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Non-vanishing of $n$-th derivatives of twisted elliptic $L$-functions in the critical point


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par JACEK POMYKALA

ABSTRACT. Let \( E \) be a modular elliptic curve over \( \mathbb{Q} \) and \( L^{(n)}(s, E) \) denote the \( n \)-th derivative of its Hasse-Weil L-series. We estimate the number of twisted elliptic curves \( E_d, d \leq D \) such that \( L^{(n)}(1, E_d) \neq 0 \).

1. Introduction

Let \( E = E/\mathbb{Q} \) be a modular elliptic curve over \( \mathbb{Q} \) with conductor \( N \) defined by the Weierstrass equation \( y^2 = f(x) \) and let \( \mathcal{D} \) be defined as follows:

\[
\mathcal{D} = \{ d \text{ - square-free} : 0 < d \equiv -\nu^2 \pmod{4N} \text{ for some } \nu \text{ prime to } 4N \}.
\]

For any \( d \in \mathcal{D} \) we consider the twisted elliptic curve \( E_d \) given by the equation \( -dy^2 = w(x) \). We denote by \( L(s, E) \) and \( L(s, E_d) \) the Hasse-Weil L-functions associated to the curves \( E \) and \( E_d \), respectively.

The celebrated Birch and Swinnerton-Dyer conjecture (see [B-S]) asserts that the rank of the group of rational points of \( E/\mathbb{Q} \) is equal to the vanishing order of the associated Hasse-Weil function \( L(s, E) \) at \( s = 1 \). Kolyvagin [Ko] has proved that \( E/\mathbb{Q} \) has rank equal to zero if:

1) \( L(E, 1) \neq 0 \),

2) There exists \( d \in \mathcal{D} \) such that \( L(s, E_d) \) has a simple zero at \( s = 1 \).
$E/\mathbb{Q}$ has rank equal to one if:

1') $L(s, E)$ has a simple zero at $s = 1$.

2') There exists $d \in \mathbb{D}$ such that $L(1, E_d) \neq 0$.

The condition 2') is true according to Waldspurger's theorem (see [Wa]). The condition 2) has been proved to hold for infinitely many $d \in \mathbb{D}$ (see [B-F-H], [M-M]).

Iwaniec [Iw] has also proved a quantitative result on this condition. Let

$$N(D) = \#\{d \in \mathbb{D}, d \leq D : L'(1, E_d) \neq 0\}.$$ 

He has obtained the estimate

$$N(D) \gg D^{\frac{3}{2} - \varepsilon}$$

with arbitrary $\varepsilon > 0$.

The above exponent is improved to $1 - \varepsilon$ in [P-P]. Here we will generalize this result to the $n$-th derivative of $L(s, E_d)$ where $n$ is an arbitrary non-negative integer.

In this connection let $w = w(E)$ be the sign in the functional equation (see (1)). We define

$$N_n(D) = \#\{d \in \mathbb{D}, d \leq D : L^{(n)}(1, E_d) \neq 0\}.$$ 

We will prove

**Theorem 1.** Let $\varepsilon$ be an arbitrary positive real number and $n$ be a fixed non-negative integer. Then we have as $D$ tends to infinity

$$N_n(D) \gg (n + |w - 1|)D^{1 - \varepsilon}.$$ 

where the constant implied in the symbol $\gg$ depends on $n$ and $\varepsilon$.

Our result is based on a recent large sieve type estimates over fundamental discriminants obtained by Heath-Brown [HB] (see Theorem 3 of [P-P]) and the method applied by Iwaniec in [Iw].
2. Outline of the proofs

For $\text{Re } s > 1$ the corresponding $L$-functions are given by

$$ L(s, E) = \sum_{k=1}^{\infty} a_k k^{-s}, \quad L(s, E_d) = \sum_{k=1}^{\infty} \chi_d(k) a_k k^{-s}, $$

where $\chi_d(\cdot) = \left( \frac{-d}{\cdot} \right)$ (the Kronecker symbol) is a real character to modulus $d$ prime to $4N$. They have the analytic prolongation to the whole complex plane, where they satisfy the equations

$$ \left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(s, E) = w \left( \frac{\sqrt{N}}{2\pi} \right)^{2-s} \Gamma(2-s)L(E, 2-s), \tag{1} $$

$$ \left( \frac{d\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(s, E_d) = w_d \left( \frac{d\sqrt{N}}{2\pi} \right)^{2-s} \Gamma(2-s)L(2-s, E_d), \tag{2} $$

where $w$ and $w_d = \pm 1$ are suitable constants depending on the reduction of $E$ at the primes dividing $N$, which satisfy the equality

$$ w_d = w \chi_d(-N). \tag{3} $$

From (1) and (2) it follows that

$$ w = (-1)^{\text{ord } L(s, E)_{s=1}}, \quad w_d = (-1)^{\text{ord } L(s, E_d)_{s=1}}. $$

Theorem 1 is a consequence of the following two theorems.

**Theorem 2.** For any integer $n \geq 0$ we have the asymptotical equality

$$ \sum_{d \leq D}^{\ast} L^{(n)}(1, E_d)F \left( \frac{d}{Y} \right) \sim c_n L(1) \int F(t) dt \ D(\ln D)^n, \quad \text{as } D \to \infty, $$

where $F$ is a smooth function compactly supported in $\mathbb{R}^+$ with positive mean-value and the star (*) above means that the summation is restricted to $d \in \mathbb{D}$. The (nonzero) constant $L(1)$ is described in $[Iw]$, while

$$ c_n = \begin{cases} 
\frac{1}{2} (1 - w)c & \text{if } n = 0 \\
 w(-2)^{n-1}c & \text{if } n \geq 1 
\end{cases} \tag{4} $$
\[
c = \frac{3}{\pi^2 N} \prod_{p \mid 4N} \left( 1 - \frac{1}{p^2} \right)^{-1} \times \\
\times \# \{ d \pmod{4N} : d \equiv -\nu^2 \pmod{4N}, (\nu, 4N) = 1 \}.
\]

The proof follows the idea exploited in [Iw] (cf. also [M-M] and [P-S]). We postpone it to sect. 3.

Along the same lines as Theorem 3 of [P-P] we obtain

**Theorem 3.** Let \( \varepsilon > 0 \) and \( n \) be a fixed non-negative integer. Then we have

\[
\sum_{d \leq D}^* |L^{(n)}(1, E_d)|^2 \ll_\varepsilon D^{1+\varepsilon}
\]

Now from Theorem 2 and Theorem 3 we obtain by an application of the Cauchy-Schwarz inequality that

\[
(n+|w-1|)D(\ln D)^n \ll \sum_{d \leq D}^* |L^{(n)}(1, E_d)| \leq \left( \sum_{d \leq D}^* |L^{(n)}(1, E_d)|^2 \right)^{1/2} N_n(D)^{1/2}
\]

hence

\[
N_n(D) \gg (n+|w-1|)D^{1-2\varepsilon}
\]

and Theorem 1 follows.

3. **Proof of Theorem 2**

For \( n \geq 0 \) we introduce the approximate functions

\[
A_n(X, \chi_d) = \sum_{m=1}^{\infty} a_m \left( \frac{\chi_d(m)}{m} \right) V_n \left( \frac{2\pi m}{X} \right)
\]

(6)

with

\[
V_n(X) = \frac{1}{2i} \int_{(3/4)} \frac{\Gamma(s)}{s^n} X^{-s} ds,
\]

(7)
where we integrate over the line  \( \text{Re } s = \frac{3}{4} \).

By the functional equation (2) we have

\[
L(1 + s, E_d) = w_d \left( \frac{d \sqrt{N}}{2\pi} \right)^{-2s} L(1 - s, E_d) \frac{\Gamma(1 - s)}{\Gamma(1 + s)}.
\]

Therefore

\[
\frac{1}{2\pi i} \int_{(-3/4)} L(1 + s, E_d) \frac{\Gamma(s)}{s^n} \left( \frac{X}{2\pi} \right)^s ds
\]

\[
= \frac{w_d}{2\pi i} \int_{(-3/4)} L(1 - s, E_d) \frac{\Gamma(1 - s)}{\Gamma(1 + s)} \frac{\Gamma(s)}{s^n} \left( \frac{2\pi X}{d^2 N} \right)^s ds
\]

\[
= -\frac{w_d}{2\pi i} \int_{(-3/4)} L(1 - s, E_d) \frac{\Gamma(-s)}{s^n} \left( \frac{2\pi X}{d^2 N} \right)^s ds
\]

\[
= (-1)^{n+1} \frac{w_d}{2\pi i} \int_{(3/4)} L(1 + s, E_d) \frac{\Gamma(s)}{s^n} \left( \frac{2\pi}{d^2 N/X} \right)^{-s} ds
\]

\[
= (-1)^{n+1} w_d A_n \left( \frac{d^2 N}{X}, \chi_d \right).
\]

Hence defining

\[
G_n(s, d, X) = L(1 + s, E_d) \frac{\Gamma(s)}{s^n} \left( \frac{X}{2\pi} \right)^s
\]

we obtain by the Cauchy theorem

\[
\text{res}_{s=0} G_n(s, d, X) = \frac{1}{2\pi i} \int_{(3/4)} G_n(s, d, X) ds - \frac{1}{2\pi i} \int_{(-3/4)} G_n(s, d, X) ds
\]

\[
= A_n(X, \chi_d) + (-1)^n w_d A_n \left( \frac{d^2 N}{X}, \chi_d \right).
\]

By the definition of \( \mathbb{D} \) we have \( w_d = w \chi_d(-N) = -w \) for any \( d \in \mathbb{D} \). Hence letting \( X = d \sqrt{N} \) we obtain

\[
\text{res}_{s=0} G_n(s, d, d \sqrt{N}) = \begin{cases} 
2A_n(d \sqrt{N}, \chi_d) & \text{if } n \equiv \frac{w + 1}{2} \pmod{2} \\
0 & \text{if } n \equiv \frac{w - 1}{2} \pmod{2}
\end{cases}
\]  \( (8) \)
On the other hand the residuum of $G_n$ may be expressed in terms of the derivatives $L^{(k)}(s, E_d)$, $k = 0, 1, 2, \ldots$, by means of the Laurent expansions
\[
\Gamma(s) = s^{-1} + \gamma_0 + \gamma_1 s + \ldots ,
\]
\[
\left( \frac{X}{2\pi} \right)^s = \sum_{k=0}^{\infty} b_k s^k ,
\]
where $-\gamma_0$ is the Euler constant and $b_k = b_k(X) = \frac{1}{k!} \left( \ln \frac{X}{2\pi} \right)^k$.

Namely we prove

**Lemma.** For any $n \geq 0$ we have

\[
\text{res}_{s=0} G_n(s, d, X) = b_0 \frac{L^{(n)}(1, E_d)}{n!}
+ \sum_{l=0}^{n-1} b_{l+1} + \sum_{m=0}^{l} \frac{\gamma_m b_{l-m}}{(n-1-l)!} \frac{L^{(n-1-l)}(1, E_d)}{(n-1-l)!} .
\]

**Proof.** We have

\[
G_n(s, d, X) = s^{-n} \left( \sum_{k=0}^{\infty} \frac{L^{(k)}(1, E_d)}{k!} s^k \right) \left( s^{-1} + \sum_{k=0}^{\infty} \gamma_k s^k \right) \left( \sum_{k=0}^{\infty} b_k s^k \right)
= s^{-n} \left( \sum_{k=0}^{\infty} \frac{L^{(k)}(1, E_d)}{k!} s^k \right) \left( \sum_{k=0}^{\infty} b_k s^{k-1} + \sum_{k=0}^{\infty} f_k s^k \right),
\]

where

\[
f_k = \sum_{l=0}^{k} b_l \gamma_{k-l} .
\]

The expression in the second bracket is equal to

\[
\frac{b_0}{s} + \sum_{k=0}^{\infty} b_{k+1} s^k + \sum_{k=0}^{\infty} f_k s^k = \frac{b_0}{s} + \sum_{k=0}^{\infty} (f_k + b_{k+1}) s^k .
\]

Hence

\[
G_n(s, d, X) = s^{-n} \left( \frac{b_0}{s} \sum_{k=0}^{\infty} \frac{L^{(k)}(1, E_d)}{k!} s^k + \sum_{k=0}^{\infty} c_k s^k \right) ,
\]
where
\[ c_k = \sum_{l=0}^{k} (f_l + b_{l+1}) \frac{L^{(k-l)}(1, E_d)}{(k-l)!}. \]

Therefore
\[
\res_{s=0} G_n(s, d, X) = b_0 \frac{L(n)(1, E_d)}{\eta!} + c_{n-1}
= b_0 \frac{L(n)(1, E_d)}{\eta!} + \sum_{l=0}^{n-1} (b_{l+1} + f_l) \frac{L^{(n-l-1)}(1, E_d)}{(n-1-l)!},
\]

as required.

Since \( b_l(X) = \frac{1}{l!} \left( \ln \frac{X}{2\pi} \right)^l \) we obtain asymptotically
\[
\res_{s=0} G_n(s, d, X) \sim \sum_{l=0}^{n} b_l(X) \frac{L^{(n-l)}(1, E_d)}{(n-l)!}, \quad \text{as } X \to \infty. \tag{9}
\]

By (8) and the Lemma we obtain the formula
\[
\sum_{l=0}^{n} b_l(d\sqrt{N}) \frac{L^{(n-l)}(1, E_d)}{(n-l)!} \sim \begin{cases} 
2A_n(d\sqrt{N}, \chi_d) & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\
0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}, 
\end{cases} \tag{10}
\]

as \( d \to \infty \). Next we sum both sides of the above equality multiplied by the weighted function \( F\left( \frac{d}{D} \right) \) over the numbers \( d \in \mathbb{D} \). The contribution of the right-hand side of (10) is evaluated on the basis of the results on square-free sieve obtained in \S7-\S9 of [Iw]. Precisely we have
\[
\sum_{d \in \mathbb{D}} 2A_n(d\sqrt{N}, \chi_d) F\left( \frac{d}{D} \right) \sim cD \int F(t) B_n(tD\sqrt{N}) \, dt, \tag{11}
\]
where
\[
B_n(X) = \int_{(3/4)} L(s+1) \frac{\Gamma(s)}{s^n} \left( \frac{X}{2\pi} \right)^s \, ds,
\]
the constant $c$ is defined by (5) and the series $L(s)$ is defined in §9 of [Iw].

In order to find the asymptotical behaviour of the right-hand side of (11) we denote
\[
\bar{G}_n(s, X) = L(s + 1) \frac{\Gamma(s)}{s^n} \left( \frac{X}{2\pi} \right)^s
\]
and apply the contour integration to obtain
\[
B_n(X) = \frac{1}{2\pi i} \int_{(3/4)} \bar{G}_n(s, X) \, ds
\]
\[
= \text{res}_{s=0} \bar{G}_n(s, X) + \frac{1}{2\pi i} \int_{(-1/4)} \bar{G}_n(s, X) \, ds
\]
\[
= \text{res}_{s=0} \bar{G}_n(s, X) + O(X^{-1/4}).
\]

Applying the asymptotical equality (9) with $G_n$ replaced by $\bar{G}_n$ we obtain
\[
B_n(X) \sim \sum_{l=0}^{n} b_l(X) \frac{L^{(n-l)}(1)}{(n-l)!} \sim b_n(X) L(1) \sim \frac{1}{n!} (\ln X)^n L(1), \quad \text{as } X \to \infty.
\]

Hence the right-hand side of (11) is asymptotically equal to
\[
L(1) \frac{c_0}{n!} D \int F(t) \ln^n(tD\sqrt{N}) \, dt \sim L(1) \frac{c_0}{n!} \int F(t) \, dt \, D(\ln D)^n.
\]

Therefore in view of (10) we obtain the asymptotic equality
\[
\sum_{d \leq D} \sum_{l=0}^{n} b_l(d\sqrt{N}) \frac{L^{(n-l)}(1, E_d)}{(n-l)!} F\left( \frac{d}{D} \right) \sim
\]
\[
\begin{cases}
  \frac{c}{n!} L(1) \int F(t) \, dt \, D(\ln D)^n & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\
  0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}.
\end{cases}
\]

Hence we obtain immediately that the constant $c_0$ in Theorem 2 is equal to $\left( \frac{1-w}{2} \right) c$, where $c$ is the constant defined by (5).

Assuming that
\[
\sum_{d \in \mathbb{D}} L^{(k)}(1, E_d) F\left( \frac{d}{D} \right) \sim c_k L(1) \int F(t) \, dt D(\ln D)^k
\]
where $c_k$ are some constants we see that they have to satisfy the equality

$$
\sum_{k=0}^{n} \binom{n}{k} c_k = \begin{cases} 
  c & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\
  0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}.
\end{cases} \tag{12}
$$

To complete the proof of Theorem 2 it remains to prove that (12) holds with the constants $c_k$ defined by (4). Indeed we have

$$
\sum_{k=0}^{n} \binom{n}{k} c_k = \left( \frac{1-w}{2} \right) c + \sum_{k=1}^{n} \binom{n}{k} c_k
$$

$$
= c \left( \frac{1-w}{2} - w \left( \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} (-2)^k \right) \right)
$$

$$
= c \left( \frac{1-w}{2} - \frac{w}{2}((-1)^n - 1) \right)
$$

$$
= \begin{cases} 
  c & \text{if } n \equiv \frac{w+1}{2} \pmod{2} \\
  0 & \text{if } n \equiv \frac{w-1}{2} \pmod{2}
\end{cases}
$$

as it is claimed. This completes the proof of Theorem 2.

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References


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