Linear fractional transformations of continued fractions with bounded partial quotients


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Linear Fractional Transformations of Continued Fractions with Bounded Partial Quotients

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RéSUMÉ. Soit $\theta$ un nombre réel de développement en fraction continue

$$\theta = [a_0, a_1, a_2, \ldots] ,$$

et soit

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

une matrice d'entiers tel que $\det M \neq 0$. Si $\theta$ est à quotients partiels bornés, alors $a_{\theta+b}/(c\theta+d) = [a_0^*, a_1^*, a_2^*, \ldots]$ est aussi à quotients partiels bornés. Plus précisément, si $a_j \leq K$ pour tout $j$ suffisamment grand, alors $a_j^* \leq |\det(M)|(K+2)$ pour tout $j$ suffisamment grand. Nous donnons aussi une borne plus faible qui est valable pour tout $a_j^*$ avec $j \geq 1$. Les démonstrations utilisent la constante d’approximation diophantienne homogène $L_\infty(\theta) = \limsup_{q \to \infty}(q||q\theta||)^{-1}$. Nous montrons que

$$\frac{1}{|\det(M)|} L_\infty(\theta) \leq L_\infty \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)| L_\infty(\theta).$$

ABSTRACT. Let $\theta$ be a real number with continued fraction expansion

$$\theta = [a_0, a_1, a_2, \ldots] ,$$

and let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a matrix with integer entries and nonzero determinant. If $\theta$ has bounded partial quotients, then $a_{\theta+b}/(c\theta+d) = [a_0^*, a_1^*, a_2^*, \ldots]$ also has bounded partial quotients. More precisely, if $a_j \leq K$ for all sufficiently large $j$, then $a_j^* \leq |\det(M)|(K+2)$ for all sufficiently large $j$. We also give a weaker bound valid for all $a_j^*$ with $j \geq 1$. The proofs use the homogeneous Diophantine approximation constant $L_\infty(\theta) = \limsup_{q \to \infty}(q||q\theta||)^{-1}$. We show that

$$\frac{1}{|\det(M)|} L_\infty(\theta) \leq L_\infty \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)| L_\infty(\theta).$$
1. INTRODUCTION.

Let $\theta$ be a real number whose expansion as a simple continued fraction is

$$\theta = [a_0, a_1, a_2, \ldots] ,$$

and set

$$K(\theta) := \sup_{i \geq 1} a_i ,$$

where we adopt the convention that $K(\theta) = +\infty$ if $\theta$ is rational. We say that $\theta$ has bounded partial quotients if $K(\theta)$ is finite. We also set

$$K_\infty(\theta) := \limsup_{i \geq 1} a_i ,$$

with the convention that $K_\infty(\theta) = +\infty$ if $\theta$ is rational. Certainly $K_\infty(\theta) \leq K(\theta)$, and $K_\infty(\theta)$ is finite if and only if $K(\theta)$ is finite.

A survey of results about real numbers with bounded partial quotients is given in [17]. The property of having bounded partial quotients is equivalent to $\theta$ being a badly approximable number, which is a number $\theta$ such that

$$\liminf_{q \to \infty} q||q\theta|| > 0 ,$$

in which $||x|| = \min(x - \lfloor x \rfloor, \lfloor x \rfloor - x)$ denotes the distance from $x$ to the nearest integer and $q$ runs through integers.

This note proves two quantitative versions of the theorem that if $\theta$ has bounded partial quotients and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then $\psi = \frac{a\theta + b}{c\theta + d}$ also has bounded partial quotients.

The first result bounds $K_\infty(\frac{a\theta + b}{c\theta + d})$ in terms of $K_\infty(\theta)$ and depends only on $|\det(M)|$:

**Theorem 1.1.** Let $\theta$ have a bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then

$$\frac{1}{|\det M|} K_\infty(\theta) - 2 \leq K_\infty\left(\frac{a\theta + b}{c\theta + d}\right) \leq |\det M|(K_\infty(\theta) + 2) .$$

The second result upper bounds $K\left(\frac{a\theta + b}{c\theta + d}\right)$ in terms of $K(\theta)$, and depends on the entries of $M$:
**Theorem 1.2.** Let \( \theta \) have bounded partial quotients. If \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is an integer matrix with \( \det(M) \neq 0 \), then
\[
K \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)|(K(\theta) + 2) + |c(c\theta + d)|.
\]

The last term in (1.4) can be bounded in terms of the partial quotient \( a_0 \) of \( \theta \), since
\[
|c\theta + d| \leq |c|(|a_0| + 1) + |d| \leq |ca_0| + |c| + |d|.
\]

Theorem 1.2 gives no bound for the partial quotient \( a_0 := \lfloor \frac{a_0}{d} \rfloor \) of \( \frac{a_0}{d} \).

Chowla [3] proved in 1931 that \( K(\frac{a}{d}) < 2ad(K(\theta) + 1)^3 \), a result rather weaker than Theorem 1.2.

We obtain Theorem 1.1 and Theorem 1.2 from stronger bounds that relate Diophantine approximation constants of \( \theta \) and \( \frac{a\theta + b}{c\theta + d} \), which appear below as Theorem 3.2 and Theorem 4.1, respectively. Theorem 3.2 is a simple consequence of a result of Cusick and Mendès France [5] concerning the Lagrange constant of \( \theta \) (defined in Section 2).

The continued fraction of \( \frac{a\theta + b}{c\theta + d} \) can be directly computed from that for \( \theta \), as was observed in 1894 by Hurwitz [9], who gave an explicit formula for the continued fraction of \( 2\theta \) in terms of that of \( \theta \). In 1912 Châtelet [2] gave an algorithm for computing the continued fraction of \( \frac{a\theta + b}{c\theta + d} \) from that of \( \theta \), and in 1947 Hall [7] also gave a method. Let \( \mathcal{M}(n, \mathbb{Z}) \) denote the set of \( n \times n \) integer matrices. Raney [15] gave for each \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}(2, \mathbb{Z}) \) with \( \det(M) \neq 0 \) an explicit finite automaton to compute the additive continued fraction of \( \frac{a\theta + b}{c\theta + d} \) from the additive continued fraction of \( \theta \).

In connection with the bound of Theorem 1.1, Davenport [6] observed that for each irrational \( \theta \) and prime \( p \) there exists some integer \( 0 \leq a < p \) such that \( \theta' = \theta + \frac{a}{p} \) has infinitely many partial quotients \( a_n(\theta') \geq p \). Mendès France [13] then showed that there exists some \( \theta' = \theta + \frac{a}{p} \) having the property that a positive proportion of the partial quotients of \( \theta' \) have \( a_n(\theta') \geq p \).

Some other related results appear in Mendès France [11,12]. Basic facts on continued fractions appear in [1,8,10,18].

### 2. BADLY APPROXIMABLE NUMBERS

Recall that the continued fraction expansion of an irrational real number
\[ \theta = [a_0, a_1, \ldots] \] is determined by
\[ \theta = a_0 + \theta_0, \quad 0 < \theta_0 < 1, \]
and for \( n \geq 1 \) by the recursion
\[ \frac{1}{\theta_{n-1}} = a_n + \theta_n, \quad 0 < \theta_n < 1. \]

The \textit{n-th complete quotient} \( \alpha_n \) of \( \theta \) is
\[ \alpha_n := \frac{1}{\theta_n} = [a_n, a_{n+1}, a_{n+2}, \ldots]. \]

The \textit{n-th convergent} \( \frac{p_n}{q_n} \) of \( \theta \) is
\[ \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n], \]
whose denominator is given by the recursion \( q_{-1} = 0, q_0 = 1, \) and \( q_{n+1} = a_{n+1}q_n + q_{n-1}. \) It is well known (see [8, §10.7]) that
\[ ||q_n\theta|| = |q_n\theta - p_n| = \frac{1}{q_n\alpha_{n+1} + q_{n-1}}. \]

Since \( a_{n+1} \leq \alpha_{n+1} < a_{n+1} + 1 \) and \( q_{n-1} \leq q_n, \) this implies that
\[ \frac{1}{a_{n+1} + 2} < q_n||q_n\theta|| \leq \frac{1}{a_{n+1}}, \]
for \( n \geq 0. \)

We consider the following Diophantine approximation constants. For an irrational number \( \theta \) define its \textit{type} \( L(\theta) \) by
\[ L(\theta) = \sup_{q \geq 1} (q||q\theta||)^{-1}, \]
and define the \textit{homogeneous Diophantine approximation constant} or \textit{Lagrange constant} \( L_{\infty}(\theta) \) of \( \theta \) by
\[ L_{\infty}(\theta) = \limsup_{q \geq 1} (q||q\theta||)^{-1}. \]
We use the convention that if $\theta$ is rational, then $L(\theta) = L_{\infty}(\theta) = +\infty$. (N.B.: some authors study the reciprocal of what we have called the Lagrange constant.)

The best approximation properties of continued fraction convergents give

\begin{equation}
L(\theta) = \sup_{n \geq 0} \left( q_n ||q_n \theta|| \right)^{-1}
\end{equation}

and

\begin{equation}
L_{\infty}(\theta) = \limsup_{n \geq 0} \left( q_n ||q_n \theta|| \right)^{-1}.
\end{equation}

The set of values taken by $L_{\infty}(\theta)$ over all $\theta$ is called the Lagrange spectrum [4]. It is well known that $L_{\infty}(\theta) \geq \sqrt{5}$ for all $\theta$. If $\theta = [a_0, a_1, a_2, \ldots]$, then another formula for $L_{\infty}(\theta)$ is

\begin{equation}
L_{\infty}(\theta) = \limsup_{j \to \infty} ([a_j, a_{j+1}, \ldots] + [0, a_{j-1}, a_{j-2}, \ldots, a_1]);
\end{equation}

see [4, p. 1].

There are simple relations between these quantities and the partial quotient bounds $K(\theta)$ and $K_{\infty}(\theta)$, cf. [16, pp. 22–23].

**Lemma 2.1.** For any irrational $\theta$ with bounded partial quotients, we have

\begin{equation}
K(\theta) \leq L(\theta) \leq K(\theta) + 2.
\end{equation}

**Proof.** This is immediate from (2.2) and (2.3). \qed

**Lemma 2.2.** For any irrational $\theta$ with bounded partial quotients

\begin{equation}
K_{\infty}(\theta) \leq L_{\infty}(\theta) \leq K_{\infty}(\theta) + 2.
\end{equation}

**Proof.** This is immediate from (2.2) and (2.4). \qed

Although we do not use it in the sequel, we note that both inequalities in (2.7) can be slightly improved. Since $q_n \leq (a_n + 1)q_{n-1}$, (2.1) yields

\[ q_n ||q_n \theta|| \leq \frac{1}{a_{n+1} + \frac{q_{n-1}}{q_n}} \leq \frac{1}{a_{n+1} + 1/(a_n + 1)}. \]
Since \( a_n \leq K_\infty(\theta) \) from some point on, this and (2.4) yield

\[
L_\infty(\theta) \geq K_\infty(\theta) + \frac{1}{K_\infty(\theta) + 1}.
\]

Next, from (2.1) we have

\[
g_n ||q_n\theta|| = \frac{q_n}{\alpha_{n+1}q_n + q_{n-1}} = \frac{1}{a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n}}.
\]

Hence

\[
(q_n ||q_n\theta||)^{-1} = a_{n+1} + \frac{1}{\alpha_{n+2}} + \frac{q_{n-1}}{q_n}.
\]

Let \( K = K_\infty(\theta) \). Then for all \( n \) sufficiently large we have

\[
\alpha_{n+2} \geq 1 + \frac{1}{K + 1} = \frac{K + 2}{K + 1},
\]

so

\[
(q_n ||q_n\theta||)^{-1} \leq K + \frac{K + 1}{K + 2} + 1 = K + 2 - \frac{1}{K + 2}.
\]

We conclude that

\[
L_\infty(\theta) \leq K_\infty(\theta) + 2 - \frac{1}{K_\infty(\theta) + 2}.
\]

3. LAGRANGE CONSTANTS AND PROOF OF THEOREM 1.1.

An integer matrix \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) with \( \det(M) \neq 0 \), acts as a linear fractional transformation on a real number \( \theta \) by

\[
M(\theta) := \frac{a\theta + b}{c\theta + d}.
\]

Note that \( M_1(M_2(\theta)) = M_1M_2(\theta) \).
Lemma 3.1. If $M$ is an integer matrix with $\det(M) = \pm 1$, then the Lagrange constants of $\theta$ and $M(\theta)$ are related by

$$L_\infty(M(\theta)) = L_\infty(\theta).$$

Proof. This is well-known, cf. [14] and [5, Lemma 1], and is deducible from (2.5).

The main result of Cusick and Mendès France [5] yields:

Theorem 3.2. For any integer $m \geq 1$, let

$$G_m = \{ M \in M(2,\mathbb{Z}) : |\det(M)| = m \}.$$

Then for any irrational number $\theta$,

$$\sup_{M \in G_m} (L_\infty(M(\theta))) = mL_\infty(\theta).$$

and

$$\inf_{M \in G_m} (L_\infty(M(\theta))) \geq \frac{1}{m} L_\infty(\theta).$$

Proof. Theorem 1 of [5] states that

$$\max_{\substack{a, b, d \in \mathbb{Z} \cap \mathbb{Z} \backslash \{0\} \mid \gcd(a, d) = 1 \atop ad = m, 0 \leq b < d}} \left( L_\infty \left( \frac{a\theta + b}{d} \right) \right) = mL_\infty(\theta).$$

Let $GL(2, \mathbb{Z})$ denote the group of $2 \times 2$ integer matrices with determinant $\pm 1$. We need only observe that for any $M$ in $G_m$ there exists some $\tilde{M} \in GL(2, \mathbb{Z})$ such that $\tilde{M}M = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ with $a'd' = m$ and $0 \leq b' < d'$. For if so, and $\psi = \frac{a\theta + b}{c\theta + d}$, then Lemma 3.1 gives

$$L_\infty(\psi) = L_\infty(\tilde{M}(\psi)) = L_\infty(\tilde{M}M(\theta)) = L_\infty \left( \frac{a'\theta + b'}{d'} \right),$$

whence (3.4) implies (3.2). To construct $\tilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we must have

$$Ca + Dc = 0.$$
Take $C = \frac{\text{lcm}(a,c)}{a}$ and $D = \frac{\text{lcm}(a,c)}{c}$. Then $\gcd(C,D) = 1$, so we may complete this row to a matrix $\tilde{M} \in GL(2, \mathbb{Z})$. Multiplying this by a suitable matrix $\begin{pmatrix} \pm 1 & \pm 1 \\ 0 & c \end{pmatrix}$ yields the desired $\tilde{M}$.

The lower bound (3.3) follows from the upper bound (3.2). We use the adjoint matrix

$$M' = \text{adj}(M) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},$$

which has $M'M = \text{det}(M)I = mI$ and $\text{det}(M') = \text{det}(M)$. If $\theta' = M(\theta)$, then

$$M'(\theta') = M'(M(\theta)) = M'M(\theta) = \theta.$$ We prove by contradiction. Suppose (3.3) were false, so that for some $M \in G_m$ and some $\theta$ we have

$$L_\infty(M(\theta)) < \frac{1}{m} L_\infty(\theta).$$

This states that

$$mL_\infty(\theta') < L_\infty(M'(\theta')),$$

which contradicts (3.2) for $\theta'$, since $\text{det}(M') = \text{det}(M) = m$. \(\square\)

**Remark.** The lower bound (3.3) holds with equality for some values of $\theta$ and not for other values. If for given $\theta$ we choose an $M \in G_m$ which gives equality in (3.2), so that $L_\infty(M(\theta)) = mL_\infty(\theta)$, then equality holds in (3.3) for $\theta' = \text{adj}(M)(\theta)$. However, if $L_\infty(\theta) = \sqrt{5}$, as occurs for $\theta = \frac{1+\sqrt{5}}{2}$, then $L_\infty(M(\theta)) \geq L_\infty(\theta)$ for all $M$; hence (3.3) does not hold with equality when $m \geq 2$.

**Proof of Theorem 1.1.** Theorem 3.2 gives $L_\infty(M(\theta)) \leq \text{det}(M)L_\infty(\theta)$. Now apply Lemma 2.2 twice to get

$$K_\infty(M(\theta)) \leq L_\infty(M(\theta)),$$

$$\leq |\text{det}(M)|L_\infty(\theta),$$

$$\leq |\text{det}(M)|(K_\infty(\theta) + 2).$$

(3.5)

To obtain the lower bound, we use the adjoint $M' = \text{adj}(M) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, and apply (3.5) with $M'$ and $\theta' = M(\theta)$ to obtain

$$K_\infty(\theta) = K_\infty(M'(M(\theta))) \leq |\text{det}(M')|(K_\infty(M(\theta))) + 2).$$
Since $|\det(M)| = |\det(M')|$, this yields

$$K_\infty(M(\theta)) \geq \frac{1}{|\det(M)|} K_\infty(\theta) - 2. \quad \square$$

4. NUMBERS OF BOUNDED TYPE AND PROOF OF THEOREM 1.2

Recall that the type $L(\theta)$ of $\theta$ is the smallest real number such that $q||q\theta|| \geq \frac{1}{L(\theta)}$ for all $q \geq 1$.

**Theorem 4.1.** Let $\theta$ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then

$$L \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)|L(\theta) + |c(c\theta + d)|.$$

**Proof.** Set $\psi = \frac{a\theta + b}{c\theta + d}$. Suppose first that $c = 0$ so that $|\det(M)| = |ad| > 0$. Then $L(\psi) \geq \frac{1}{x}$, where

$$x := q||q\psi|| = q||q \left( \frac{a\theta + b}{d} \right)|| = q|q \left( \frac{a\theta + b}{d} \right) - p|. $$

We have

$$|ad|x = |aq| |aq\theta + (bq - dp)|$$

$$\geq |aq| |aq\theta|| \geq \frac{1}{L(\theta)}.$$

For any $\epsilon > 0$ we may choose $q$ in (4.2) so that $\frac{1}{x} \geq L(\psi) - \epsilon$. Then

$$|\det(M)|L(\theta) = |ad||L(\theta)| \geq \frac{1}{x} \geq L(\psi) - \epsilon.$$

Letting $\epsilon \to 0$ yields (4.1) when $c = 0$.

Suppose now that $c \neq 0$. Again $L(\psi) \geq \frac{1}{x}$ where

$$x := q||q\psi|| = q|q \left( \frac{a\theta + b}{c\theta + d} \right) - p|. $$
We have

\[ |c\theta + d| x = q |(qa - pc)\theta - (pd - qb)| , \]

so that

\[ |c\theta + d| \left| \frac{qa - pc}{q} \right| x = |qa - pc| |(qa - pc)\theta - (pd - qb)| \geq |qa - pc| |(qa - pc)\theta|| . \]

(4.6)

We first treat the case \( qa - pc = 0 \). Now

\[
\begin{bmatrix}
  a & -c \\
  -b & d
\end{bmatrix}
\begin{bmatrix}
  q \\
  p
\end{bmatrix}
= \begin{bmatrix}
  qa - pc \\
  pd - qb
\end{bmatrix} \neq \begin{bmatrix}
  0 \\
  0
\end{bmatrix},
\]

since \( \det \begin{bmatrix}
  a & -c \\
  -b & d
\end{bmatrix} = \det(M) \neq 0 \). Thus if \( qa - pc = 0 \) then \( |pd - qb| \geq 1 \), hence (4.5) gives

\[ |c\theta + d| x = q |pd - qb| \geq 1 . \]

(4.7)

It follows that \( qa - pc \neq 0 \) provided that

\[ \frac{1}{x} > |c\theta + d| . \]

(4.8)

We next treat the case when \( qa - pc \neq 0 \). Now from the definition of \( L(\theta) \) we see

\[ |qa - pc| |(qa - pc)\theta|| \geq \frac{1}{L(\theta)} . \]

(4.9)

Given \( \epsilon > 0 \), we may choose \( q \) so that \( \frac{1}{x} \geq L(\psi) - \epsilon \), and we obtain from (4.6) and (4.9) that

\[ |c\theta + d| \left| \frac{qa - pc}{q} \right| L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon . \]

(4.10)

However, the bound

\[ \left| q \left( \frac{a\theta + b}{c\theta + d} \right) - p \right| \leq \frac{1}{2} \]
implies that
\[
\left| \frac{qa - pc}{c} \right| = \left| q \left( \frac{a}{c} \right) - p \right| \leq \left| q \left( \frac{a\theta + b}{c\theta + d} \right) - q \left( \frac{a}{c} \right) \right| + \left| q \left( \frac{a}{c} \right) - p \right| \leq q|\text{det}(M)| \left| \frac{1}{c(c\theta + d)} \right| + \frac{1}{2}.
\]

Multiplying this by $\frac{\epsilon}{q}$ and applying it to the left side of (4.10) yields

\begin{equation}
(4.11) \quad L \left( \frac{a\theta + b}{c\theta + d} \right) - \epsilon \leq |\text{det}(M)|L(\theta) + \frac{1}{2} \left| \frac{c(c\theta + d)}{q} \right|.
\end{equation}

Letting $\epsilon \to 0$ and using $q \geq 1$ yields

\begin{equation}
(4.12) \quad L \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\text{det}(M)|L(\theta) + \frac{1}{2} |c(c\theta + d)|,
\end{equation}

provided that (4.8) holds. Now (4.8) fails to hold only if

\begin{equation}
(4.13) \quad L \left( \frac{a\theta + b}{c\theta + d} \right) \leq |c\theta + d|.
\end{equation}

The last two inequalities imply (4.1) when $c \neq 0$. \qed

**Proof of Theorem 1.2.** Applying Theorem 4.1 and Lemma 2.1 gives

\[
K \left( \frac{a\theta + b}{c\theta + d} \right) \leq L \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\text{det}(M)|L(\theta) + |c(c\theta + d)| \leq |\text{det}(M)|(K(\theta) + 2) + |c(c\theta + d)|,
\]

which is the desired bound. \qed

**Remarks.** (1). The proof method of Theorem 4.1 can also be used to directly prove the bounds

\begin{equation}
(4.14) \quad \frac{1}{|\text{det}(M)|}L_\infty(\theta) \leq L_\infty(M(\theta)) \leq |\text{det}(M)|L_\infty(\theta),
\end{equation}

of Theorem 3.2, from which Theorem 1.1 can be easily deduced. The lower bound in (4.14) follows from the upper bound as in the proof of Theorem 3.2. We sketch a proof of the upper bound in (4.14) for the case
\( \psi = \frac{a\theta + b}{c\theta + d} \) with \( c \neq 0 \). For any \( \epsilon^* > 0 \) and all sufficiently large \( q^* \geq q^*(\epsilon^*) \), we have

\[
(4.15) \quad q^*||q^*\theta|| \geq \frac{1}{L_\infty(\theta) + \epsilon^*}.
\]

We choose \( q = q_n(\psi) \) for sufficiently large \( n \), and note that

\[
q^* = |q_n(\psi)a - p_n(\psi)c| \to \infty
\]
as \( n \to \infty \), since \( \psi \) is irrational. We can then replace (4.9) by (4.15), and then deduce (4.11) with \( L(\theta) \) replaced by \( L_\infty(\theta) + \epsilon^* \). Letting \( q \to \infty \), \( \epsilon \to 0 \) and \( \epsilon^* \to 0 \) in that order yields the upper bound in (4.14).

(2). For a given matrix \( M \) consider the set of attainable ratios

\[
(4.16) \quad \mathcal{V}(M) := \left\{ \frac{L_\infty(M(\theta))}{L_\infty(\theta)} : \theta \text{ has bounded partial quotients} \right\}.
\]

By Lemma 3.1 the set \( \mathcal{V}(M) \) depends only on its \( SL(2, \mathbb{Z}) \)-double coset

\[
[M] = \{N_1MN_2 : N_1, N_2 \in SL(2, \mathbb{Z})\}.
\]

Theorem 3.2 shows that

\[
(4.17) \quad \mathcal{V}(M) \subseteq \left[ \frac{1}{|\det(M)|}, \frac{1}{|\det(M)|} \right].
\]

It is an interesting open problem to determine the set \( \mathcal{V}(M) \). Both \( |\det(M)| \) and \( \frac{1}{|\det(M)|} \) lie in \( \mathcal{V}(M) \), as follows from Theorem 3.2 and the remark following it.

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**References**


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