (36) and (22), we get

$$\sigma = \sum_{d_1, \dots, d_{y-x-r}=0}^{q-1} \frac{1}{A_n^{y-x-r}} \sum_{b_{x+r+1} \in S_{c_r, d_1}} \dots \sum_{b_{y+1} \in S_{d_{y-x-r}, c_1}} 1.$$

Applying (7) and (11), we obtain

$$\sigma = \sum_{d_1, \dots, d_{y-x-r}=0}^{q-1} p_{c_r, d_1}(n) p_{d_1, d_2}(n) \dots p_{d_{y-x-r}, c_1}(n) = p_{c_r, c_1}^{(y-x-r)}(n).$$

It follows from (58), that

$$\sigma_1(x, y) = (p_{c_1}(n) p_{c_1, c_2}(n) \dots p_{c_{r-1}, c_r}(n))^2 p_{c_r, c_1}^{(y-x-r)}(n) / p_{c_1}(n).$$

Similarly for $x < y$. According (14) the lemma is proved. ■

LEMMA 8. Let $n > k_1$, $|y - x| \leq r$. Then

$$\sigma_1(x, y) \leq \mu_n[\gamma_r, \gamma_r + 1/q^r] \left(\frac{1+p}{2}\right)^{|y-x|-1}.$$

Proof. Let $y \geq x$.

As in the proof of Lemma 6 and Lemma 7, we get

$$\sigma_1(x, y) \leq p_{c_1}(n) p_{c_1, c_2}(n) \dots p_{c_{y-x-1}, c_{y-x}}(n) p_{c_{y-x}, c_1}(n) p_{c_1, c_2}(n) \dots p_{c_{r-1}, c_r}(n).$$

It follows from (12), that

$$\sigma_1(x, y) \leq p_{c_1}(n) p_{c_1, c_2}(n) \dots p_{c_{r-1}, c_r}(n) \left(\frac{1+p}{2}\right)^{y-x-1}.$$

Similarly for $x < y$. According to (14) the lemma is proved. ■

LEMMA 9. Let $n > k_1$. Then

$$(60) \quad B(r, c) = O(A_{2n} \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}]).$$

Proof. Applying (47) and Lemma 6, we obtain

$$B(r, c) = \sum_{x, y=1}^{A_{2n}} \sigma(x, y),$$

where

$$\sigma(x, y) = e^{2\pi i \frac{m(x-y)}{A_{2n}}} (\sigma_1(x, y) - \mu_n^2[\gamma_r, \gamma_r + \frac{1}{q^r}]).$$

Let

$$(61) \quad B(r, c) = B_1 + B_2 + B_3, \quad \text{where} \quad B_1 = \sum_{1 \leq x, y \leq A_{2n}, |y-x| \leq r} \sigma(x, y),$$

$$B_2 = \sum_{1 \leq x, y \leq A_{2n}, y-x > r} \sigma(x, y), \quad B_3 = \sum_{1 \leq x, y \leq A_{2n}, x-y > r} \sigma(x, y).$$

According to Lemma 8, (12) and (14) we obtain

$$(62) \quad |B_1| \leq \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}] \sum_{1 \leq x, y \leq A_{2n}, |y-x| \leq r} (\frac{1+p}{2})^{|y-x|-1} =$$

$$= O(A_{2n} \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}]).$$

It follows from Lemma 7 that

$$B_2 = \mu_n^2[\gamma_r, \gamma_r + 1/q^r] \sum_{x=1}^{A_{2n}} \sum_{y=x+r}^{A_{2n}} e^{2\pi i \frac{m(x-y)}{A_{2n}}} (p_{c_r c_1}^{(y-x-r)}(n) / p_{c_1}(n) - 1).$$

Changing the variable y to $y_1 = y - x - r$ and applying Lemma 3, we obtain

$$B_2 = O(A_{2n} \mu_n^2[\gamma_r, \gamma_r + 1/q^r]).$$

Similarly estimate is valid for B_3 .

Hence and from (61)-(62) we obtain the assertion of the lemma. ■

Proof of Lemma 2. Substituting (60) into (50) and bearing in mind (5), we deduce

$$E_n = O(\sum_{m=0}^{A_{2n}-1} \frac{(nq)^{1/2}}{m+1} (\sum_{r=1}^n \sum_{c_1=0}^{q-1} \dots \sum_{c_r=0}^{q-1} A_{2n} \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}])^{1/2} =$$

$$O(\sqrt{A_{2n} n} \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1}) = O(p^{-n} n^2).$$

Lemma 2 is proved. ■

Remark. By a similar method and the method in [12] a Markov normal vector for the multidimensional case can be constructed. By the method

in [12] one can reduce the logarithmic multiplier in (32) to $O(\log N^{3/2})$. To reduce the logarithmic multiplier further see [15].

Problem. According to [12-14] the Borel and Bernoulli normal numbers exist with discrepancy $O(N^{-2/3+\epsilon})$. It would be interesting to know whether Markov normal numbers exist with discrepancy $O(N^{-c})$ where $c > 1/2$.

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