KEN YAMAMURA

Maximal unramified extensions of imaginary quadratic number fields of small conductors

Journal de Théorie des Nombres de Bordeaux, tome 9, no 2 (1997), p. 405-448

<http://www.numdam.org/item?id=JTNB_1997__9_2_405_0>
Maximal unramified extensions of imaginary quadratic number fields of small conductors

par Ken Yamamura

RéSUMÉ. Nous déterminons la structure du groupe de Galois Gal(K_{ur}/K) de l'extension maximale non ramifiée K_{ur} de chaque corps quadratique imaginaire de conducteur \leq 420 (\leq 719 sous GRH). Pour tous ces corps K, l'extension K_{ur} coïncide avec K, ou avec le corps de classes de Hilbert de K, ou avec le second corps de classes de Hilbert de K ou avec le troisième corps de classes de Hilbert de K. Les bornes d'Odlyzko sur les discriminants et les informations sur la structure des groupes de classes obtenues par l'action du groupe de Galois sur les groupes de classes sont ici essentielles. Nous utilisons aussi des relations sur le nombre de classes et un ordinateur pour le calcul du nombre de classes de corps de bas degré pour obtenir le nombre de classes de corps de degré plus élevé. Nous utilisons aussi des résultats sur les tours de corps de classes, ainsi que notre connaissance des 2-groupes d'ordre \leq 2^6 et des groupes linéaires sur des corps finis.

Abstract. We determine the structures of the Galois groups Gal(K_{ur}/K) of the maximal unramified extensions K_{ur} of imaginary quadratic number fields K of conductors \leq 420 (\leq 719 under the Generalized Riemann Hypothesis). For all such K, K_{ur} is K, the Hilbert class field of K, the second Hilbert class field of K, or the third Hilbert class field of K. The use of Odlyzko's discriminant bounds and information on the structure of class groups obtained by using the action of Galois groups on class groups is essential. We also use class number relations and a computer for calculation of class numbers of fields of low degrees in order to get class numbers of fields of higher degrees. Results on class field towers and the knowledge of the 2-groups of orders \leq 2^6 and linear groups over finite fields are also used.

1. INTRODUCTION

Let K be an algebraic number field (of finite degree) and K_{ur} its maximal unramified extension. Then the Galois group Gal(K_{ur}/K) can be

1991 Mathematics Subject Classification. Primary 11R32, 11R11.

Key words and phrases. Maximal unramified extension, imaginary quadratic number field, discriminant bounds, class field tower.

Manuscrit reçu le 17 avril 1997
both finite and infinite and in general it is quite difficult to determine the structure of this group. If \( K \) has sufficiently small root discriminant, then \( K_{ur} = K \), that is, \( K \) has no nontrivial unramified extension. This is the case, for example, for the imaginary quadratic number fields with class number one, the cyclotomic number fields with class number one, the real abelian number fields of prime power conductors \( \leq 67 \) (see [57, Appendix]). For some fields \( K \) with small root discriminant, we can determine \( \text{Gal}(K_{ur}/K) \). The purpose of this paper is to determine the structure of \( \text{Gal}(K_{ur}/K) \) of imaginary quadratic number fields \( K \) of small conductors. For imaginary quadratic number fields \( K \) of conductors \( \leq 420 \) (\( \leq 719 \) under the Generalized Riemann Hypothesis (GRH)) we determine and tabulate them for \( K \) with \( K_{ur} \neq K_1 \), where \( K_1 \) denotes the Hilbert class field of \( K \). (If \( K_{ur} = K_1 \), then \( \text{Gal}(K_{ur}/K) = \text{Gal}(K_1/K) \cong \text{Cl}(K) \), the class group of \( K \) by class field theory.) For all such \( K \), \( K_{ur} \) is one of \( K, K_1, K_2, \) or \( K_3 \), where \( K_2 \) (resp. \( K_3 \)) is the second (resp. third) Hilbert class field of \( K \). In other words, \( K_{ur} \) coincides with the top of the class field tower of \( K \) and the length of the tower is at most three. If possible, we give also simple expressions of \( K_1 \) and \( K_2 \). Also for \( K = \mathbb{Q}(\sqrt{d}) \) with \( 723 \leq |d| < 1000 \), we tabulate \( \text{Gal}(K_{ur}/K) \) except for some \( d \).

From now on let \( K = \mathbb{Q}(\sqrt{d}) \) be an imaginary quadratic number field with discriminant \( d < 0 \). J. Martinet stated in [34] that if \( |d| < 250 \), then \( K_{ur} = K_1 \) except for 7 fields, for which he gave the structure of \( \text{Gal}(K_{ur}/K) \). (We note that \( \text{Gal}(K_{ur}/K) \cong H_{24} \) for \( K = \mathbb{Q}(\sqrt{-248}) \) in [34] is false\(^1\).) He also stated that this fact is proved by using the methods which J. Masley [35] (and later F. J. van der Linden [50]) used for calculation of class numbers of real abelian number fields of small conductors. They used Odlyzko's discriminant bounds and information on the structure of class groups obtained by using the action of Galois groups on class groups. In addition to their methods, we use a computer for calculation of class numbers of fields of low degrees (we use KANT) and then use class number relations to get class numbers of fields of higher degrees (see §3). Results on class field towers [5, 25, 29, 31, 49] and the knowledge of the 2-groups of orders \( \leq 2^6 \) [17] and linear groups over finite fields (see §4) are also used.

We know that if \( |d| \leq 499 \) \((|d| \leq 2003 \) under GRH\), then the degree \([K_{ur} : K]\) is finite (see §2). For these \( d \), we want to determine \( \text{Gal}(K_{ur}/K) \). The key fact is that any unramified (finite) extension \( L \) of \( K \) has the same root discriminant as \( K \): \( rd_L = |d_L|^{|L:Q|} = rd_K = \sqrt{|d|} \). Thus, if we have \( rd_K < B(2N) \), where \( B(2N) \) denotes the lower bound for the root

\(^1\)The referee pointed out that this was corrected by Martinet in a supplement to his paper which he distributed along with the original reprint.
discriminants of the totally imaginary number fields of (finite) degrees $\geq 2N$, then we get $[K_{ur} : K] < N$. We do not know the real values of $B(2N)$ (except for $N \leq 4$), however, some lower bounds for $B(2N)$ are known. The best known unconditional lower bounds for $B(2N)$ can be found in the tables due to F. Diaz y Diaz [12]. If we assume the truth of GRH, much better lower bounds can be obtained. The best known conditional (GRH) lower bounds are found in the unpublished tables due to A. M. Odlyzko [38], which are copied in Martinet’s expository paper [34]. Let $K_l$ be the top of the class field tower of $K$: $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ ($K_{i+1}$ is the Hilbert class field of $K_i$), that is, $l$ is the smallest number with $K_l = K_1$. If we cannot get $[K_{ur} : K_l] < 60$, which implies $K_{ur} = K_l$, from available lower bounds for $B(2N)$, we need to judge whether $K_l$ has an unramified nonsolvable Galois extension and this is quite difficult. For the fields $\mathbb{Q}(\sqrt{-423})$ and $\mathbb{Q}(\sqrt{-723})$, we have $h(K_1) = 1$, that is, $l = 1$ and we cannot get $[K_{ur} : K_1] < 60$ from available lower bounds for $B(2N)$ (even under GRH for $\mathbb{Q}(\sqrt{-723})$). For $|d| \leq 420$ ($|d| \leq 719$ under GRH), we get $[K_{ur} : K_1] < 60$ and our main problem is to determine the degree $[K_1 : \mathbb{Q}]$. In general, it is difficult to determine $[K_2 : \mathbb{Q}]$, because it is very hard to calculate the class number $h(K_1)$ of $K_1$. (Of course, for $K$ with small $\text{Cl}(K)$, we can calculate $h(K_1)$ with the help of a computer.) Now let $K_g$ be the genus field of $K$, that is, the maximal unramified abelian extension of $K$ which is abelian over $\mathbb{Q}$. If $d$ is the discriminant of $K$ and $d = d_1d_2\cdots d_t$ is the factorization of $d$ into the product of fundamental prime discriminants, then $K_g = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \ldots, \sqrt{d_t})$, and we have

$$\mathbb{Q} \subset K \subseteq K_g \subseteq K_1 \subseteq (K_g)_1 \subseteq K_2,$$

which implies

$$[K_2 : \mathbb{Q}] = [K_2 : (K_g)_1][(K_g)_1 : \mathbb{Q}] = [K_2 : (K_g)_1]h(K_g)[K_g : \mathbb{Q}].$$

As $K_g$ is a multi-quadratic number field, $h(K_g)$ can be calculated by the method in [51], and we may expect that $[K_2 : (K_g)_1]$ is small for fields we consider on the ground of the following proposition (§4, Proposition 2).

**Proposition.** Let $L$ be the Hilbert class field of the genus field $K_g$ of an imaginary abelian number field $K$. Then for any prime number $p$ with $p \nmid [L : \mathbb{Q}]$, the $p$-class group $\text{Cl}^{(p)}(L)$ of $L$ is trivial or noncyclic.

For all $K$ with $|d| < 1000$ such that $h(K_g) > h(K)/[K_g : K]$, which is equivalent to $(K_g)_1 \nsubseteq K_1$, we have $K_2 = (K_g)_1$. For $h(K_g) > h(K)/[K_g : K]$, $h(K)$ must necessarily be even. Now $h(K)$ is even if and only if $d$ has
(at least) two distinct prime factors, however, for most $K$, this inequality holds. In fact, if a quadratic subfield $\neq K$ of $K_g$ has class number divisible by an odd prime $p$, then we have $h(K_g) \geq ph(K)/[K_g : K]$. Thus, the following problem arises naturally:

**Problem.** Characterize the imaginary quadratic number fields $K$ with $K_2 = (K_g)_1 \neq K_1$.

The author expects that this problem can be settled group-theoretically and a similar problem can be posed for real quadratic number fields. We will discuss this problem in the Appendix 2.

For 12 fields $K$ with $|d| < 1000$ for which we have verified $K_{ur} \supsetneq (K_g)_1$, there exists an $S_4$-extension $M$ of $Q$ such that $K_gM/K_g$ is an unramified $A_4$-extension, where $S_4$ (resp. $A_4$) denotes the symmetric (resp. alternating) group of degree four, and therefore $K_{ur} \supsetneq (K_g)_1 M \supsetneq (K_g)_1$. Except for $Q(\sqrt{-856})$ and $Q(\sqrt{-996})$, we can characterize such $K$ simply as $d_E | d$ for some quartic number field $E$: If the discriminant of $d$ of $K$ is divisible by the discriminant $d_E$ of a quartic number field $E$, then $K_g$ has an unramified $A_4$-extension. Then the normal closure of $E$ is an $S_4$-extension of $Q$ unramified at all finite primes over its quadratic subfield $Q(\sqrt{d_E})$. Since $d_E | d$, $KQ(\sqrt{d_E})/K$ is unramified by genus theory and therefore $K_gM/K_g$ is an unramified $A_4$-extension. Also for the fields $Q(\sqrt{-856})$ and $Q(\sqrt{-996})$, we can find $S_4$-extensions of $Q$ that give their unramified $S_4$-extensions by composition. (For details, see §7.) Therefore, data for quartic number fields are useful for our study. Thus, $K = Q(\sqrt{d})$ with $|d| < 1000$, can be classified simply as follows:

\[
\begin{align*}
\text{d} & \neq -856, -996 \quad \text{d} | d \\
\text{d} & = -856, -996 \quad \text{d} = -856, -996 \\
\text{d} & = \text{prime} \quad \text{d} = d_E \quad \text{d} = \text{composite} \\
\end{align*}
\]

Here, by “$d_E | d$" (resp. “$d_E \nmid d$”) we mean “$d$ is divisible by $d_E$ for some quartic number field” (resp. “$d$ is never divisible by the discriminant $d_E$ for any quartic number field”), and in the factorization of $d = d'd_E$, $d'$ denotes a fundamental quadratic discriminant. We expect that $K_{ur} = K_l$ holds for all $K$ with $|d| < 1507$, because we expect that $Q(\sqrt{-1507})$ is the first $K$ having an unramified nonsolvable Galois extension (see below). This actually holds for all $K$ with $|d| < 1000$ and $l \geq 2$. We also expect that
for all fields $K$ with $d_E \nmid d \neq -856, -996$, $K_{ur} = (K_g)_1$ holds and that all inequalities for $l$ are equalities. If our expectation is true, the classification above can be replaced by the following:

\[
\begin{aligned}
\begin{cases}
    d \neq -856, -996 & \quad h(K_g) = \frac{h(K)}{[K_g:K]} \\
    h(K_g) > \frac{h(K)}{[K_g:K]} & \quad \{ \begin{array}{l}
        h(K) = 1 \ldots K_{ur} = K \\
        h(K) > 1 \ldots K_{ur} = K_2 \\
        \ldots K_{ur} = K_3
    \end{array} \}
\end{cases} \\
\begin{cases}
    d = -856, -996 & \quad \ldots K_{ur} = K_2 \\
    d = d_E & \quad \ldots K_{ur} = K_3 \\
    d = d'd_E & \quad K_{ur} \nsubseteq (K_g)_1
\end{cases}
\end{aligned}
\]

Though, there are possible exceptions, the author think that this classification is meaningful because this is complete for $|d| \leq 719$ and even if an exception would occur for $|d| > 719$, modification is easy.

For most $K$ we considered, $K_{ur} = K_1$ is verified. Thus, the following natural question arises: What is the first imaginary quadratic number field having an unramified nonsolvable Galois extension? (What is the first $K$ with $K_{ur} \neq K_1$?) Recent data for quintic number fields [3, 44] enable us to give a partial answer (§8, Proposition 8):

**Proposition.** The field $\mathbb{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified $A_5$-extension which is normal over $\mathbb{Q}$ in the sense that none of $\mathbb{Q}(\sqrt{d})$ of discriminants $d$ with $0 > d > -1507$ has such an extension.

We expect that the field $\mathbb{Q}(\sqrt{-1507})$ gives the answer to the question above.

For the determination of the structure of $\text{Gal}(K_{ur}/K)$, the results on 2-class field towers due to H. Kisilevsky [25], F. Lemmermeyer [29, 31], and E. Benjamin, F. Lemmermeyer, and C. Snyder [5] are very helpful. They give us information on the structure of the Galois group $\text{Gal}(K_2^{(2)}/K)$ of the second Hilbert 2-class field $K_2^{(2)}$ of $K$ over $K$ in many cases.

Now we explain the notations in our table. In the simple expressions of $K_1$ and $K_2$, $\alpha_i, \beta_i$ and $\gamma_i$ denote any algebraic numbers generating the $i$th cubic number field of signature $(1,1)$, the $i$th quartic number field of signature $(2,1)$ with Galois group isomorphic to $S_4$, and the $i$th quintic number field of signature $(1,2)$ with Galois group isomorphic to $D_5$, respectively, where we consider that the number fields of each signature and each type (of Galois group of normal closure) are numbered up to conjugacy by absolute values of discriminants. (Here we do not need to consider nonisomorphic fields with same discriminants.)
G denotes the Galois group Gal(K_{ur}/K). As usual, C_n is the cyclic group of order n, V_4 is the four group, that is, V_4 = C_2^2 = C_2 \times C_2, D_n (n \geq 3) is the dihedral group of order 2n, Q_{4n} (n \geq 2) is the generalized quaternion group of order 4n, and SD_{8n} (n \geq 2) is the semi-dihedral group of order 8n:

\[
D_n = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle,
\]
\[
Q_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle,
\]
\[
SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, b^{-1}ab = a^{2n-1} \rangle.
\]

I_{2n}^m (m \geq 2, n \geq 3) denotes the group of order 2mn given by
\[
\langle a, b \mid a^{2m} = b^n = 1, a^{-1}ba = b^{-1} \rangle.
\]

M_{2n} (n \geq 4) denotes the modular group of order 2^n given by
\[
\langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{2^{n-2}+1} \rangle.
\]

\overline{A}_4 is the double cover of A_4 : \overline{A}_4 \cong SL(2, 3).

For some 2-groups we use designations given in the table by M. Hall and J. K. Senior [17]. We note that T. W. Sag and J. W. Wamsley give minimal presentations for all 2-groups of orders \leq 2^6 [42]. For simplicity, for some of them we use the following designations used in [5]. \Gamma_{m,t}^1 denotes the group of order 2^{m+t+1} given by
\[
\langle a, b \mid a^4 = b^{2m} = 1, c = [a, b], a^2 = c^{2^{t-1}}, [a, c] = c^2, [ab, c] = 1 \rangle.
\]

\Gamma_{m,t}^2 denotes the group of order 2^{m+t+1} given by
\[
\langle a, b \mid a^4 = 1, c = [a, b], a^2 = b^{2m} = c^{2^{t-1}}, [a, c] = c^2, [ab, c] = 1 \rangle.
\]

We note that \Gamma_{3,2}^2 = 64\Gamma_{3p}, \Gamma_{1,3}^1 = 64\Gamma_{8c2}, \Gamma_{1,2}^1 = 64\Gamma_{3n2}, \Gamma_{2,3}^2 = 64\Gamma_{8e}.

The organization of this paper is as follows: In Section 2, we review how to obtain an upper bound for \[K_{ur} : K\] by using discriminant bounds. In Section 3, we review some results on class number relations. In Section 4, we describe the information on the structure of class groups that can be obtained by considering the action of Galois groups on class groups. In Section 5, we review some results on class field towers. In Section 6, we describe how to determine Gal(K_{ur}/K) for selected values of \[d\]. In Section 7, we describe fields having an unramified extension not contained
in \((K_2)_1\). In Section 8, we describe unramified nonsolvable Galois extensions of imaginary quadratic number fields. In the Appendix 1, we explain how to calculate class numbers of \(S_4\)-extensions of \(\mathbb{Q}\). In the Appendix 2, we discuss the problem of characterizing quadratic number fields \(K\) with \(K_2 = (K_2)_1 \neq K_1\).

Acknowledgements. The author thanks Prof. R. Schoof for useful advices [43]. He also thanks Dr. F. Lemmermeyer for information on his results [32].

Table of imaginary quadratic number fields \(K = \mathbb{Q}(\sqrt{d}), |d| \leq 719\) with \(K_{ur} \neq K_1\)

<table>
<thead>
<tr>
<th>(-d)</th>
<th>(Cl(K))</th>
<th>(K_1)</th>
<th>(K_2)</th>
<th>(l)</th>
<th>(G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>115</td>
<td>(C_2)</td>
<td>(K(\sqrt{5}))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(D_3)</td>
</tr>
<tr>
<td>120</td>
<td>(V_4)</td>
<td>(K(\sqrt{-3}, \sqrt{5}))</td>
<td>(K_1(\sqrt{2 \times 2 + \sqrt{5}}(2 + \sqrt{5})))</td>
<td>2</td>
<td>(Q_8)</td>
</tr>
<tr>
<td>155</td>
<td>(C_4)</td>
<td>(K(\sqrt{(-1 + 5\sqrt{5})/2}))</td>
<td>(K_1(\alpha_2))</td>
<td>2</td>
<td>(Q_{12})</td>
</tr>
<tr>
<td>184</td>
<td>(C_4)</td>
<td>(K(\sqrt{-3 + 4\sqrt{2}}))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(Q_{12})</td>
</tr>
<tr>
<td>195</td>
<td>(V_4)</td>
<td>(K(\sqrt{-3}, \sqrt{5}))</td>
<td>(K_1(\gamma_1))</td>
<td>2</td>
<td>(D_5)</td>
</tr>
<tr>
<td>235</td>
<td>(C_2)</td>
<td>(K(\sqrt{5}))</td>
<td>(K_1(\alpha_2))</td>
<td>2</td>
<td>(I_{3}^{3})</td>
</tr>
<tr>
<td>248</td>
<td>(C_8)</td>
<td>(K(\sqrt{-3}, \sqrt{5}))</td>
<td>(K_1(\gamma_1))</td>
<td>2</td>
<td>(Q_{16})</td>
</tr>
<tr>
<td>255</td>
<td>(C_6 \times C_2)</td>
<td>(K(\sqrt{5}, \sqrt{9 + \sqrt{85}}/2))</td>
<td>(K_1(\sqrt{5 + 2\sqrt{3}}(2 + \sqrt{5})))</td>
<td>2</td>
<td>(Q_8 \times C_3)</td>
</tr>
<tr>
<td>260</td>
<td>(C_4 \times C_2)</td>
<td>(K(\sqrt{5}, \sqrt{8 + \sqrt{65}}))</td>
<td>(K_1(\gamma_1))</td>
<td>2</td>
<td>(M_{16})</td>
</tr>
<tr>
<td>276</td>
<td>(C_4 \times C_2)</td>
<td>(K(\sqrt{-1}, \sqrt{13 + 8\sqrt{3}}))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(Q_{12} \times C_2)</td>
</tr>
<tr>
<td>280</td>
<td>(V_4)</td>
<td>(K(\sqrt{-7}, \sqrt{5}))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(Q_{16})</td>
</tr>
<tr>
<td>283</td>
<td>(C_3)</td>
<td>(K(\alpha_31))</td>
<td>(K_1(\beta_1))</td>
<td>3</td>
<td>(A_4)</td>
</tr>
<tr>
<td>295</td>
<td>(C_8)</td>
<td>(K(\alpha_4))</td>
<td>(K_1(\alpha_4))</td>
<td>2</td>
<td>(I_{3}^{3})</td>
</tr>
<tr>
<td>299</td>
<td>(C_8)</td>
<td>(K(\alpha_1))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(I_{3}^{3})</td>
</tr>
<tr>
<td>312</td>
<td>(V_4)</td>
<td>(K(\sqrt{-3}, \sqrt{2}))</td>
<td>(K_1(\alpha_2))</td>
<td>2</td>
<td>(Q_{16})</td>
</tr>
<tr>
<td>331</td>
<td>(C_3)</td>
<td>(K(\alpha_36))</td>
<td>(K_1(\beta_2))</td>
<td>3</td>
<td>(A_4)</td>
</tr>
<tr>
<td>340</td>
<td>(V_4)</td>
<td>(K(\sqrt{-1}, \sqrt{5}))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(SD_{16})</td>
</tr>
<tr>
<td>355</td>
<td>(C_4)</td>
<td>(K(\sqrt{-3 + 4\sqrt{5}}))</td>
<td>(K_1(\alpha_2))</td>
<td>2</td>
<td>(Q_{28})</td>
</tr>
<tr>
<td>372</td>
<td>(V_4)</td>
<td>(K(\sqrt{-1}, \sqrt{-3}))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(D_6)</td>
</tr>
<tr>
<td>376</td>
<td>(C_8)</td>
<td></td>
<td>(K_1(\gamma_1))</td>
<td>2</td>
<td>(I_{3}^{5})</td>
</tr>
<tr>
<td>391</td>
<td>(C_{14})</td>
<td>(K_1(\alpha_1))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(D_3 \times C_7)</td>
</tr>
<tr>
<td>395</td>
<td>(C_8)</td>
<td>(K_1(\gamma_1))</td>
<td>(K_1(\gamma_2))</td>
<td>2</td>
<td>(I_{3}^{5})</td>
</tr>
<tr>
<td>403</td>
<td>(C_2)</td>
<td>(K(\sqrt{13}))</td>
<td>(K_1(\alpha_1))</td>
<td>2</td>
<td>(D_3)</td>
</tr>
<tr>
<td>408</td>
<td>(V_4)</td>
<td>(K(\sqrt{-3}, \sqrt{2}))</td>
<td>(K_1(\sqrt{(-5 + \sqrt{17})/2}))</td>
<td>2</td>
<td>(D_4)</td>
</tr>
<tr>
<td>415</td>
<td>(C_{10})</td>
<td>(K(\sqrt{5}, \gamma_{18}))</td>
<td>(K_1(\alpha_6))</td>
<td>2</td>
<td>(D_3 \times C_5)</td>
</tr>
<tr>
<td>420</td>
<td>(C_2)</td>
<td>(K(\sqrt{-1}, \sqrt{-3}, \sqrt{5}))</td>
<td>(K_1(\alpha_6))</td>
<td>2</td>
<td>(32\Gamma_{4,c_3})</td>
</tr>
<tr>
<td>$-d$</td>
<td>$\text{Cl}(K)$</td>
<td>$K_1$</td>
<td>$K_2$</td>
<td>$l$</td>
<td>$G$</td>
</tr>
<tr>
<td>------</td>
<td>----------------</td>
<td>-------</td>
<td>-------</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>435</td>
<td>$V_4$</td>
<td>$K(\sqrt{3}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$Q_{16} \times C_3$</td>
</tr>
<tr>
<td>440</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{2}, \sqrt{5}, \alpha_{50})$</td>
<td></td>
<td>2</td>
<td>$Q_{16} \times C_3$</td>
</tr>
<tr>
<td>455</td>
<td>$C_{10} \times C_2$</td>
<td>$K(\sqrt{-7}, \sqrt{5}, \gamma_{21})$</td>
<td></td>
<td>2</td>
<td>$Q_8 \times C_5$</td>
</tr>
<tr>
<td>472</td>
<td>$C_6$</td>
<td>$K(\sqrt{2}, \alpha_4)$</td>
<td>$K_1(\alpha_4)$</td>
<td>2</td>
<td>$D_3 \times C_3$</td>
</tr>
<tr>
<td>483</td>
<td>$V_4$</td>
<td>$K(\sqrt{-3}, \sqrt{-7})$</td>
<td>$K_1(\alpha_1)$</td>
<td>2</td>
<td>$D_6$</td>
</tr>
<tr>
<td>491</td>
<td>$C_9$</td>
<td></td>
<td>$K_1(\beta_3)$</td>
<td>3</td>
<td>$Q_8 \times C_9$</td>
</tr>
<tr>
<td>515</td>
<td>$C_6$</td>
<td>$K(\sqrt{5}, \alpha_{60})$</td>
<td>$K_1(\gamma_3)$</td>
<td>2</td>
<td>$D_5 \times C_3$</td>
</tr>
<tr>
<td>520</td>
<td>$V_4$</td>
<td>$K(\sqrt{-2}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$Q_{24}$</td>
</tr>
<tr>
<td>527</td>
<td>$C_{18}$</td>
<td></td>
<td>$K_1(\alpha_2)$</td>
<td>2</td>
<td>$D_3 \times C_9$</td>
</tr>
<tr>
<td>535</td>
<td>$C_{14}$</td>
<td></td>
<td>$K_1(\alpha_9)$</td>
<td>2</td>
<td>$D_3 \times C_7$</td>
</tr>
<tr>
<td>552</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-3}, \sqrt{-1} + 2\sqrt{6})$</td>
<td>$K_1(\alpha_1)$</td>
<td>2</td>
<td>$Q_{12} \times C_2$</td>
</tr>
<tr>
<td>555</td>
<td>$V_4$</td>
<td>$K(\sqrt{-3}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$Q_{32}$</td>
</tr>
<tr>
<td>563</td>
<td>$C_9$</td>
<td></td>
<td>$K_1(\beta_4)$</td>
<td>3</td>
<td>$Q_8 \times C_9$</td>
</tr>
<tr>
<td>564</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt{1} + 4\sqrt{3})$</td>
<td>$K_1(\gamma_1)$</td>
<td>2</td>
<td>$Q_{20} \times C_2$</td>
</tr>
<tr>
<td>568</td>
<td>$C_4$</td>
<td>$K(\sqrt{-1} + 6\sqrt{2})$</td>
<td></td>
<td>2</td>
<td>$Q_{28}$</td>
</tr>
<tr>
<td>580</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{5}, \sqrt{12} + \sqrt{145})$</td>
<td></td>
<td>2</td>
<td>$32\Gamma_3 f \times C_3$</td>
</tr>
<tr>
<td>595</td>
<td>$V_4$</td>
<td>$K(\sqrt{-7}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$Q_{40}$</td>
</tr>
<tr>
<td>611</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{13}, \gamma_{28})$</td>
<td>$K_1(\gamma_1)$</td>
<td>2</td>
<td>$D_5 \times C_5$</td>
</tr>
<tr>
<td>632</td>
<td>$C_8$</td>
<td></td>
<td>$K_1(\gamma_2)$</td>
<td>2</td>
<td>$I_8^5$</td>
</tr>
<tr>
<td>635</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{5}, \gamma_{31})$</td>
<td>$K_1(\gamma_5)$</td>
<td>2</td>
<td>$D_5 \times C_5$</td>
</tr>
<tr>
<td>643</td>
<td>$C_3$</td>
<td>$K(\alpha_{72})$</td>
<td>$K_1(\beta_3)$</td>
<td>3</td>
<td>$A_4$</td>
</tr>
<tr>
<td>644</td>
<td>$C_8 \times C_2$</td>
<td></td>
<td>$K_1(\alpha_1)$</td>
<td>2</td>
<td>$D_3 \times C_8$</td>
</tr>
<tr>
<td>651</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-7}, \sqrt{(13 + \sqrt{217})/2})$</td>
<td>$K_1(\alpha_2)$</td>
<td>2</td>
<td>$D_3 \times C_4$</td>
</tr>
<tr>
<td>655</td>
<td>$C_{12}$</td>
<td>$K(\sqrt{7} + 6\sqrt{5}, \alpha_{75})$</td>
<td>$K_1(\gamma_6)$</td>
<td>2</td>
<td>$Q_{20} \times C_3$</td>
</tr>
<tr>
<td>660</td>
<td>$C_2^3$</td>
<td>$K(\sqrt{-1}, \sqrt{-3}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$64\Gamma_{15} f_2$</td>
</tr>
<tr>
<td>663</td>
<td>$C_8 \times C_2$</td>
<td></td>
<td></td>
<td>2</td>
<td>$\Gamma_{3,2}^2$</td>
</tr>
<tr>
<td>664</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{2}, \gamma_{32})$</td>
<td></td>
<td>2</td>
<td>$D_3 \times C_5$</td>
</tr>
<tr>
<td>667</td>
<td>$C_4$</td>
<td>$K(\sqrt{-13} + 3\sqrt{29})/2$</td>
<td>$K_1(\alpha_1)$</td>
<td>2</td>
<td>$Q_{12}$</td>
</tr>
<tr>
<td>680</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{-2}, \sqrt{5}, \alpha_{80})$</td>
<td></td>
<td>2</td>
<td>$Q_{16} \times C_3$</td>
</tr>
<tr>
<td>687</td>
<td>$C_{12}$</td>
<td>$K(\sqrt{(11 + \sqrt{229})/2}, \alpha_{81})$</td>
<td>$K_1(\alpha_{13})$</td>
<td>3</td>
<td>$(A_4 \times C_4) \times C_3$</td>
</tr>
<tr>
<td>695</td>
<td>$C_{24}$</td>
<td></td>
<td></td>
<td>2</td>
<td>$I_8^3 \times C_3$</td>
</tr>
<tr>
<td>696</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{2}, \sqrt{99 + 13\sqrt{58}})$</td>
<td></td>
<td>2</td>
<td>$Q_{24} \times C_3$</td>
</tr>
<tr>
<td>708</td>
<td>$V_4$</td>
<td>$K(\sqrt{-1}, \sqrt{-3})$</td>
<td>$K_1(\alpha_4)$</td>
<td>2</td>
<td>$D_6$</td>
</tr>
<tr>
<td>715</td>
<td>$V_4$</td>
<td>$K(\sqrt{-11}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$Q_{16} \times C_5$</td>
</tr>
</tbody>
</table>
### Table of $\text{Gal}(K_{ur}/K)$ for $723 \leq |d| < 1000$ (not complete)

<table>
<thead>
<tr>
<th>$-d$</th>
<th>$\text{Cl}(K)$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$l$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>723</td>
<td>$C_4$</td>
<td>$K(\sqrt{(-7 + 3\sqrt{241})/2})$</td>
<td></td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>724</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{-1}, \gamma_{37})$</td>
<td></td>
<td>1</td>
<td>$C_{10}$</td>
</tr>
<tr>
<td>727</td>
<td>$C_{13}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{13}$</td>
</tr>
<tr>
<td>728</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{-7}, \sqrt{2}, \alpha_{86})$</td>
<td></td>
<td>2</td>
<td>$Q_{16} \times C_3$</td>
</tr>
<tr>
<td>731</td>
<td>$C_{12}$</td>
<td>$K(\sqrt{-5 + 2\sqrt{17}}, \alpha_{87})$</td>
<td></td>
<td>2</td>
<td>?</td>
</tr>
<tr>
<td>739</td>
<td>$C_5$</td>
<td>$K(\gamma_{38})$</td>
<td></td>
<td>1</td>
<td>$C_5$</td>
</tr>
<tr>
<td>740</td>
<td>$C_8 \times C_2$</td>
<td></td>
<td></td>
<td>2</td>
<td>$M_{32}$</td>
</tr>
<tr>
<td>743</td>
<td>$C_{21}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{21}$</td>
</tr>
<tr>
<td>744</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{-2}, \sqrt{63 + 8\sqrt{62}})$</td>
<td>$K_1(\alpha_2)$</td>
<td>2</td>
<td>$D_3 \times C_6$</td>
</tr>
<tr>
<td>751</td>
<td>$C_{15}$</td>
<td>$K(\gamma_{91}, \gamma_{39})$</td>
<td>$K_1(\beta_8)$</td>
<td>3</td>
<td>$A_4 \times C_5$</td>
</tr>
<tr>
<td>755</td>
<td>$C_{12}$</td>
<td>$K(\sqrt{13 + 8\sqrt{5}}, \alpha_{92})$</td>
<td></td>
<td>2</td>
<td>$Q_{28} \times C_3$</td>
</tr>
<tr>
<td>759</td>
<td>$C_{12} \times C_2$</td>
<td>$K(\sqrt{-3}, \sqrt{(-5 + \sqrt{69})/2}, \alpha_{94})$</td>
<td>$K_1(\alpha_1)$</td>
<td>2</td>
<td>$D_3 \times C_{12}$</td>
</tr>
<tr>
<td>760</td>
<td>$V_4$</td>
<td>$K(\sqrt{2}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$Q_{32} \times C_3$</td>
</tr>
<tr>
<td>763</td>
<td>$C_4$</td>
<td>$K(\sqrt{(-9 + \sqrt{109})/2})$</td>
<td></td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>767</td>
<td>$C_{22}$</td>
<td></td>
<td>$K_1(\alpha_4)$</td>
<td>2</td>
<td>$D_3 \times C_{11}$</td>
</tr>
<tr>
<td>771</td>
<td>$C_6$</td>
<td>$K(\sqrt{16 + \sqrt{257}})$</td>
<td></td>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>772</td>
<td>$C_4$</td>
<td>$K(\sqrt{(-7 + \sqrt{193})/2})$</td>
<td></td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>776</td>
<td>$C_{20}$</td>
<td>$K(\sqrt{(-5 + \sqrt{97})/2}, \gamma_{40})$</td>
<td></td>
<td>1</td>
<td>$C_{20}$</td>
</tr>
<tr>
<td>779</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{-19}, \gamma_{41})$</td>
<td></td>
<td>1</td>
<td>$C_{10}$</td>
</tr>
<tr>
<td>787</td>
<td>$C_5$</td>
<td>$K(\gamma_{42})$</td>
<td></td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>788</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{-1}, \gamma_{43})$</td>
<td></td>
<td>1</td>
<td>$C_{10}$</td>
</tr>
<tr>
<td>791</td>
<td>$C_{32}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{32}$</td>
</tr>
<tr>
<td>795</td>
<td>$V_4$</td>
<td>$K(\sqrt{-3}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$Q_{40}$</td>
</tr>
<tr>
<td>799</td>
<td>$C_{16}$</td>
<td></td>
<td>$K_1(\gamma_1)$</td>
<td>2</td>
<td>$I_{16}^5$</td>
</tr>
<tr>
<td>803</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{-11}, \gamma_{44})$</td>
<td></td>
<td>1</td>
<td>$C_{10}$</td>
</tr>
<tr>
<td>804</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt{22 + 2\sqrt{67}})$</td>
<td></td>
<td>1</td>
<td>$C_6 \times C_2$</td>
</tr>
<tr>
<td>807</td>
<td>$C_{14}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{14}$</td>
</tr>
<tr>
<td>808</td>
<td>$C_6$</td>
<td>$K(\sqrt{-2}, \alpha_{98})$</td>
<td></td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>811</td>
<td>$C_7$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_7$</td>
</tr>
<tr>
<td>815</td>
<td>$C_{30}$</td>
<td>$K(\sqrt{5}, \alpha_{100}, \gamma_{45})$</td>
<td></td>
<td>1</td>
<td>$C_{30}$</td>
</tr>
<tr>
<td>820</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt{6 + \sqrt{41}})$</td>
<td></td>
<td>2</td>
<td>$\Gamma_{2,3}$</td>
</tr>
<tr>
<td>823</td>
<td>$C_9$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_9$</td>
</tr>
<tr>
<td>824</td>
<td>$C_{20}$</td>
<td>$K(\sqrt{5 + 8\sqrt{2}, \gamma_{46}})$</td>
<td></td>
<td>2</td>
<td>$Q_{20} \times C_5$</td>
</tr>
<tr>
<td>827</td>
<td>$C_7$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_7$</td>
</tr>
<tr>
<td>831</td>
<td>$C_{28}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{28}$</td>
</tr>
<tr>
<td>835</td>
<td>$C_6$</td>
<td>$K(\sqrt{5, \alpha_{102}})$</td>
<td></td>
<td>2</td>
<td>$D_{11} \times C_3$</td>
</tr>
<tr>
<td>836</td>
<td>$C_{10} \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt{-11}, \gamma_{47})$</td>
<td></td>
<td>1</td>
<td>$C_{10} \times C_2$</td>
</tr>
<tr>
<td>839</td>
<td>$C_{33}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{33}$</td>
</tr>
<tr>
<td>840</td>
<td>$C_2$</td>
<td>$K(\sqrt{-2}, \sqrt{-3}, \sqrt{5})$</td>
<td></td>
<td>2</td>
<td>$32\Gamma_4 d$</td>
</tr>
<tr>
<td>843</td>
<td>$C_6$</td>
<td>$K(\sqrt{5 + \sqrt{281}}/2)$</td>
<td></td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>$-d$</td>
<td>$\text{Cl}(K)$</td>
<td>$K_1$</td>
<td>$K_2$</td>
<td>$l$</td>
<td>$G$</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>851</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{-23}, \gamma_{49})$</td>
<td>$K_1(\alpha_4)$</td>
<td>2</td>
<td>$D_3 \times C_5$</td>
</tr>
<tr>
<td>852</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt{2} + 5\sqrt{3})$</td>
<td>?</td>
<td>2</td>
<td>$Q_{28} \times C_2$</td>
</tr>
<tr>
<td>856</td>
<td>$C_6$</td>
<td>$K(\sqrt{2}, \alpha_{105})$</td>
<td>$K_1(\alpha_9)$</td>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>859</td>
<td>$C_7$</td>
<td>?</td>
<td>1</td>
<td>$C_7$</td>
<td></td>
</tr>
<tr>
<td>863</td>
<td>$C_{21}$</td>
<td>?</td>
<td>1</td>
<td>$C_{21}$</td>
<td></td>
</tr>
<tr>
<td>868</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt{9 + 4\sqrt{7}})$</td>
<td>$K_1(\alpha_2)$</td>
<td>2</td>
<td>$Q_{12} \times C_2$</td>
</tr>
<tr>
<td>871</td>
<td>$C_{22}$</td>
<td>?</td>
<td>1</td>
<td>$C_{22}$</td>
<td></td>
</tr>
<tr>
<td>872</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{-2}, \gamma_{50})$</td>
<td>?</td>
<td>1</td>
<td>$C_{10}$</td>
</tr>
<tr>
<td>879</td>
<td>$C_{22}$</td>
<td>?</td>
<td>1</td>
<td>$C_{22}$</td>
<td></td>
</tr>
<tr>
<td>883</td>
<td>$C_3$</td>
<td>$K(\alpha_{109})$</td>
<td>?</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>884</td>
<td>$C_8 \times C_2$</td>
<td>$\Gamma_3^1$</td>
<td>2</td>
<td>$\Gamma_{3,2}^1$</td>
<td></td>
</tr>
<tr>
<td>887</td>
<td>$C_{29}$</td>
<td>?</td>
<td>1</td>
<td>$C_{29}$</td>
<td></td>
</tr>
<tr>
<td>888</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{2}, \sqrt{43 + 5\sqrt{74}})$</td>
<td>$K_1(\gamma_9)$</td>
<td>2</td>
<td>$Q_{32} \times C_3$</td>
</tr>
<tr>
<td>895</td>
<td>$C_{16}$</td>
<td>?</td>
<td>2</td>
<td>$I_8^{16}$</td>
<td></td>
</tr>
<tr>
<td>899</td>
<td>$C_{14}$</td>
<td>?</td>
<td>2</td>
<td>$D_3 \times C_7$</td>
<td></td>
</tr>
<tr>
<td>903</td>
<td>$C_8 \times C_2$</td>
<td>?</td>
<td>1</td>
<td>$C_8 \times C_2$</td>
<td></td>
</tr>
<tr>
<td>904</td>
<td>$C_8$</td>
<td>?</td>
<td>1</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>907</td>
<td>$C_3$</td>
<td>$K(\alpha_{112})$</td>
<td>1</td>
<td>1</td>
<td>$C_{31}$</td>
</tr>
<tr>
<td>911</td>
<td>$C_{31}$</td>
<td>?</td>
<td>1</td>
<td>$C_{31}$</td>
<td></td>
</tr>
<tr>
<td>915</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-3}, \sqrt{(-1 + \sqrt{61})/2})$</td>
<td>$K_1(\alpha_2)$</td>
<td>2</td>
<td>$\Gamma_{2,3}^2$</td>
</tr>
<tr>
<td>916</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{-1}, \gamma_{52})$</td>
<td>?</td>
<td>3</td>
<td>?</td>
</tr>
<tr>
<td>919</td>
<td>$C_{19}$</td>
<td>?</td>
<td>1</td>
<td>$C_{19}$</td>
<td></td>
</tr>
<tr>
<td>920</td>
<td>$C_{10} \times C_2$</td>
<td>$K(\sqrt{2}, \sqrt{5}, \gamma_{53})$</td>
<td>?</td>
<td>2</td>
<td>$(Q_{16} \times C_3) \times C_5$</td>
</tr>
<tr>
<td>923</td>
<td>$C_{10}$</td>
<td>$K(\sqrt{13}, \gamma_{74})$</td>
<td>?</td>
<td>2</td>
<td>$D_7 \times C_5$</td>
</tr>
<tr>
<td>932</td>
<td>$C_{12}$</td>
<td>$K(\sqrt{13 + \sqrt{233}})/2, \alpha_{115}$</td>
<td>?</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>935</td>
<td>$C_{14} \times C_2$</td>
<td>?</td>
<td>2</td>
<td>$Q_{16} \times C_7$</td>
<td></td>
</tr>
<tr>
<td>939</td>
<td>$C_8$</td>
<td>?</td>
<td>1</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>943</td>
<td>$C_{16}$</td>
<td>?</td>
<td>1</td>
<td>$I_4^{16}$</td>
<td></td>
</tr>
<tr>
<td>947</td>
<td>$C_5$</td>
<td>$K(\gamma_{55})$</td>
<td>?</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>948</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt{80 + 9\sqrt{79}})$</td>
<td>$K_1(\gamma_2)$</td>
<td>2</td>
<td>$D_5 \times C_6$</td>
</tr>
<tr>
<td>951</td>
<td>$C_{26}$</td>
<td>?</td>
<td>1</td>
<td>$C_{26}$</td>
<td></td>
</tr>
<tr>
<td>952</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-7}, \sqrt{9 + 10\sqrt{2}})$</td>
<td>?</td>
<td>2</td>
<td>?</td>
</tr>
<tr>
<td>955</td>
<td>$C_4$</td>
<td>$K(\sqrt{-9 + 13\sqrt{5}})/2$</td>
<td>?</td>
<td>2</td>
<td>$Q_{52}$</td>
</tr>
<tr>
<td>959</td>
<td>$C_{36}$</td>
<td>?</td>
<td>1</td>
<td>$C_{36}$</td>
<td></td>
</tr>
<tr>
<td>964</td>
<td>$C_{12}$</td>
<td>$K(\sqrt{-15 + \sqrt{241}})/2, \alpha_{119}$</td>
<td>?</td>
<td>1</td>
<td>$C_{11}$</td>
</tr>
<tr>
<td>967</td>
<td>$C_{11}$</td>
<td>?</td>
<td>1</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>971</td>
<td>$C_{15}$</td>
<td>$K(\alpha_{120}, \gamma_{57})$</td>
<td>?</td>
<td>1</td>
<td>?</td>
</tr>
<tr>
<td>979</td>
<td>$C_8$</td>
<td>?</td>
<td>1</td>
<td>?</td>
<td></td>
</tr>
</tbody>
</table>
Supplements. We expect that all inequalities for $l$ in our table are equalities. For this expectation we give the following supplemental data.

For $K = \mathbb{Q}(\sqrt{-731})$, $\text{Gal}(K_2/K) \cong A_4 \times C_4$. If $K_{ur} \neq K_2$, $\text{Gal}(K_{ur}/K_2) \cong C_2$, or $C_4$.

For $K = \mathbb{Q}(\sqrt{-771})$, $\text{Gal}(K_3/K) \cong S_4 \times C_3$. If $K_{ur} \neq K_3$, $\text{Gal}(K_{ur}/K_3) \cong C_2$, or $C_4$.

For $K = \mathbb{Q}(\sqrt{-856})$, $\text{Gal}(K_2/K) \cong D_3 \times C_3$. $K_2$ has an unramified $V_4$-extension $L$ which is an $S_4 \times C_3$-extension of $K$. If $K_{ur} \neq L$, then $[K_{ur} : L] = 2$, or 4.

For $K = \mathbb{Q}(\sqrt{-916})$, $\text{Gal}(K_2/K) \cong D_3 \times C_5$. Put $L = K\mathbb{Q}(\sqrt{229})_{w-ur}$, where $\mathbb{Q}(\sqrt{229})_{w-ur}$ denotes the maximal extension of $\mathbb{Q}(\sqrt{229})$ unramified at all finite primes, which is an $S_4$-extension of $\mathbb{Q}$. (Cf. For $K = \mathbb{Q}(\sqrt{-687})$, $K_{ur} = K_1\mathbb{Q}(\sqrt{229})_{w-ur}$. See §7. The index “$w$-ur” means “weakly-unramified.”) $L$ is an unramified $V_4$-extension of $K_2$ which is an $S_4 \times C_5$-extension of $K$. If $K_{ur} \neq L$, then $[K_{ur} : L] = 2$, or 4.

For $K = \mathbb{Q}(\sqrt{-952})$, $\text{Gal}(K_2/K) \cong \Gamma_{2,2} \times C_5$. (Note that $\Gamma_{2,2} \cong C_4 \wr C_2 (= 32\Gamma_3 e_\Gamma)$ If $K_{ur} \neq K_2$, $\text{Gal}(K_{ur}/K_2) \cong C_2$.

For $K = \mathbb{Q}(\sqrt{-984})$, $\text{Gal}(K_2/K) \cong Q_8 \times C_3$. If $K_{ur} \neq K_2$, $\text{Gal}(K_{ur}/K_2) \cong C_2^2$, or $C_5^2$.

For $K = \mathbb{Q}(\sqrt{-987})$, $\text{Gal}(K_2/K) \cong Q_{20} \times C_2$. If $K_{ur} \neq K_2$, $\text{Gal}(K_{ur}/K_2) \cong C_2^4$.

For $K = \mathbb{Q}(\sqrt{-996})$, $\text{Gal}(K_2/K) \cong D_3 \times C_6$. $K_2$ has an unramified $V_4$-extension $L$ which is an $S_4 \times C_6$-extension of $K$. We have $[K_{ur} : L] \leq 32$ and the odd part of $\text{Cl}(L)$ is trivial or isomorphic to $C_3^2, C_3^3, \text{ or } C_5^2$.

---

<table>
<thead>
<tr>
<th>$-d$</th>
<th>$\text{Cl}(K)$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$l$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>983</td>
<td>$C_{27}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{27}$</td>
</tr>
<tr>
<td>984</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{2}, \sqrt[3]{9 + \sqrt{82}})$</td>
<td>$\geq 2$</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>987</td>
<td>$C_4 \times C_2$</td>
<td>$K(\sqrt{-3}, \sqrt{(-1 + 3\sqrt{21})/2})$</td>
<td>$\geq 2$</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>991</td>
<td>$C_{17}$</td>
<td></td>
<td></td>
<td>1</td>
<td>$C_{17}$</td>
</tr>
<tr>
<td>995</td>
<td>$C_8$</td>
<td></td>
<td></td>
<td>2</td>
<td>$I_9^8$</td>
</tr>
<tr>
<td>996</td>
<td>$C_6 \times C_2$</td>
<td>$K(\sqrt{-1}, \sqrt[3]{82 + 9\sqrt{83}})$</td>
<td>$K_1(\alpha_2)$</td>
<td>$\geq 3$</td>
<td>?</td>
</tr>
</tbody>
</table>
2. UPPER BOUND FOR \([K_{ur} : K]\)

We review here how to obtain an upper bound for \([K_{ur} : K]\) for a given number field \(K\) by using discriminant bounds.

In this section \(K\) denotes an algebraic number field of finite degree. If we denote by \(n_K\) the degree of \(K\), the \(n_K\)th root of the absolute value of the discriminant \(d_K\) of \(K\) is called root discriminant of \(K\) and denoted by \(rd_K\):

\[ rd_K = |d_K|^{1/n_K}. \]

The following lemma is fundamental for our study.

**Lemma 1.** Let \(L/K\) be a finite extension of algebraic number fields. Then \(rd_L = rd_K\) if and only if \(L/K\) is unramified at all finite primes.

This can be easily proved by the transitive law of discriminants.

The following estimates for \(rd = rd_L\) are known (see [39]): Unconditionally we have

\[ rd \geq (4\pi e^{1+C})^{r_1/n}(4\pi e^C)^{2r_2/n} - O(n^{-3/2}) \]

\[ = (60.8395 \ldots)^{r_1/n}(22.3816 \ldots)^{2r_2/n} - O(n^{-3/2}), \]

as \(n \to \infty\), where \(C = 0.5772\ldots\) denotes Euler's constant, and \(n = n_L\) and \(r_1\) (resp. \(r_2\)) denotes the number of real (resp. imaginary) primes of \(L\). From this, for all imaginary quadratic number fields \(K = \mathbb{Q}(\sqrt{d}), d < 0\) with \(|d| \leq 499, [K_{ur} : K] < \infty\). Under GRH we have

\[ rd \geq (8\pi e^{C+\pi/2})^{r_1/n}(8\pi e^C)^{2r_2/n} - O((\log n)^{-2}) \]

\[ = (215.3325 \ldots)^{r_1/n}(44.7632 \ldots)^{2r_2/n} - O((\log n)^{-2}), \]

as \(n \to \infty\). From this, for all \(K = \mathbb{Q}(\sqrt{d}), d < 0\) with \(|d| \leq 2003, [K_{ur} : K] < \infty\).

By Lemma 1 we immediately obtain the following proposition, which tells us how to get an upper bound for \([K_{ur} : K]\).

**Proposition 1.** Let \(B(n, r_1, r_2)\) (where \(n \in \mathbb{N} \ni n = r_1 + 2r_2, r_1, r_2\) nonnegative integers) be the lower bound for the root discriminants of algebraic number fields \(F\) of finite degrees (\(\geq n\)) such that \(r_i(F)/n_L = r_i/n\) for \(i = 1, 2\), where \(r_1(F)\) (resp. \(r_2(F)\)) is the number of real (resp. imaginary) primes of \(F\). Suppose that \(K\) has an unramified normal extension \(L\) of degree \(m\).
Let $H$ be a positive integer.

(i) If $rd_K < B(Hmn_K, Hmr_1(K), Hmr_2(K))$, then $[K_{ur} : L] < H$ and therefore $h(L) < H$. In particular, if $rd_K < B(2mn_K, 2mr_1(K), 2mr_2(K))$, then $K_{ur} = L$.

(ii) If $h(L) = 1$ and $rd_K < B(60mn_K, 60mr_1(K), 60mr_2(K))$, then $K_{ur} = L$.

We note that the number 60 in (ii) is the minimal order of the finite nonsolvable groups, that is the order of $A_5$, the alternating group of degree five.

Thus, the knowledge of good lower bounds for $B(n, r_1, r_2)$ is very important for our study. The best known unconditional lower bounds for $B(n, r_1, r_2)$ can be found in the tables due to F. Diaz y Diaz [12]. If we assume GRH, much better lower bounds can be obtained. The best known conditional (GRH) lower bounds are found in the unpublished tables due to A. M. Odlyzko [38], which are copied in Martinet's expository paper [34].

All the imaginary quadratic number fields $K = \mathbb{Q}(\sqrt{d})$, $-d = 3, 4, 7, 8, 11, 19, 43, 67, 163$ satisfy the condition in (ii) for $L = K$. In fact by the table in [12] we have $rd_K \leq \sqrt{163} < B(60 \cdot 2, 0, 60 \cdot 1)$. Therefore none of these fields has any nontrivial unramified extension.

Let $K$ be an imaginary quadratic number field of small conductor. What unramified normal extension can we take as $L$ when we apply Proposition 1? Since known lower bounds for $B(n, r_1, r_2)$ grow large as $n$ grows large if $r_2/n$ is fixed, we need to take an extension $L$ of large degree. By the reason described in the Introduction it seems that the best choice is $L = (K_g)_1$, the Hilbert class field of the genus field $K_g$ of $K$ except when we can easily get an unramified extension of larger degree. In most cases we cannot conclude $K_{ur} = L$ only by (i) of Proposition 1, that is, we get only $[K_{ur} : L] < H$ for some $H \geq 3$. Therefore we need to show $h(L) = 1$ by class number calculation or using criteria for class number divisibility.

3. CLASS NUMBER RELATIONS

In this section we review some results on class number relations on normal extensions of algebraic number fields. We will use them to calculate class numbers of fields of large degrees from class numbers of subfields.

Let $L/K$ be a (finite) normal extension of algebraic number fields and $G$ its Galois group. For any subgroup $H$ of $G$, we denote by $1^G_H$ the induced character of $G$ from the principal character of $H$. S. Kuroda [27] and R. Brauer [6] proved independently the following relation among the class
numbers $h(M)$, the regulators $R(M)$, and the numbers $w_2(M)$ of roots of unity of 2-power orders of intermediate fields $M$ of $L/K$: If we have a linear relation among $1_H^G$

$$\sum_{H \leq G} a_H 1_H^G = 0,$$

then we have the following relation

$$(\dagger) \quad \prod_{H \leq G} \left( \frac{h(L^H) R(L^H)}{w_2(L^H)} \right)^{a_H} = 1.$$  

Here, $L^H$ denotes the intermediate field of $L/K$ corresponding to $H$ by Galois theory.

We want to get $h(L)$ from the knowledge of $h(K)$ and $h(M)$ for intermediate fields $M$. When an arbitrary finite group $G$ is given, there does not exist necessarily such a nontrivial (that is, $a_H \neq 0$ for some $H$) linear relation. Even if there exists a nontrivial linear relation, it is difficult to use $(\dagger)$ in this form because it contains regulators. However, in some cases the product $\prod_{H} R(L^H)^{a_H}$ can be simplified into the form $[E_L : E]/q$ by elementary calculations, where $E_L$ is the group of units in $L$, $E$ is its subgroup generated by $E_M$ for some subfields $M$ of $L$, and $q$ is some power of a prime, and moreover $[E_L : E]$ is also some power of the prime dividing $q$. Such a simplified relation is often very useful for calculation of class numbers of fields of high degrees.

Let $G = \langle a, b \mid a^{2n} = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ be the dihedral group of order $2n$ ($n \geq 2$) (the four group if $n = 2$). Then we have the following relation:

$$1_G^{(1)} + 2 \cdot 1_G^{(b)} = 1_G^{(a)} + 1_G^{(ab)}.$$  

From this and some elementary calculations, we get the following lemmas.

**Lemma 2.** ([27, 28]) Let $L/K$ be a $V_4$-extension of algebraic number fields and $F, M$ and $N$ its three intermediate fields. Then we have the following class number relation:

$$h(L) = \frac{[E_L : E_F E_M E_N]}{2^2 + r + v - t} \cdot \frac{h(F) h(M) h(N)}{h(K)^2},$$

where $r$ is the $\mathbb{Z}$-rank of $E_K$, $t$ is the number of infinite primes ramified in $L/K$, and $v \in \{0, 1\}$ (for the definition of $v$, see [28]). Moreover, the index $[E_L : E_F E_M E_N]$ is a nonnegative power of 2.
The special case $K = \mathbb{Q}$ and $L \supset \mathbb{Q}(\sqrt{-1})$ of this is a classical result due to Dirichlet, which we can consider as a prototype. By this lemma we can calculate class numbers of some fields of degree 24.

**Lemma 3.** ([18, 19, 36]) Let $K$ be an imaginary quadratic number field or the rational number field $\mathbb{Q}$, and $p$ an odd prime number. Let $L$ be a $D_p$-extension of $K$. Let $M$ and $M'$ be any two intermediate fields of $L/K$ with $[M : K] = [M' : K] = p$ and $N$ the unique quadratic subextension of $L/K$. Then we have the following class number relation:

$$h(L) = \frac{[E_L : E_ME_M'\cdot E_N]}{p^b} \cdot \frac{h(M)^2h(N)}{h(K)^2},$$

where $b = 1$ if $K = \mathbb{Q}$ and $L$ is imaginary, and $b = 2$ otherwise. Moreover, the index $[E_L : E_ME_M'\cdot E_N] = p^a$ with $0 \leq a \leq b$. Furthermore, if $L/N$ is unramified, then $a = b - 1$, that is,

$$h(L) = \frac{1}{p} \cdot \frac{h(M)^2h(N)}{h(K)^2}.$$

Let $K$ be an imaginary quadratic number field whose $p$-class group is cyclic of order $p$. Then the Hilbert $p$-class field of $K$ is a $D_p$-extension of $\mathbb{Q}$. If $p = 3$, or $5$, by Lemma 3 we can compute its class number by calculating the class number of its subfield of degree $p$. For this purpose, we use data for cubic fields and quintic fields. Data for number fields of degrees $n$ with $3 \leq n \leq 7$ and small discriminant (in absolute value) are available by anonymous ftp from megrez.math.u-bordeaux.fr (147.210.16.17).

**Lemma 4.** ([9]) Let $L$ be an imaginary $D_4$-extension of $\mathbb{Q}$. Let $M$ and $N$ be the two nonisomorphic nonnormal quartic subfields and $K$ its quadratic subfield such that $L/K$ is cyclic. Then we have the following class number relation:

$$h(L) = \frac{[E_L : E_ME_N\cdot E_K]}{2^v} \cdot h(M)h(N)h(K).$$

Here, $v = 3$ if $L$ is a CM-field and $v = 2$ otherwise. Moreover, the index $[E_L : E_ME_N\cdot E_K]$ is a nonnegative power of 2.

If $K$ is an imaginary quadratic number field with cyclic 2-class group of order $\geq 4$, then by this lemma the class number of the unique unramified cyclic quartic extension $L$ of $K$ can be computed from the class numbers of two nonnormal quartic subfields of $L$. In this case, $L$ is a $D_4$-extension of $\mathbb{Q}$.
and all of its nonnormal quartic subfields have odd class number. Therefore we have

\[ h(L) = h(M)h(N)h(K)/4, \]

because the exact 2-power dividing \( h(L) \) is \( h^{(2)}(K)/4 \), where \( h^{(2)}(K) \) is the 2-class number of \( K \).

4. THE ACTION OF GALOIS GROUPS ON CLASS GROUPS

The action of Galois groups on class groups can often be used to obtain useful information on the structure of class groups.

We first review the following, often called \( p \)-rank theorem.

**Lemma 5.** (See [53, Theorem 10.8]) Let \( L/K \) be a finite cyclic extension of degree \( n \) of algebraic number fields. Let \( p \) be a prime number with \( p \nmid n \) and assume that all fields \( E \) with \( K \subseteq E \subseteq L \) have trivial \( p \)-class group. Then for each positive integer \( a \), the \( p^a \)-rank of the \( p \)-class group of \( L \) is a multiple of the order \( f \) of \( p \) modulo \( n \). In particular, if \( p \mid h(L) \), then \( p^f \mid h(L) \), and if \( h(L) < p^f \), then \( p \nmid h(L) \).

The following proposition gives us more information in a sense.

**Proposition 2.** Let \( K \) be an imaginary abelian number field and \( K_g \) its genus field, that is, the maximal unramified abelian extension of \( K \) which is abelian over \( Q \). Let \( L \) be the Hilbert class field of \( K_g \). Then for any prime number \( p \) with \( p \nmid [L : Q] \), the \( p \)-class group \( \text{Cl}^{(p)}(L) \) of \( L \) is trivial or noncyclic.

**Proof.** We first note that both \( L \) and its Hilbert \( p \)-class field \( M \) are normal over \( Q \). The Galois group \( \Gamma = \text{Gal}(L/Q) \) acts by conjugation on \( \text{Gal}(M/L) \), which is isomorphic to \( \text{Cl}^{(p)}(L) \) by class field theory. This action induces a group homomorphism

\[ \rho : \Gamma \rightarrow \text{Aut}(\text{Cl}^{(p)}(L)). \]

Now we assume that \( \text{Cl}^{(p)}(L) \) is cyclic. Then \( \text{Aut}(\text{Cl}^{(p)}(L)) \) is abelian and therefore the kernel of \( \rho \) contains the derived group of \( \Gamma \). Hence if we let \( F \) be the field corresponding to \( \rho(\Gamma) \), \( F \) is abelian over \( Q \) and therefore \( F \) is contained in \( K_g \). Since \( \text{Ker}(\rho) = \text{Gal}(L/F) \) acts on \( \text{Gal}(M/L) \) trivially, by the assumption \( p \nmid [L : Q] = |\Gamma| \) we conclude that \( \text{Gal}(M/F) \) is factored as

\[ \text{Gal}(M/F) = \text{Gal}(M/L) \times H, \text{ with } H \cong \text{Ker}(\rho). \]
This implies that the class number of \( F \) is divisible by \( |\text{Gal}(M/L)| = |\text{Cl}^{(p)}(L)| \) and therefore \( |\text{Cl}^{(p)}(L)| \mid h(K_g) \). Hence \( \text{Cl}^{(p)}(L) \) must be trivial.

**Remark.** As described in the Introduction, if \( K \) is an imaginary quadratic number field, we can easily calculate the class number of \( K_g \). Thus, this will be a powerful tool.

The group homomorphism \( \rho \) above often gives a useful information on \( \text{Cl}^{(l)}(L) \) also for prime divisors \( l \) of \( [L : \mathbb{Q}] \). The same argument works in more general situation. The referee pointed out that this proposition is a special case of a result due to O. Grün with proofs (essentially the same as above) by L. Holzer and A. Scholz (Jaresber. DMV 44 (1934), 74-75).

Let \( L/K \) be a finite Galois extension of algebraic number fields with Galois group \( \Gamma \). Let \( p \) be a prime number and \( A \) the \( p \)-class group of \( L \). The action of \( \Gamma \) on \( V = A^{p^n - 1} / A^{p^n} \) induces a group homomorphism

\[
\Gamma \to \text{Aut}(V) \cong \text{GL}(r, p),
\]

where \( r \) is the \( p^n \)-rank of \( A \). Thus, the knowledge of the structure of linear groups over finite fields is often useful for the study of the structure of class groups. Later we use the following facts.

**Lemma 6.** ([22, II, Satz 7.3a]) Let \( p \) be a prime and \( f \) and \( n \) positive integers. \( \text{GL}(f, p^n) \) has a cyclic subgroup \( S \) of order \( p^n - 1 \). (\( S \) is called Singer cycle.) For any subgroup \( T \) of \( S \) whose order is not a divisor of \( p^n - 1 \) for any \( e < f \), its normalizer in \( \text{GL}(f, p^n) \) is the semi-direct product of \( S \) with a cyclic group \( C \) of order \( f \) such that the action of one generator of \( C \) on \( S \) is \( p^n \)-th powering.

**Remark.** Let \( q \) be a prime number \( \neq p \) and \( f \) the order of \( p \) modulo \( q \). Then any Sylow \( q \)-subgroup of \( \text{GL}(f, p) \) is a subgroup of a Singer cycle and its normalizer in \( \text{GL}(f, p) \) has order \( f(p^f - 1) \).

**Lemma 7.** ([8]) Let \( q \) be a power of an odd prime and \( W \) a Sylow 2-subgroup of \( \text{GL}(2, q) \). If \( q \equiv 1 \pmod{4} \), then \( W \) is isomorphic to \( C_2 \wr C_2 \), the wreath product of \( C_2 \) by \( C_2 \), where \( 2^a \) is the exact power of 2 dividing \( q - 1 \), while if \( q \equiv 3 \pmod{4} \), then \( W \) is a semi-dihedral group of order \( 2^{2t+2} \), where \( 2^t \) is the exact power of 2 dividing \( q + 1 \).

**Remark.** By this lemma Sylow 2-subgroups of \( \text{GL}(2, 3) \) and \( \text{GL}(2, 5) \) are isomorphic to \( SD_{16} \) and \( C_4 \wr C_2 = 32 \Gamma_3 e \), respectively. We note that
$C_4 \wr C_2$ has three maximal subgroups isomorphic to $C_4^2$, $D_4 \rtimes C_4$, and $SD_{16}$. Here, $D_4 \rtimes C_4$ denotes the central product of $D_4$ and $C_4$.

Later for an unramified Galois extension $L$ of an imaginary quadratic number field which is normal over $\mathbb{Q}$, we consider the action of the Galois group $\text{Gal}(L/\mathbb{Q})$ on the class group $\text{Cl}(L)$ of $L$. Then an important problem is to determine the image of the induced group homomorphism $\text{Gal}(L/\mathbb{Q}) \to \text{Aut}(\text{Cl}(L))$. Thus, we prepare the following.

**Lemma 8.** Let $K$ be a quadratic number field and $L$ its finite extension which is normal over $\mathbb{Q}$. Suppose that $L/K$ is unramified at all finite primes. Then the Galois group of $L/\mathbb{Q}$ is generated by elements of order two not contained in the Galois group of $L/K$. In particular, $\text{Gal}(L/\mathbb{Q})$ is isomorphic to none of the 2-groups $C_4, Q_8, SD_{16}$, and $C_4 \wr C_2$.

The former assertion of this is a special case of [11, Corollary 16.31], which is easily deduced from Čebotarev monodromy theorem. None of the 2-groups $C_4, Q_8, SD_{16}$, and $C_4 \wr C_2$ is generated by elements of order two. We note that if a quadratic number field has an unramified quaternionic extension which is normal over $\mathbb{Q}$, then its Galois group over $\mathbb{Q}$ is isomorphic to $D_4 \rtimes C_4$ ([30]).

5. RESULTS ON CLASS FIELD TOWER

We also use results on class field tower. We review here some results. The following is well known.

**Lemma 9.** (See [49, Theorem I].) Let $K$ be an algebraic number field of finite degree and $p$ any prime number. If the $p$-class group, i.e., the $p$-part of the class group of $K$ is cyclic, then the $p$-class group of the Hilbert $p$-class field of $K$ is trivial. Moreover, if $p = 2$ and the 2-class group of $K$ is isomorphic to $V_4$, then the 2-class group of the second Hilbert 2-class field, that is, the Hilbert 2-class field of the Hilbert 2-class field, of $K$ is trivial.

This can be easily proved by using group theory.

All fields we consider have cyclic $p$-class group for any odd prime $p$. In fact, a noncyclic $p$-class group for an odd prime $p$ first occurs as $\text{Cl}(\mathbb{Q}(\sqrt{-3299}))$, which is isomorphic to $C_9 \times C_3$ ([7]). On the other hand, noncyclic 2-class groups often occur as the 2-class group of $\mathbb{Q}(\sqrt{d}), d < 0$ with small $|d|$. But for all fields we consider, the 4-ranks of the class groups are at most one. The first example for bigger 4-rank is $d = -2379$. 
In the rest of this section, we review some results on the 2-class field
towers of imaginary quadratic number fields $K$:

$$K = K_0^{(2)} \subseteq K_1^{(2)} \subseteq K_2^{(2)} \subseteq \cdots \quad (K_i^{(2)} \text{ is the Hilbert 2-class field of } K_i^{(2)}).$$

By the lemma above, if the 2-class group $\text{Cl}^{(2)}(K)$ of $K$ is cyclic, then the
2-class field tower of $K$ terminates with $K_1^{(2)}$, that is, $K_2^{(2)} = K_1^{(2)}$. When
does $K_2^{(2)} = K_1^{(2)}$ hold? F. Lemmermeyer [29] gave a necessary condition:
If $K_2^{(2)} = K_1^{(2)}$, then $\text{Cl}^{(2)}(K)$ is cyclic, or isomorphic to $C_2 \times C_2^n (m \geq 1)$. He also obtained sufficient criteria in terms of the factorization of $d$.

If $\text{Cl}^{(2)}(K) \cong V_4$, then $K_3^{(2)} = K_2^{(2)}$ by the lemma above. We know that
any nonabelian finite groups of order $2^n$ with $n > 2$ whose abelianization is
isomorphic to $V_4$ is isomorphic to $D_{2n-1}, Q_{2n},$ or $SD_{2n}$. Thus, if $\text{Cl}^{(2)}(K) \cong V_4,$ then $\text{Gal}(K_2^{(2)}/K) \cong V_4, D_{2n-1}, Q_{2n},$ or $SD_{2n},$ where $n$ is determined by
$2^{n-2} \parallel h(K_1^{(2)})$ (and therefore $n$ can be easily calculated, because $K_1^{(2)} = K_3^{(2)}$). H. Kisilevsky [25] characterized the occurrence of these groups in
terms of the factorization of the discriminant $d$ of $K$.

When does $K_3^{(2)} = K_2^{(2)}$ hold? The complete answer has not been given
yet. However, Lemmermeyer [29] determined when $\text{Gal}(K_2^{(2)}/K)$ is meta-
cyclic. (Of course, if $\text{Gal}(K_2^{(2)}/K)$ is metacyclic, $K_3^{(2)} = K_2^{(2)}$ by Lemma
9.) He proved that if $\text{Gal}(K_2^{(2)}/K)$ is nonabelian but metacyclic, then
$\text{Cl}^{(2)}(K) \cong V_4$, or $d$ is of the form $-4pq$, where $p$ and $q$ are primes congru-
ent to 5 modulo 8:
(i) If $(p/q) = 1$, then $\text{Gal}(K_2^{(2)}/K)$ is isomorphic to the group given by

$$\langle \sigma, \tau \mid \tau^{2^{n+1}} = 1, \sigma^4 = \tau^{2^n}, \sigma^{-1} \tau \sigma = \tau^{-1} \rangle,$$

or

$$\langle \rho, \sigma \mid \rho^{2^{n+1}} = \sigma^{2^m} = 1, \sigma^{-1} \rho \sigma = \rho^{-1} \rangle,$$

according as the norm of the fundamental unit of the real quadratic number
field $\mathbb{Q}(\sqrt{pq})$ is $-1$ or 1. Here, $n$ is determined by $2^n \parallel h(\mathbb{Q}(\sqrt{pq}))$ (in
both cases), and $m$ is determined by $2^{m+n} \parallel h(\mathbb{Q}(\sqrt{-1}, \sqrt{pq}))$. In [5],
these groups are denoted by $MC^-_n$ and $MC^+_n,m$, respectively. We use these
notations in the Appendix 2.
(ii) If $(p/q) = -1$, then $\text{Gal}(K_2^{(2)}/K) \cong M_{2^{m+2}}$, where $m$ is determined by
$2^{m+1} \parallel h(K)$. Note that the presentation of $\text{Gal}(K_2^{(2)}/K)$ given in [29] is

$$\langle \rho, \sigma \mid \rho^2 = \sigma^{2^m}, \rho^4 = 1, \sigma^{-1} \rho \sigma = \rho^{-1} \rangle,$$
and that if we put $\tau = \rho \sigma^{2m-1}$, then $\sigma^{2m+1} = \tau^2 = 1$ and $\tau^{-1}\sigma\tau = \sigma^{2m+1}$.

Recently E. Benjamin, F. Lemmermeyer, and C. Snyder characterized $K$ whose 2-class fields have cyclic 2-class group [5]. They proved that $\text{Gal}(K_{2}(2)/K)$ is nonmetacyclic and its derived group is cyclic, if and only if $d$ is of one of the following forms: $d = -4pp'$, where $p$ and $p'$ are primes with $p \equiv 1, p' \equiv 5 \pmod{8}$ and $(p/p') = 1, (p/p')_4(p'/p)_4 = -1; d = -rpp'$, where $-r$ is a negative discriminant $\neq -4$, and $p$ and $p'$ are positive prime discriminants with $(p/p') = (r/p) = 1$ and $(r/p') = (p/p')_4(p'/p)_4 = -1$. Moreover, $\text{Gal}(K_{2}(2)/K)$ is isomorphic to $\Gamma_{m,t}^1$ or $\Gamma_{m,t}^2$, according as $d = -4pp'$ (in this case $2t = h(2)(\mathbb{Q}(-4p))$, or $d = -rpp'$ (in this case $2t = h(2)(\mathbb{Q}(-,rp))$, and in both cases $2^m = h(2)(K)/2$.

6. DETERMINATION OF $\text{Gal}(K_{ur}/K)$

In this section, we determine $\text{Gal}(K_{ur}/K)$. Our procedure for each $K = \mathbb{Q}(\sqrt{d})$ is as follows:

(1) We take a normal unramified extension $L$ of $K$ of large degree for which we can obtain an upper bound for $[K_{ur} : L]$ less than 60 by discriminant bounds (unconditional for $|d| < 420$ and conditional for $|d| > 420$).

(2) We show $h(L) = 1$ and conclude $K_{ur} = L$ by Proposition 1. (For fields with cyclic class group of odd order prime to 15, we can conclude $K_{ur} = K$, from $[K_{ur} : K_1] < 168$ (see below).)

(3) We determine the structure of $\text{Gal}(K_{ur}/K)$. (This is not difficult for most fields.)

As described in §2, for most $K$, we take $L = (K_g)_1$. The exceptional cases are the following two cases:

(E1) $d$ is divisible by the discriminant $d_E$ of a quartic number field $E$ and the quotient $d/d_E$ is a fundamental quadratic discriminant or 1. In this case the normal closure $M$ of $E$ is an unramified $A_4$-extension of the quadratic number field $\mathbb{Q}(\sqrt{d_E})$ and the composite field $KM$ is not contained in $(K_g)_1$. This case is divided into the following three subcases:

(a) $d = d_E = -p$, where $p$ is a prime $\equiv 3 \pmod{4}$: $d = -283, -331, -491, -563, -643, -751$. In this case, $K$ has an unramified $\widetilde{A}_4$-extension, which yields a quaternionic extension of $K_1 = (K_g)_1$ by composition.

(b) $d = d_E$ is a composite: $d = -731 = 17 \cdot (-43)$. In this case, $K$ has an unramified $A_4$-extension, which yields a $V_4$-extension of $K_1 = (K_g)_1$ by composition.

(c) $d = d'd_E$, where $d'$ is a fundamental quadratic prime discriminant:
$d = -687 = (-3) \cdot 229, d = -771 = (-3) \cdot 257, d = -916 = (-4) \cdot 229$. In this case, $K$ has an unramified $S_4$-extension, which yields a $V_4$-extension of $(K_g)_1$ by composition.

(E2) Though $K$ does not satisfy the condition in (E1), we can check that $K$ has an unramified $S_4$-extension: $d = -856, -996$.

These exceptional cases will be treated in the next section. In the rest of this section, we treat the other fields. Let $L = (K_g)_1$. Then we will show $K_{ur} = L = (K_g)_1$.

We can treat fields with same class group similarly. (Of course, for some fields, additional consideration is needed.) None of $K$ with $h(K) = 1$ has any nontrivial unramified extension. For the fields $K$ with $h(K) = 2$, we have determined $\text{Gal}(K_{ur}/K)$ in [58].

We first treat fields with odd class number $\geq 3$. Note that $K_g = K$ if and only if $h(K)$ is odd. We note that except for fields listed in (a) no fields with odd class number appear in our table as fields with $l \geq 2$. For all $K$ with odd class number and $|d| < 1000$ except for $d = -787, -827, -859, -883, -947, -967, -971, -991$, we get $[K_{ur} : K_1] < 60$ by discriminant bounds. Therefore our task is to prove $h(K_1) = 1$. We have the following.

**Proposition 3.** Let $K$ be an imaginary quadratic number field with cyclic class group of odd order $n \geq 7$. Assume that for all intermediate fields $E$ of $K_1/K$, we have $E_1 = K_1$. Then if $h(K_1) > 1$ (this is equivalent to $K_2 \neq K_1$), we have $h(K_1) \geq 2^6 = 64$. (This is valid also for real quadratic number fields. In the case where $K$ is imaginary and $h(K)$ is a prime ($\geq 7$), $h(K_1) \geq 64$ can be improved to $h(K_1) \geq 13^2 = 169$. See the remark after the proof.)

**Proof.** Let $l$ be a prime factor of $h(K) = n$. Then by the assumption there exists a subextension $E$ of $K_1/K$ such that $h(E) = l$ and $E_1 = K_1$. Therefore $l \mid h(K_1)$ by Lemma 9.

Let $p$ be a prime factor of $h(K_1)$ and $r$ the $p$-rank of the $p$-class group of $K_1$. We will show $p^r \geq 2^6 = 64$. Since by Lemma 5 and Proposition 2 we have $r \geq 2$ and $p^r \equiv 1 \pmod{n}$, it suffices to eliminate the following four possibilities: (a) $p = 2, r = 3$ if $n = 7$; (b) $p = 2, r = 4$ if $n = 15$; (c) $p = 2, r = 5$ if $n = 31$; (d) $p = 3, r = 3$ if $n = 13$. Note that for all these possibilities, $r$ is the order of $p$ mod $n$.

Put $\Gamma = \text{Gal}(K_1/Q) \cong D_n$ and $\Lambda = \text{Gal}(K_1/K) \cong C_n$. The action of $\Gamma$ on $\text{Cl}(K_1)/\text{Cl}(K_1)^P \cong F^*_p$, induces a group homomorphism $\Gamma \to \text{GL}(r, p)$.
If \( \Lambda \cap \text{Ker}(\rho) \neq \{1\} \), then we would have \( p^r \mid h(F) \) for the field \( F \) corresponding to \( \Lambda \cap \text{Ker}(\rho) \), which contradicts the assumption. (Cf. the proof of Proposition 2.) Therefore \( \Lambda \cap \text{Ker}(\rho) = \{1\} \) and \( \rho(\Lambda) \cong \Lambda \cong C_n \). Since \( \text{Im}(\rho) \) is a factor group of \( \Gamma \cong D_n \), \( \text{Im}(\rho) \cong D_n \). Since \( \Lambda \) is a normal subgroup of \( \Gamma \), \( \text{Im}(\rho) \) is contained in the normalizer \( N \) of \( \rho(\Lambda) \) in \( \text{GL}(r,p) \). Therefore \( N \) must have a subgroup of isomorphic to \( D_n \).

Assume that \( n \) is a prime and that \( r \) is the order of \( p \) mod \( n \). Then \( \rho(\Lambda) \) is a subgroup of a Singer cycle of \( \text{GL}(r,p) \) satisfying the condition of Lemma 6. Therefore \( N \cong C_{pr-1} \rtimes C_r \). Since \( N \) has a subgroup of isomorphic to \( D_n \), \( r \) must be even. This eliminates the possibilities (a), (c), and (d).

We know \( \text{GL}(4,2) = \text{PSL}(4,2) \cong A_8 \) [22, II, Satz 6.14, (5)]. Since this group does not have a subgroup isomorphic to \( D_{15} \), \( r \neq 4 \) if \( p = 2 \) and \( n = 15 \). This completes the proof.

**Remark.** Regrettably the same argument does not work for \( h(K) = 5 \): The Singer cycles of \( \text{GL}(4,2) \cong A_8 \) have a subgroup isomorphic to \( D_5 \). However, the class number relation (Lemma 3) enables us to compute \( h(K_1) \) by calculating class numbers of quintic subfields of \( K_1 \). For all \( K \) with \( h(K) = 5 \) (all such \( K \) have discriminant \( \leq 2683 \) [2, 52]), we have \( h(K_1) = 1 \) and therefore \( K_2 = K_1 \). We note that we can find in [20] the class numbers of the unramified cyclic quintic extensions \( F \) of \( K \) with \( 5 \mid h(K) \) and \( |d| < 1000 \). For all such \( K \), \( h(F) = h(K)/5 \). By Lemma 3 we can check that for all \( K \) with \( 3 \mid h(K) \) and \( |d| < 1000 \) except for \( d \) listed in (a), we have \( h(F) = h(K)/3 \), where \( F \) is the unique unramified cyclic cubic extension of \( K \).

F. Hajir checked the parities of \( h(K_1) \) of \( K \) with \( \text{Cl}(K) \cong C_q \) (\( q \) an odd prime \( \leq 19 \)) and \( |d| < 15000 \) by using elliptic units [16]: For all such \( K \) except for \( d = -283 \) (\( q = 3 \)), \(-331 \) (\( q = 3 \)), \(-643 \) (\( q = 3 \)), and \(-14947 \) (\( q = 17 \)), \( h(K_1) \) is odd (\( 2^8 = 256 \mid h(K_1) \) for \( d = -14947 \)). All \( K \) with \( h(K) = 7 \) have discriminant \( \leq 5923 \) [2, 52] and therefore \( h(K_1) \) is odd for all such \( K \). By this reason in the case where \( (K \text{ is imaginary and}) h(K) = q \) (\( q \) an odd prime \( \geq 7 \)), \( h(K_1) \geq 64 \) can be improved to \( h(K_1) \geq 48^2 = 169 \). (Note that \( \text{GL}(2,13) \) has a subgroup isomorphic to \( D_7 \).) Moreover, in the case where \( h(K) \geq 11 \) and \( h(K) \neq 21,63 \), \( h(K_1) \geq 64 \) can be improved to \( h(K_1) \geq 2^8 = 256 \) (without assuming the imaginarity of \( K \)). (Note that \( \text{GL}(8,2) \) has a subgroup isomorphic to \( D_{17} \).)

Suppose \( \text{Cl}(K) \cong C_n \), where \( n \) is an odd integer prime to \( 15 \). The estimate \( h(K_1) \geq 169 \) under the assumption \( h(K_1) > 1 \) is important. By this, if \( [K_{ur} : K_1] < 168 \), then we can conclude \( K_{ur} = K_1 \) under the assumption that \( K \) does not have an unramified \( A_5 \)-extension which is
normal over \( \mathbb{Q} \). (Though there exist (probably infinitely) many imaginary quadratic number fields with cyclic class group of odd order prime to 15 having such an extension, the maximal discriminant of such fields is \(-2083\). \((h(\mathbb{Q}(\sqrt{-2083}) = 7. \text{ See §8.})\)) For this conclusion, we need only to show that \( K_1 \) does not have an unramified \( A_5 \)-extension. (Note that the second minimal order of nonabelian simple groups is 168.) Suppose that \( K_1 \) has such an extension \( M \). Then the Galois group \( \text{Gal}(M/K) \) is an extension of \( A_n \). Since any such extension is the direct product \( A_5 \times C_n \), \( K \) has an unramified \( A_5 \)-extension and this extension must be normal over \( \mathbb{Q} \), for otherwise its normal closure has degree \( 2 \cdot 60^2 = 7200 \). This contradicts the assumption. Thus, \([K_{ur} : K_1] < 168 \) implies \( K_{ur} = K_1 \). For \( K = \mathbb{Q}(\sqrt{-287}), \mathbb{Q}(\sqrt{-859}), \mathbb{Q}(\sqrt{-967}), \mathbb{Q}(\sqrt{-991}) \), we do not get \([K_{ur} : K_1] < 60 \) but get \([K_{ur} : K_1] < 168 \) by Odlyzko’s (conditional) discriminant bounds and therefore \( K_{ur} = K_1 \) (under GRH).

Now we treat fields \( K \) with even class number. For such \( K \), \( K_g \neq K \). First we calculate \( h(K_g) \). We have two cases:

(i) \( h(K_g) = h(K)/[K_g : K] \).

(ii) \( h(K_g) > h(K)/[K_g : K] \).

The equality \( h(K_g) = h(K)/[K_g : K] \) is equivalent to \((K_g)_1 = K_1 \).

(This holds trivially in the case of odd class number.) If \( K_{ur} = K_1 \), then we have \((K_g)_1 = K_1 \). We expect the converse, that is, if \((K_g)_1 = K_1 \), then \( K_{ur} = K_1 \) holds for fields considered. In fact, we can show this for most \( K \).

We first treat fields with nontrivial cyclic 2-class group. For such \( K \), its discriminant \( d \) is the product of two prime discriminants: \( d = d_1d_2 \) and \( K_g = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \). Since \( \frac{1}{2} h(K) | h(K_g) \) but \( h(K) \nmid h(K_g) \) by Lemma 9, we have \( h(K_g) = h(K)h(\mathbb{Q}(\sqrt{d_1}))h(\mathbb{Q}(\sqrt{d_2}))/2 \) by Lemma 2. (Note that since \( d_1 \) and \( d_2 \) are prime discriminants, both \( h(\mathbb{Q}(\sqrt{d_1})) \) and \( h(\mathbb{Q}(\sqrt{d_2})) \) are odd. More precisely, we have

\[
\text{Cl}(K_g) \cong \text{Cl}(K)^2 \times \text{Cl}(\mathbb{Q}(\sqrt{d_1})) \times \text{Cl}(\mathbb{Q}(\sqrt{d_2})),
\]

where \( \text{Cl}(K)^2 \neq \text{Cl}(K) \times \text{Cl}(K) \) but \( \text{Cl}(K)^2 = \{c^2 \mid c \in \text{Cl}(K)\} \), because the odd part of the class group of a biquadratic bicyclic number field is isomorphic to the direct product of those of its quadratic subfields (see [28]). Hence in this case \( h(K_g) = h(K)/[K_g : K] \) is equivalent to \( h(\mathbb{Q}(\sqrt{d_1})) = h(\mathbb{Q}(\sqrt{d_2})) = 1 \). We take \( d_2 \) as \( d_2 > 0 \). How often does \( h(\mathbb{Q}(\sqrt{d_2})) = 1 \) occur? H. Cohen and H. W. Lenstra, Jr. [10] posed heuristic conjectures (the so-called Cohen-Lenstra heuristics) on the distribution of class groups of algebraic number fields. Their conjectures state that the
probability that a real quadratic number field with prime discriminant has class number one is about 0.75446 and this has been numerically supported by A. G. Stephens and H. C. Williams [48]. The first two discriminants of real quadratic number fields with prime discriminant and class number > 1 are 229 and 257. These are also discriminants of quartic number fields! The Cohen-Lenstra heuristics state also that the probability that an imaginary quadratic number field has class group whose odd part is cyclic is about 0.97757. Thus, for simplicity, we consider here fields with $h(Q(\sqrt{d})) = 1$ and $Cl(Q(\sqrt{d})) \cong C_n$ (n an odd integer). Then $h(K_g) = h(K)/[K_g : K]$ is equivalent to $n = 1$. When $n = 1$, under some condition by considering Galois action as in Proposition 3, we get an estimate for $h(K_1)$. When $n > 1$, we can easily determine the structure of $Gal((K_g)_1/K)$. We first give an estimate for $h(K_1)$ if $n = 1$ and then determine $Gal((K_g)_1/K)$ if $n > 1$.

**Proposition 4.** Let $K$ be an imaginary quadratic number field with non-trivial cyclic 2-class group.

(i) Assume $Cl(K) \cong C_{2m}$ ($m \geq 3$) and that the unique unramified cyclic quartic extension $E$ of $K$ has class number $2^{m-2}$.

The latter assumption implies $h(K_g) = h(K)/2$. If $h(K_1) > 1$, then we have $h(K_1) \geq 7^2 = 49$. (The latter assumption implies $h(K_g) = h(K)/2$.) If $h(K_1) > 1$, then we have $h(K_1) \geq 2^6 = 64$. (These are valid also for real quadratic number fields if $K_g$ is the unique unramified quartic extension of $K$.)

Proof. We give only a sketch.

(i) Put $\Gamma = Gal(K_1/Q)$. Then $\Gamma \cong D_{2m}$. By Lemmas 7 and 8 any group homomorphisms $\Gamma \to GL(2,3)$ and $\Gamma \to GL(2,5)$ have image of order at most eight and this implies that neither $Cl(K_1) \cong C_2^2$ nor $Cl(K_g) \cong C_2^2$ can occur by the assumption. (Cf. the proof of Proposition 2. Note that the Sylow 2-subgroups of $GL(2,7)$ are isomorphic to $SD_{32}$ and they have a subgroup isomorphic to $D_8$.) Hence we get the conclusion.

(ii) Put $F = Q(\sqrt{d_1})$. Then $Gal(K_1/F) \cong D_q$. Thus, the same argument as in the proof of Proposition 3 works.

Remark. Note that when $K$ has an unramified cyclic quartic extension $E$, we can calculate $h(E)$ by using Lemma 4. The fields with $Cl(K) \cong C_{2m}$ ($m \geq 2$) and $|d| < 1000$ are divided into two types: fields $K$ with $h(E) = 2^{m-2}$ and fields $K$ with $h(K_g) = 2^{m-1}q$ ($q$ an odd prime). For most of the fields of the former type, we can easily check $K_{ur} = K_1$ by using the proposition above. For the fields of the latter type, we can check $K_{ur} = (K_g)_1$. Then we consider the Galois group.
Also when $K$ has an unramified cyclic octic extension $F$, we can calculate $h(F)$ by using Lemma 2, however we must calculate the factorization of a rational prime in its nonnormal quartic subfield and find a suitable generator of a prime factor. (For the construction of $F$, see [54].) For example, let $K = \mathbb{Q}(\sqrt{-904})$. Then $\text{Cl}(K) \cong C_8$. First we calculate the class number of the unique unramified cyclic quartic extension $E$ of $K$. Its nonisomorphic nonnormal subfields are $\mathbb{Q}(\sqrt{9 + 4\sqrt{-2}})$ and $\mathbb{Q}(\sqrt{9 + \sqrt{113}})/2)$. Both of them have class number one and therefore $h(E) = 2$. Put $\alpha = \sqrt{9 + 4\sqrt{-2}}$ and $N = \mathbb{Q}(\alpha)$. Then $K_1$ is a $V_4$-extension of $N$. To obtain $h(K_1)$, we calculate the class number of an intermediate field ($\neq E$) of $K_1/N$. By using KANT we get the factorization of the rational prime 113 in the ring of integers of $N$ and calculate a generator of each prime factor. Then as such an intermediate field we can take $N(\sqrt{\beta})$, where $\beta = 104 + 48\sqrt{-2} - (235 + 108\sqrt{-2})\alpha$. KANT gives $h(N(\sqrt{\beta})) = 1$ and therefore $h(K_1) = 1$ by Lemma 2.

**Proposition 5.** Let $K$ be an imaginary quadratic number field with cyclic class group of order $2^m$ ($m \geq 2$). Assume that the class group of its genus field $K_g$ is cyclic of order $2^{m-1}n$, where $n$ is an odd integer $\geq 3$. Then the Galois group of the Hilbert class field $L$ of $K_g$ over $K$ is isomorphic to $I_n^2$:

$$\text{Gal}(L/K) \cong I_n^2.$$ 

**Proof.** In this case, $L$ is a cyclic extension of $K_1$ of degree $n$ and therefore $\text{Gal}(L/K)$ is isomorphic to a semi-direct product of $C_n$ by $C_2^m$. Thus, it suffices to show that for each prime factor $p$ of $n$, the action of $\text{Gal}(K_1/K) \cong C_2^m$ on the Sylow $p$-subgroup of $\text{Gal}(L/K_1)$, which is isomorphic to $\text{Gal}(K_1 F/K_1) \cong C_q$, is given by inversion, that is, $\text{Gal}(K_1 F/K) \cong I_q^{2m}$, where $F$ is the Hilbert $p$-class field of $K_g$ and $q = [F : K_g]$. Since $\text{Gal}(K_1 F/K) \cong C_q \rtimes C_2^m$, we can present it as

$$\langle a, b \mid a^{2^m} = b^q = 1, \ a^{-1}ba = b^i \rangle$$

for some $i$ with $q \nmid i$. By the assumption $\text{Cl}(K_g) \cong C_2^{m-1}$, $a^2$ must commute with $b$ and therefore $i = \pm 1$. Since $\text{Gal}(K_1 F/K)$ is nonabelian, $i = -1$, that is, $\text{Gal}(K_1 F/K) \cong I_q^{2m}$.

**Remark.** All $K$ with $\text{Cl}(K) \cong C_4$ satisfy $h(K_1) = 1$ or $h(K_g) = 2q$ ($q$ an odd prime). (S. Arno [1] verified that the known list of imaginary quadratic number fields with class number four is complete. Therefore we can easily check this.) Note that $I_q^4 \cong Q_{4q}$.

Now we consider fields $K = \mathbb{Q}(\sqrt{d_1d_2})$ with $2 \mid h(K)$, $h(\mathbb{Q}(\sqrt{d_2})) = 1$ and $\text{Cl}(F) \cong C_n$ ($n$ an odd integer $\geq 3$), where $F = \mathbb{Q}(\sqrt{d_1})$. For such $K$,
Cl(K_2) \cong Cl_{\text{odd}}(K) \times C_n$, where $Cl_{\text{odd}}(K)$ is the odd part of $Cl(K)$ and $Gal(KF_1/K) \cong Gal(F_1/Q) \cong D_n$. Hence $Gal((K_2)_1/K) \cong Gal(KF_1/K) \times Cl_{\text{odd}}(K) \cong D_n \times Cl_{\text{odd}}(K)$. Thus, groups of this form occur as $G$.

For the other fields, the 2-rank of the 2-class group is larger than or equal to 2, and the length of the 2-class field tower often becomes two. Therefore an important problem is to determine the Galois group $Gal(K_2^{(2)} / K)$. For this purpose we do not have sufficient knowledge of the 2-class field tower. In fact, we cannot get the structure of $Gal(K_2^{(2)} / K)$ for $K = Q(\sqrt{-660})$ and $K = Q(\sqrt{-840})$ without additional consideration. For some fields many computer calculations of class numbers are needed. In the rest of this section, we will describe details for selected values of $d$. For simplicity, we denote by $B(2N)$ the lower bound for the root discriminants of the totally imaginary number fields of (finite) degrees $\geq 2N$ instead of $B(2N, 0, N)$.

$d = -420$. J. Martinet writes in [34] that R. Schoof communicated to him an unconditional proof for $[K_{ur} : Q] = 64$, but that proof is not given there. The author was informed by Schoof with a complete proof that there had been a gap in the proof he had communicated to Martinet and that this gap was filled by F. Lemmermeyer a few years ago [43]. Recently in [31] he gives an unconditional proof for $[K_{ur} : Q] = 64$. He also determines the structure of $Gal(K_2^{(2)} / K)$ for some fields $K$ with $Cl(K) \cong C_2^3$ from which we get $Gal(K_{ur}/K) \cong \Gamma_1 \cong 32\Gamma_4 c_3$ for $K = Q(\sqrt{-420})$, where $\Gamma_n$ is the designation used in [31] denoting the group of order $2^{n+4}$ given by

$$\langle \rho, \sigma, \tau \mid \rho^4 = \sigma^{2^{n+1}} = \tau^4 = 1, \rho^2 = \sigma^2 \tau^2, \quad [\sigma, \tau] = 1, \quad [\rho, \sigma] = \sigma^2, \quad [\rho, \tau] = \rho^2 \rangle.$$

We can give another proof for $Gal(K_{ur}/K) \cong 32\Gamma_4 c_3$, but we do not give it here because the situation is similar to the case $K = Q(\sqrt{-840})$.

$d = -660$. F. Lemmermeyer told the author the structure of $Gal(K_{ur}/K)$ and sketch of his proof [32]. We give here another more computational proof.

We have $Cl(K) \cong C_2^3$, $K_1 = Q(\sqrt{-1}, \sqrt{-3}, \sqrt{-11}, \sqrt{5})$, and $h(K_1) = 8$. We will show $K_{ur} = K_2$. Since $rd_K = \sqrt{660} = 25.6904 \ldots$ and $rd_K < B(2 \cdot 8 \cdot 8 \cdot 4)$ ([38]), we have $[K_{ur} : K_2] \leq 3$. By Proposition 2, $h(K_2) \neq 3$. Before proving $h(K_2) \neq 2$, we determine the structure of $Cl(K_1)$ and $Gal(K_2/K)$.

We first show that $Cl(K_1) \cong C_4 \times C_2$. We have $Cl(Q(\sqrt{3}, \sqrt{-55})) \cong C_8 \times C_2$. Therefore $Cl(K_1)$ contains an element of order at least four. Hence the 2-rank of $Cl(K_1)$ is at most two, that is, $Cl(K_1) \cong C_8$ or $C_4 \times C_2$. Assume $Cl(K_1) \cong C_8$. Put $G = Gal(K_2/K)$. Then $G$ is a group of order
Therefore $G$ belongs to the family $\Gamma_8$ or $\Gamma_{19}$. We note that any quadratic subextensions of $K_1/K$ has class number eight or sixteen. In fact, we have

$$\text{Cl}(\mathbb{Q}(\sqrt{-1}, \sqrt{165})) \cong \text{Cl}(\mathbb{Q}(\sqrt{-3}, \sqrt{55}))$$

$$\cong \text{Cl}(\mathbb{Q}(\sqrt{-5}, \sqrt{33})) \cong \text{Cl}(\mathbb{Q}(\sqrt{11}, \sqrt{-15})) \cong C_4 \times C_2,$$

$$\text{Cl}(\mathbb{Q}(\sqrt{5}, \sqrt{-33})) \cong C_4 \times V_4, \quad \text{Cl}(\mathbb{Q}(\sqrt{-11}, \sqrt{15})) \cong C_2^3,$$

$$\text{Cl}(\mathbb{Q}(\sqrt{3}, \sqrt{-55})) \cong C_8 \times C_2.$$

Therefore $G$ does not have an abelian subgroup of index two. Hence $G$ must be isomorphic to $64 \Gamma_{19}a_1$ or $64 \Gamma_{19}a_2$. Now we put $H_1 = \text{Gal}(K_2/\mathbb{Q}(\sqrt{3}, \sqrt{-55}))$. Then $H_1$ is a group of order 32 such that

$$H_1/H'_1 \cong C_8 \times C_2, \quad H'_1 \cong C_2.$$

Therefore $H_1$ is isomorphic to $32 \Gamma_{2j_1}, 32 \Gamma_{2j_2}$, or $32 \Gamma_{2k}$. However, neither $64 \Gamma_{19}a_1$ nor $64 \Gamma_{19}a_2$ has a subgroup isomorphic to any one of these groups. This is a contradiction. Hence $\text{Cl}(K_1) \cong C_4 \times C_2$.

Next, we determine $G = \text{Gal}(K_2/K)$. $G$ is a group of order 64 such that

$$G/G' \cong C^3_2, \quad G' \cong C_4 \times C_2.$$

Therefore $G$ belongs to the family $\Gamma_{14}, \Gamma_{15}, \Gamma_{16}$, or $\Gamma_{24}$. Since $G$ has a subgroup isomorphic to $32 \Gamma_{2j_1}, 32 \Gamma_{2j_2},$ or $32 \Gamma_{2k},$ $G$ belongs to the family $\Gamma_{15}$.

In order to know subgroups of $G$ of order 32, we determine three such groups $H_1$, $H_2 = \text{Gal}(K_2/\mathbb{Q}(\sqrt{5}, \sqrt{-33}))$, and $H_3 = \text{Gal}(K_2/\mathbb{Q}(\sqrt{-11}, \sqrt{15}))$.

First, we determine $H_1$ exactly. For this, we determine the class groups of the three intermediate fields of $K_1 = \mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{5}, \sqrt{-11})/\mathbb{Q}(\sqrt{3}, \sqrt{-55})$, that is, of $\mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{-55})$, $\mathbb{Q}(\sqrt{3}, \sqrt{-5}, \sqrt{11})$, and $\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{-11})$. We can execute this by computer. We have

$$\text{Cl}(\mathbb{Q}(\sqrt{-1}, \sqrt{-3}, \sqrt{-55})) \cong \text{Cl}(\mathbb{Q}(\sqrt{3}, \sqrt{-5}, \sqrt{11})) \cong C_8 \times C_2,$$

and

$$\text{Cl}(\mathbb{Q}(\sqrt{3}, \sqrt{5}, \sqrt{-11})) \cong C_4 \times V_4.$$
Thus, $H_1$ contains two subgroups isomorphic to $C_8 \times C_2$ and one subgroup isomorphic to $C_4 \times V_4$. Hence $H_1 \cong 32\Gamma_2 f_1$.

Next, we determine $H_2$. For this, we calculate the class groups of some quadratic subextensions of $K_1/Q(\sqrt{5}, \sqrt{-33})$, that is, of $Q(\sqrt{3 \pm 2\sqrt{5}}, \sqrt{-33})$. (Note that the Hilbert class field of $Q(\sqrt{-55})$ is $Q(\sqrt{-11}, \sqrt{3 \pm 2\sqrt{5}}) = Q(\sqrt{-11}, \sqrt{3 - 2\sqrt{5}}).$) By computer calculation we have

$$\text{Cl}(Q(\sqrt{3 + 2\sqrt{5}}, \sqrt{-33})) \cong \text{Cl}(Q(\sqrt{3 - 2\sqrt{5}}, \sqrt{-33})) \cong C_4^2.$$ 

Therefore $H_2$ contains two subgroups isomorphic to $C_4^2$ and one subgroup isomorphic to $C_4 \times V_4$. Since $H_2$ is a group of order 32 such that

$$H_2/H_2' \cong C_4 \times V_4, \quad H_2' \cong C_2,$$

we have $H_2 \cong 32\Gamma_2 f_2$.

Thus, $G$ contains a subgroup isomorphic to $32\Gamma_2 f_1$ and one isomorphic to $32\Gamma_2 f_2$, and therefore $G \cong 64\Gamma_{15} f_1$ or $64\Gamma_{15} f_2$.

Now, we determine $H_3$. $H_3$ is a group of order 32 such that

$$H_3/H_3' \cong C_2^3, \quad |H_3| = 4.$$ 

Since $H_3$ is a subgroup of $64\Gamma_{15} f_1$ or $64\Gamma_{15} f_2$, $H_3 \cong 32\Gamma_4 b_1$ or $32\Gamma_4 b_2$, according as $G \cong 64\Gamma_{15} f_1$ or $64\Gamma_{15} f_2$. To judge the structure of $H_3$, we consider its maximal subgroups. Among the seven maximal subgroups of $32\Gamma_4 b_1$ (resp. $32\Gamma_4 b_2$), three are (isomorphic to) $16\Gamma_2 a_1$, two are $16\Gamma_2 c_1$, and one is $16\Gamma_2 c_1$ (resp. one is $16\Gamma_2 a_2$, two are $16\Gamma_2 c_1$, and three are $16\Gamma_2 c_2$). They correspond to the quadratic subextensions of $K_2/Q(\sqrt{-11}, \sqrt{15})$, that is, $Q(\sqrt{3}, \sqrt{5}, \sqrt{-11}), Q(\sqrt{-1}, \sqrt{-11}, \sqrt{-15}), Q(\sqrt{-3}, \sqrt{-5}, \sqrt{-11}), Q(\sqrt{3 \pm \sqrt{-11}}/2, \sqrt{15}), Q(\sqrt{-3 \pm \sqrt{-11}}/2, \sqrt{15}).$ (Note that the Hilbert class field of $Q(\sqrt{-55})$ can be expressed as $Q(\sqrt{5}, \sqrt{(3 \pm \sqrt{-11})/2}) = Q(\sqrt{5}, \sqrt{-(3 \pm \sqrt{-11})/2}).$) By computer calculation we have

$$\text{Cl}(Q(\sqrt{(3 \pm \sqrt{-11})/2}, \sqrt{15})) \cong \text{Cl}(Q(\sqrt{-(3 \pm \sqrt{-11})/2}, \sqrt{15})) \cong C_4 \times C_2.$$ 

Noting that both of the abelianizations of $16\Gamma_2 a_1$ and $16\Gamma_2 a_2$ (resp. $16\Gamma_2 c_1$ and $16\Gamma_2 c_2$) are isomorphic to $C_2^3$ (resp. $C_4 \times C_2$), we conclude that $H_3$ contains at least four subgroups isomorphic to $16\Gamma_2 c_1$ or $16\Gamma_2 c_2$. Therefore $H_3 \cong 32\Gamma_4 b_2$. Hence $G \cong 64\Gamma_{15} f_2$. 
Now we prove $h(K_2) \neq 2$. Assume $h(K_2) = 2$. If we denote by $\Gamma$ the Galois group $\text{Gal}(K_3/Q(\sqrt{-1}, \sqrt{-3}, \sqrt{-55}))$. Then $\Gamma$ is a group of order 32 such that

$$\Gamma/\Gamma' \cong C_8 \times C_2, \quad \Gamma' \cong C_2.$$ 

Therefore $\Gamma$ is isomorphic to $32\Gamma_2j_1, 32\Gamma_2j_2$, or $32\Gamma_2k$. All of these groups have only abelian maximal subgroups. This contradicts that the maximal subgroup $\text{Gal}(K_3/K_1)$ of $\Gamma$ is nonabelian. Hence $h(K_2) \neq 2$. Thus, $K_\text{ur} = K_2$.

$d = -759$. We have $\text{Cl}(K) \cong C_{12} \times C_2$, $K_g = Q(\sqrt{-3}, \sqrt{-11}, \sqrt{-23})$, and $h(K_g) = 18$. Put $L = (K_g)_1$. We will show $K_\text{ur} = L$. Since $rd_K = \sqrt{759} = 27.5499 \ldots$ and $rd_K < B(2 \cdot 4 \cdot 18 \cdot 5) ([38])$, we have $[K_\text{ur} : L] \leq 4$. We will show $h(L) = 1$.

First we show $h(L) \neq 3$. Put $F = Q(\sqrt{-23})$. Then $h(F) = 3$ and $L = K_1F_1$. Since $K_1^{(2)}/KF$ is cyclic (quartic), $K_1/K_1^{(3)}F$ is cyclic and therefore $L/K_1^{(3)}F$ is cyclic. Hence if the class number of the intermediate field $K_gK_1^{(3)}F_1$ of $L/K_1^{(3)}F_1$ is not divisible by 3, then the 3-rank of $\text{Cl}(L)$ is a multiple of 2 by Lemma 5 and then we can conclude $h(L) \neq 3$. In order to show $3 \nmid h(K_gK_1^{(3)}F_1)$, it suffices to show $3 \nmid h(K_gK_1^{(3)})$. So we calculate $h(K_gK_1^{(3)})$. For this we calculate the class numbers of the intermediate fields $K_1^{(3)}F, K_1^{(3)}(\sqrt{-3}), K_1^{(3)}(\sqrt{-11})$ of the $V_4$-extension $K_gK_1^{(3)}/K_1^{(3)}$. $K_1^{(3)}F/F$ is a $D_3$-extension and its quadratic subextension is $KF = Q(\sqrt{-23}, \sqrt{33})$. We have $h(KF) = 36$. We calculate the class number of a cubic subextension $F(\alpha_{94})$ of $K_1^{(3)}F/F$, where $\alpha_{94}^3 - \alpha_{94} + 6\alpha_{94} - 3 = 0$. Put $\lambda = \alpha_{94} + \sqrt{-23}$. Then $F(\alpha_{94}) = Q(\lambda)$ and $\lambda^8 - 2\lambda^5 + 82\lambda^4 - 110\lambda^3 + 1676\lambda^2 - 956\lambda^2 + 7047 = 0$. The function ‘polred’ of pari-gp gives a simpler generating polynomial: $Q(\lambda) = Q(\theta)$, where $\theta^6 - 3\theta^5 + \theta^4 + 3\theta^3 - 11\theta^2 + 9\theta + 27 = 0$. Then KANT gives $h(Q(\theta)) = 3$. Hence by Lemma 3 we get $h(K_1^{(3)}F) = 12$ and therefore $\text{Cl}(K_1^{(3)}F) \cong C_{12}$. Similarly we get $h(K_1^{(3)}(\sqrt{-3})) = h(K_1^{(3)}(\sqrt{-11})) = 4$. Thus, by Lemma 4 we conclude that $h(K_gK_1^{(3)})$ is 3 up to a 2-power. Hence $h(L) \neq 3$.

Next, we show $2 \nmid h(L)$. Assume that $2 \nmid h(L)$. Since $\text{Cl}(K_1^{(3)}F) \cong C_{12}$ and $K_1/K_1^{(3)}F$ is cyclic quartic, by Lemma 9 we have $2 \nmid h(K_1)$. Therefore since $L/K_1$ is cyclic cubic, $\text{Cl}(L) \cong V_4$ by Lemma 6. Now we consider the action of $\text{Gal}(L/K) \cong \text{Gal}(KF_1/K) \times C_{12} \cong D_3 \times C_{12}$ on $\text{Cl}(L)$. This induces a group homomorphism

$$\rho : D_3 \times C_{12} \to \text{Aut}(\text{Cl}(L)) \cong \text{Aut}(V_4) \cong D_3.$$
Obviously $3 \mid |\ker(\rho)|$ and therefore $\mathrm{Gal}(L/K_1)$ or $\mathrm{Gal}(L/K_1^{(2)})$ acts trivially on $\mathrm{Cl}(L)$. Hence $4 \mid h(K_1)$ or $4 \mid h(K_1^{(2)})$. We already see $2 \nmid h(K_1)$. Thus we calculate $h(K_{F_1})$. $K_{F_1}/K$ is a $D_3$-extension and $K(\alpha_1)$ is its cubic subextension. As above, we get $h(K(\alpha_1)) = 24$ and $h(K_{F_1}) = 12$ by Lemma 3. Since $K_1^{(2)}F_1/K_{F_1}$ is cyclic quartic, $2 \nmid h(K_1^{(2)})$ by Lemma 9. This is a contradiction. Hence $h(L) = 1$. Thus, $K_{ur} = L$.

$d = -840$. We have $\mathrm{Cl}(K) \cong C_2^3$, $K_1 = \mathbb{Q}(\sqrt{-2}, \sqrt{-3}, \sqrt{5}, \sqrt{-7})$, and $h(K_1) = 4$. We will show $K_{ur} = K_2$. Since $rd_{K} = \sqrt{840} = 28.9827\ldots$ and $rd_{K} < B(2 \cdot 8 \cdot 4 \cdot 11) ([38])$, we have $[K_{ur} : K_2] < 16$. We first show that $h(K_2)$ is odd. Since $\mathrm{Cl}(\mathbb{Q}(\sqrt{5}, \sqrt{-42})) \cong C_2^2$, we have $\mathrm{Cl}(K_1) \cong V_4$. Hence by Lemma 9, the 2-class group of $K_2$ is cyclic. Assume that $h(K_2)$ is even. Let $M$ be the unique quadratic subextension of $K_3/K_2$. Then obviously $M$ is normal over $K$. We put $\Lambda = \mathrm{Gal}(M/K)$. Then $\Lambda$ is a group of order 64 such that $\Lambda / \Lambda' \cong C_2^3$, $\Lambda' / \Lambda'' \cong V_4$, $\Lambda'' / \Lambda''' \cong C_2$.

However, any group of order 64 has abelian derived group (see [17]). This is a contradiction. Thus, $h(K_2)$ is odd and therefore $\mathrm{Cl}(K_2)$ is trivial or $\mathrm{Cl}(K_2) \cong C_2^2$ by Proposition 2. (Note that $K_g = K_1$ in this case.)

Assume $\mathrm{Cl}(K_2) \cong C_2^2$. Put $\Gamma = \mathrm{Gal}(K_2/Q)$. The action of $\Gamma$ on $\mathrm{Cl}(K_2)$ induces a group homomorphism

$$\rho : \Gamma \rightarrow \mathrm{Aut}(\mathrm{Cl}(K_2)) \cong \mathrm{Aut}(C_2^2) \cong GL(2, 3).$$

By Lemma 8 the image of $\rho$ is isomorphic to $D_4$, because if $|\mathrm{Im}(\rho)| \leq 4$, then $\rho$ has an abelian image and the same argument as in the proof of Proposition 2 shows that $h(K_1)$ must be divisible by 9. Let $F$ be the field corresponding to $\ker(\rho)$. Then $F$ is normal over $Q$ and $\mathrm{Gal}(F/Q) \cong D_4$. Let $E$ be the quadratic subfield of $F$ such that $F/E$ is cyclic and $E'$ the intermediate field of $F/E$. If $E'/E$ is ramified, then $F/E$ is totally ramified. Therefore $E''/E$ must be unramified because all the inertia groups of primes of $F$ have order at most two. We take the quadratic subfield $Q(\sqrt{d'})$ of $K_1$ such that the primes ramified in this field coincide with those ramified in $F$. Then $F(\sqrt{d'})/Q(\sqrt{d'})$ is unramified at all finite primes. Thus, if $Q(\sqrt{d'})$ is contained in $F$, then $F/E'$ is unramified at all finite primes, but this is impossible, because any quadratic subfield of $K_1$ has narrow class group of exponent at most two. Hence $F(\sqrt{d'})$ is a $D_4 \times C_2$-extension of $Q$. Then $F(\sqrt{d'})/E'$ is a $V_4$-extension. Let $F'$ be the third quadratic subextension of it. Then $F'$ is a $D_4$-extension of $Q$. There exists a prime $p$ that is ramified in $F/Q$ but not ramified in $E'/Q$. Then the prime divisors of $p$
in $E'$ must be ramified in both $F/E'$ and $F'/E'$. Since $F(\sqrt{d'})/E'(\sqrt{d'})$ is unramified, these divisors are ramified also in $E'(\sqrt{d'})/E'$. Thus, the inertia group of any prime divisor of $p$ in $F(\sqrt{d'})$ is isomorphic to $V_4$. This is a contradiction. Hence $\text{Cl}(K_2)$ is trivial and $K_{ur} = K_2$.

Now we determine $G = \text{Gal}(K_2/K)$. $G$ is a group of order 32 such that

$$G/G' \cong C_2^3, \ G' \cong V_4.$$ 

There are 9 (nonisomorphic) such groups, which belong to the family $\Gamma_4$ (see [17]). Among them, the groups having an abelian subgroup isomorphic to $C_2^2$ (note that $\text{Gal}(K_2/Q(\sqrt{5}, \sqrt{-42})) \cong C_2^2$) are $32\Gamma_4a_2, 32\Gamma_4a_3, 32\Gamma_4c_2, 32\Gamma_4c_3,$ and $32\Gamma_4d$. Let $N$ be any quadratic subextension of $K_1/K$ other than $Q(\sqrt{5}, \sqrt{-42})$, that is, any of the fields $Q(\sqrt{-2}, \sqrt{105}), Q(\sqrt{-3}, \sqrt{70}), Q(\sqrt{-7}, \sqrt{30}), Q(\sqrt{6}, \sqrt{-35}), Q(\sqrt{-10}, \sqrt{21}),$ and $Q(\sqrt{14}, \sqrt{-15})$, and put $H = \text{Gal}(K_2/N)$. We have $\text{Cl}(N) \cong C_4 \times C_2$. Hence $H$ is a group of order 16 such that

$$H/H' \cong C_4 \times C_2, \ H' \cong C_2.$$ 

Therefore $H$ is isomorphic to $16\Gamma_2c_1, 16\Gamma_2c_2,$ or $16\Gamma_2d$. Thus, $G$ is isomorphic to $32\Gamma_4c_3$, or $32\Gamma_4d$, according as $G$ has a quotient group isomorphic to $16\Gamma_2a_2(\cong Q_8 \times C_2)$, or not. (All these arguments work also for $K = Q(\sqrt{-420})$. We have $h(Q(\sqrt{-420}_1)) = 4$ and $\text{Gal}(K_2/Q(\sqrt{5}, \sqrt{-21})) \cong C_2^2$.)

If $K$ has an unramified $Q_8$-extension, then the composite field of it with $K_1$ is a $Q_8 \times C_2$-extension of $K$. Since $32\Gamma_4c_3$ does not have two normal subgroups with quotient isomorphic to $Q_8 \times C_2$, we need only to judge whether $K$ has an unramified $Q_8$-extension which is normal over $Q$. F. Lemmermeyer gives a criterion for this [30]. By his result $K$ does not have such an extension. Hence $G \cong 32\Gamma_4d$. (Since $Q(\sqrt{-420})$ has an unramified $Q_8$-extension which is normal over $Q$, $\text{Gal}(Q(\sqrt{-420})_{ur}/Q(\sqrt{-420})) \cong 32\Gamma_4c_3$.)

$d = -920$. We have $\text{Cl}(K) \cong C_{10} \times C_2, \ K_g = Q(\sqrt{2}, \sqrt{5}, \sqrt{-23})$, and $h(K_2) = 60$. Put $L = (K_g)_1$. We will show $K_{ur} = L$. Since $rd_K = \sqrt{920} = 30.3315 \ldots$ and $rd_K < B(2 \cdot 4 \cdot 60 \cdot 4)$ ([38]), we have $[K_{ur} : L] < 4$. Since $\text{Cl}^{(2)}(K) \cong V_4$, the 2-class field tower of $K$ terminates with $K_2^{(2)}$ and therefore $2 \nmid h(K_2^{(2)})$. Since $K_1K_2^{(2)}/K_2^{(2)}$ is cyclic quintic, $2 \nmid h(K_1K_2^{(2)})$ by Lemma 5 and since $L/K_1K_2^{(2)}$ is cyclic cubic, $2 \nmid h(L)$ by Lemma 5 again.

Next, we show $3 \nmid h(L)$. Put $F = Q(\sqrt{5}, \sqrt{-46})$. This is one of the intermediate fields of $K_g/K$. We have $\text{Cl}(F) \cong C_{40}$. Hence $K_2^{(2)}/F$ is
cyclic octic and therefore also $K_2^{(2)}(\alpha_1)/F(\alpha_1)$ is cyclic octic. Note that $h(Q(\sqrt{-23})) = 3$ and $Q(\sqrt{-23})_1 = Q(\sqrt{-23}, \alpha_1)$, where $\alpha_1^3 - \alpha_1 - 1 = 0$ and that $(K_g)_1^{(3)} = K_g(\alpha_1)$. Therefore $3 \not| h((K_g)_1^{(3)})$ by Lemma 9 and then $3 \not| h(K_2^{(2)}(\alpha_1))$ by Lemma 5. Since $L/K_2^{(2)}(\alpha_1)$ is cyclic quintic, $3 \not| h(L)$ by Lemma 5. Hence $h(L) = 1$ and therefore $K_{ur} = L = K_2$.

We easily check that $\text{Gal}(K_{ur}/K) \cong \text{Gal}(K_2^{(2)}(\alpha_1)/K) \times \text{Gal}(K_1^{(5)}/K)$. By the result of H. Kisilevsky [25], we have $\text{Gal}(K_2^{(2)}/K) \cong Q_{16}$. Therefore $\text{Gal}(K_{ur}/K) \cong (Q_{16} \times C_3) \times C_5$.

Finally, we note that the maximal unramified extension of $Q(\sqrt{-920})$ has degree 480 and is the field with class number one of the largest degree we know (under GRH). (J. Martinet gave an example of a field with class number one and degree 116 [33]. (This is an unconditional result.))

7. QUADRATIC NUMBER FIELDS HAVING AN UNRAMIFIED EXTENSION NOT CONTAINED IN $(K_g)_1$

We explain here that from some quartic number fields of type $S_4$ we can obtain quadratic number fields whose genus fields have an unramified $A_4$-extension.

First, we review the following general fact.

**Proposition 6.** ([26]. See also [56].) Let $E$ be an algebraic number field of degree $n \geq 3$. If the discriminant $d_E$ of $E$ is a fundamental quadratic discriminant, that is, a discriminant of a quadratic number field, then the normal closure $M$ of $E$ is an $S_n$-extension of $Q$ and is unramified at all finite primes over its quadratic subfield $Q(\sqrt{d_E})$.

From this, any number field $E$ of degree $n \geq 3$ whose discriminant is fundamental quadratic yields a quadratic number field $Q(\sqrt{d_E})$ having an unramified $A_n$-extension. (If $d_E > 0$ and $E$ is not totally real, then the infinite primes are ramified.) Moreover, $E$ yields infinitely many quadratic number fields having an unramified $S_n$-extension. In fact, for any quadratic number field $K \neq Q(\sqrt{d_E})$ with $d_K = d_Ed'$, where $d'$ is a fundamental quadratic discriminant, the composite field $KM$ is an unramified $S_n$-extension of $K$. (The unramifiedness of $KQ(\sqrt{d_E})/K$ follows by genus theory.) In this case, since $KM$ and $K_g$ are linearly disjoint over $KQ(\sqrt{d_E})$, $K_g$ has an unramified $A_n$-extension. (This is the case also when $K = Q(\sqrt{d_E})$.) Therefore, if $n \geq 4$, then $K_gM$ is an unramified extension not contained in $(K_g)_1$. In particular, if $n = 4$, then the (narrow) class field tower of the
Now we describe the field $K = \mathbb{Q}(\sqrt{-687})$. The factorization of $-687$ is $-687 = (-3) \cdot 229$ and 229 is the first prime discriminant of totally imaginary quartic number fields. Let $E$ be a quartic number field with discriminant $229$ (there exists only one such field up to isomorphism [14]) and $M$ its normal closure. Then by the proposition above, $M$ is an $S_4$-extension of $\mathbb{Q}$ which is an $A_4$-extension of the real quadratic number field $F := \mathbb{Q}(\sqrt{229})$ unramified at all finite primes (abbrev. weakly-unramified). We can show that the class number of $M$ is one by the method described in §8 plus some additional consideration. Since by the table in [12], $B(24, 60, 0, 12 \cdot 60) > rd_F = \sqrt{229}$, $M$ has no nontrivial weakly-unramified extension. (This is unconditional.) Hence $M$ is the maximal weakly-unramified extension of $F$. (While $F_{ur} = F_1$, $[F_1 : F] = 3$.) Since $\text{Cl}(K) \cong C_{12}$, the composite field $K_1M$ has degree 288. By the table in [38], $B(576, 0, 288) > rd_K = \sqrt{687}$ under GRH. Therefore $K_{ur} = K_1M$ under GRH. Now we determine the structure of the Galois group $G = \text{Gal}(K_{ur}/K)$. Obviously $K_{1}^{(2)}M$ and $K_{1}^{(3)}$ are linearly disjoint over $K$ and therefore $G$ is isomorphic to the direct product $H = \text{Gal}(K_{1}^{(2)}M/K)$ with $\text{Gal}(K_{1}^{(3)}/K) \cong C_3$. We consider the structure of $H$. Let $C$ be a cubic subextension of $KM/K$. Then $C$ corresponds to a Sylow 2-subgroup of $H$ and $\text{Gal}(KM/E)$ is isomorphic to a Sylow 2-subgroup of $S_4$, that is, isomorphic to $D_4$. Thus, if we put $T = \text{Gal}(K_{1}^{(2)}M/C)$, then we have

$$T/T' \cong C_4 \times C_2, \quad T' \cong C_2.$$ 

Since $\text{Gal}(K_{1}^{(2)}M/K,F_1) \cong C_2^3$, we have $T \cong 16 \Gamma_2c_1 \cong D_4 \rtimes C_4$. Hence $T$ contains a cyclic subgroup $W$ of order four containing $\text{Gal}(K_{1}^{(2)}M/KM)$ with $T \cap W = \{1\}$, and this group and $\text{Gal}(K_{1}^{(2)}M/K_{1}^{(2)})$ generate $H$. Therefore $H \cong A_4 \rtimes C_4$. (We note that $H$ is a nonsplit extension of $\text{Gal}(KM/K) \cong S_4$ by $C_2$. But $H$ is not a double cover of $S_4$.) Thus, $G \cong (A_4 \rtimes C_4) \times C_3$.

Some quartic number fields whose discriminants are fundamental quadratic yield infinitely many quadratic number fields with class field tower of length at least four.

**Proposition 7.** (See [45]) Let $E$ be a quartic number field with $d_E = -p$ $(p \equiv 3 \pmod{4}$ a prime$)$. Then $E$ is embedded into an octic number field $\tilde{E}$ whose normal closure is an $\tilde{S}_4$-extension of $\mathbb{Q}$ and is unramified over its quadratic subfield $\mathbb{Q}(\sqrt{-p})$, where $\tilde{S}_4$ is the double cover of $S_4$ which is
characterized by the property that a transposition in $S_4$ lifts to an element of order two, while a product of two disjoint transpositions lifts to an element of order four: $\bar{S}_4 \cong \text{GL}(2, 3)$.

From this, from a quartic number field $E$ with $d_E = -p \equiv 1 \pmod{4}$, we get an unramified $A_4$-extension of the imaginary quadratic number field $Q(\sqrt{-p})$. We note that from the data for quartic number fields of signature $(2, 1)$ [13], the quartic prime discriminants $-p \equiv 1 \pmod{4}$ with $p \leq 2003$ are $-283, -331, -491, -563, -643, -751, -1399, -1423, -1823, -1879$, and $-1931$. All such fields give two dimensional complex projective linear representation

$$\rho : \text{Gal}(\overline{Q}/Q) \to \text{PGL}(2, \mathbb{C})$$

of the absolute Galois group $\text{Gal}(\overline{Q}/Q)$ of $Q$ of type $S_4$ which has a lifting

$$\tilde{\rho} : \text{Gal}(\overline{Q}/Q) \to \text{GL}(2, \mathbb{C})$$

with conductor $p$ and odd determinant (see [45, §8]). (A. Jehanne [23] has studied the general embedding problem $\tilde{S}_4 \to S_4$, especially for the case where the base field is $Q$. We find in [23] a generating polynomial of $\tilde{E}$ for $p = 283, 331, 491, 563, 643, 751$. Field $E$ in Proposition 7 yields infinitely many quadratic number fields with class field tower of length at least four. Let $\tilde{M}$ be the normal closure of $E$. Then $\tilde{M}$ is an $\tilde{S}_4$-extension of $Q$. For any quadratic number field $K$ such that $d_K/d_E$ is fundamental quadratic, the composite field $K\tilde{M}$ is an unramified $\tilde{S}_4$-extension of $K$. Then $K$ has (narrow) class field tower of length at least four. For example, let $K = Q(\sqrt{-5 \cdot 283})$. Since $\text{Cl}(K) \cong C_{34} \cong C_2 \times C_{17}$, $\text{Gal}(K_1 Q(\sqrt{-283})_{ur}/K) \cong S_4 \times C_{17}$ and therefore $[K_1 Q(\sqrt{-283})_{ur} : Q] = 1632$. Even under GRH, we get only $[K_{ur} : K_1 Q(\sqrt{-283})_{ur}] < 44$ from Odlyzko’s bound, however, it is thought likely that $K_{ur} = K_1 Q(\sqrt{-283})_{ur}$ (this implies $K_{ur} = K_4 = K_1 Q(\sqrt{-283})_{ur}$), but no proof has yet been obtained. (We can prove $K_3 = K_1 Q(\sqrt{-283})_{ur}$.)

There exist many quartic number fields of type $S_4$ whose discriminants are not fundamental quadratic which yield infinitely many quadratic number fields having an unramified $S_4$-extension.

We will not consider here general such fields. We describe only such quartic number fields giving unramified $S_4$-extensions of $Q(\sqrt{-856})$ and $Q(\sqrt{-996})$.

Let $E$ be a quartic number field defined by the (irreducible) polynomial $f(X) = X^4 - 2X^3 + 5X^2 - 2X - 1$ and $M$ its normal closure. We easily
see \( \text{Gal}(M/Q) \cong S_4 \) and \( d_E = -6848 = -2^6 \cdot 107 \). Let \( K = \mathbb{Q}(\sqrt{-856}) \). Note that \(-856 = 8 \cdot (-107)\). We see that \( KM/K \) is an unramified \( S_4 \)-extension. For this, it suffices to show that \( KE/K \) is unramified. We can easily get a generating polynomial of \( KE \). For example, we can easily calculate the minimal polynomial of \( \theta + \sqrt{-214} \), where \( \theta \) is a root of \( f(X) \) with \( E = \mathbb{Q}(\theta) \). From this we get as a generating polynomial of \( KE \), \( X^8 - 2X^7 + 15X^6 + 20X^5 + 25X^4 + 228X^3 + 573X^2 + 170X + 25 \) by using the function ‘polred’ of pari-gp. Then we get \( d_{KE} = 2^{12} \cdot 107^4 = d_E^4 \) by using ‘discf’ of pari-gp. Hence \( KM/K \) is unramified. For any quadratic number field \( F \) such that \( 856 \mid d_F \) and that the localization of \( F \) at the prime divisor of 2 in \( F \) is (isomorphic to) the one of \( K \) at the prime divisor of 2 in \( K \), \( FM/F \) is an \( S_4 \)-extension unramified at all finite primes.

Let \( E' \) be a quartic number field defined by the (irreducible) polynomial \( X^4 - 3X^2 - 2X + 1 \) and \( M' \) its normal closure. We easily see \( \text{Gal}(M'/Q) \cong S_4 \) and \( d_{E'} = -1328 = -2^4 \cdot 83 \). Let \( K' = \mathbb{Q}(\sqrt{-996}) \). Note that \(-996 = (-4) \cdot (-3) \cdot (-83)\). As above we can show that \( K'M'/K' \) is an unramified \( S_4 \)-extension.

We note that the discriminants of the fields \( E \) and \( E' \) above are of the form \( f^2d' \), where \( d' \) is a fundamental discriminant with \( h(\mathbb{Q}(\sqrt{d'})) = 3 \) and \( f = 8 = d_{\mathbb{Q}(\sqrt{3})} \), or \( f = -4 = d_{\mathbb{Q}(\sqrt{-1})} \). We also note that the ramification of the prime 2 in \( M/Q \) (resp. \( M'/Q \)) occurs over the subfield of \( M \) (resp. \( M' \)) which is an \( S_3 \)-extension of \( \mathbb{Q} \).

8. Unramified nonsolvable Galois extensions of imaginary quadratic number fields

For imaginary quadratic number fields \( K \) of small conductors we considered, the maximal unramified extension \( K_{ur} \) is the top of the class field tower of \( K \). Then the following natural question arises: What is the first imaginary quadratic number field having an unramified nonsolvable Galois extension? Recent data for quintic number fields enable us to give a partial answer:

**Proposition 8.** The field \( \mathbb{Q}(\sqrt{-1507}) \) is the first imaginary quadratic number field having an unramified \( A_5 \)-extension which is normal over \( \mathbb{Q} \) in the sense that none of \( \mathbb{Q}(\sqrt{d}) \) of discriminant \( d \) with \( 0 > d > -1507 \) has such an extension. Moreover, such an extension of \( K = \mathbb{Q}(\sqrt{-1507}) \) is given by the composite field of \( K \) with the splitting field of the quintic polynomial \( X^5 - 5X^3 + 5X^2 + 24X + 4 \), which is an \( A_5 \)-extension of \( \mathbb{Q} \).

**Proof.** From the table in [2], the splitting field of the quintic polynomial
$X^5 - 5X^3 + 5X^2 + 24X + 4$ is an $A_5$-extension of $\mathbb{Q}$ and the only two primes 11 and 137 are ramified in this field with ramification index two. Therefore the composite field of $\mathbb{Q}(\sqrt{-1507})$ with it is an unramified $A_5$-extension of $\mathbb{Q}(\sqrt{-1507})$ (note that the factorization of 1507 is 1507 = $11 \cdot 137$), which is an $A_5 \times C_2$-extension of $\mathbb{Q}$. Now we show that none of $\mathbb{Q}(\sqrt{d})$ of discriminant $d$ with $0 > d > -1507$ has such an extension. Let $L$ be an unramified $A_5$-extension of a quadratic number field $F$ which is normal over $\mathbb{Q}$. Then the Galois group $\text{Gal}(L/\mathbb{Q})$ is isomorphic to $S_5$ or $A_5 \times C_2$. If $\text{Gal}(L/\mathbb{Q}) \cong S_5$, then $L$ is the normal closure of its quintic subfield $E$ and the discriminant $d_E$ of $E$ coincides with the one $d_F$ of $F$ (see [26]). Since the maximal discriminant of quintic number fields of signature $(3,1)$ (whose normal closures are $S_5$-extensions of $\mathbb{Q}$) is $-4511$ and the minimal discriminant of quintic number fields of signature $(1,2)$ is $1609$ ([44]), $\mathbb{Q}(\sqrt{-4511})$ is the first imaginary quadratic number field having an unramified $A_5$-extension which is an $S_5$-extension of $\mathbb{Q}$. Now, we assume $\text{Gal}(L/\mathbb{Q}) \cong A_5 \times C_2$ and let $M$ be the unique subfield of $L$ which is an $A_5$-extension of $\mathbb{Q}$. Since $L = MF/F$ is unramified, the primes ramified in $M$ are prime divisors of $d_F$ and their ramification indices are all two. Hence if we let $E$ be a quintic subfield of $M$, we get $d_E^2 | d_F$. From the data for quintic number fields in [44] and the table in [3], $\mathbb{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified $A_5$-extension which is an $A_5 \times C_2$-extension of $\mathbb{Q}$.

Remark. Probably, the condition "which is normal over $\mathbb{Q}$" is unnecessary. It is reasonable to expect this. In fact, if we assume that GRH is true, Odlyzko's discriminant bounds enable us to show that most of $\mathbb{Q}(\sqrt{d})$ with $0 > d > -1507$ do not have an unramified $A_5 \times A_5$-extension. (We can show that under GRH, except for $d = -1387, -1451, -1459, -1480, -1483, -1492$, and $-1499$, none of $\mathbb{Q}(\sqrt{d})$ with $0 > d > -1507$ has any unramified $A_5$-extension.) We expect that in future data for number fields of degree ten will enable us to eliminate the condition above; If an $A_5$-extension $L$ of a quadratic number field $K$ is not normal over $\mathbb{Q}$, its normal closure $\tilde{L}$ has Galois group isomorphic to $A_5 \times A_5$ over $K$. Let $E$ be a quintic subextension of $L/K$. Then $E$ has degree ten. The five conjugate fields of $E$ which are conjugate over $K$ generate $L$ and the other five conjugate fields generate another $A_5$-extension of $K$. Thus $\text{Gal}(\tilde{L}/K) \cong A_5 \times A_5$.

We expect stronger assertion that $\mathbb{Q}(\sqrt{-1507})$ is the first imaginary quadratic number field having an unramified nonsolvable Galois extension. We exhibit here two other examples of an imaginary quadratic number field having an unramified nonsolvable Galois extension. $\mathbb{Q}(\sqrt{-14731})$ has an unramified $A_6$-extension which is an $S_6$-extension of $\mathbb{Q}$. Such an extension

Ken Yamamura
is given as the splitting field of the sextic polynomial $X^6 + 3X^5 + 5X^4 + 4X^3 + 3X^2 + 2X + 1$. $\mathbb{Q}(\sqrt{-30759})$ has an unramified $\text{PSL}(2,7)$-extension which is a $\text{PSL}(2,7) \times C_2$-extension of $\mathbb{Q}$. Such an extension is given as the composite field of $\mathbb{Q}(\sqrt{-30759})$ with the splitting field of the septic polynomial $X^7 + 2X^6 - 3X^4 - X^3 - X^2 - X + 2$. The fact that this splitting field is a $\text{PSL}(2,7)$-extension of $\mathbb{Q}$ was found by K. Yamazaki [61].

As explained in §2, under GRH, if $|d| \leq 2003$, then $[K_\text{ur} : K] < \infty$ for $K = \mathbb{Q}(\sqrt{d}), d = d_K < 0$. In this range, we only know one other field having an unramified nonsolvable Galois extension, namely, the field with $d = -1959 = (-3) \cdot 653$. This is the second imaginary quadratic number field having an unramified $A_5$-extension which is normal over $\mathbb{Q}$. Let $L$ be the splitting field of the quintic polynomial $X^5 + X^3 + 62X^2 + 104X + 169$. Then $L$ is an imaginary $A_5$-extension of $\mathbb{Q}$, in which (except for the infinite prime) only the finite prime 653 is ramified and its ramification index is two ([3]). Thus, for any quadratic number field $K$ whose discriminant is a multiple of 653, the compositum $KL$ is an $A_5$-extension of $K$ unramified at all finite primes. (If $K$ is real, the infinite primes are ramified in $KL/K$.)

We note that $F = \mathbb{Q}(\sqrt{653})$ is the first real quadratic number field having an weakly-unramified $A_5$-extension which is normal over $\mathbb{Q}$. $F$ has narrow class number one. Therefore $F$ has no nontrivial weakly-unramified abelian extension. Under GRH from Odlyzko's bound we get $[F_\text{w-ur} : FL] \leq 3$ and therefore $\text{Gal}(F_\text{w-ur}/F)$ is isomorphic to $A_5$, or a nonsplit extension of $A_5$ by $C_2$ or $C_3$.

The third imaginary quadratic number field having an unramified $A_5$-extension which is normal over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{-2083})$. Moreover, the $A_5$-extension of $K = \mathbb{Q}(\sqrt{-2083})$ is given by the composite field of $K$ with the splitting field of the quintic polynomial $X^5 + 8X^3 + 7X^2 + 172X + 53$, which is an $A_5$-extension of $\mathbb{Q}$.

Now we return to the field $\mathbb{Q}(\sqrt{-1507})$. We have $\text{Cl}(K) \cong C_4$. The Hilbert class field $K_1$ of $K$ is $K(\sqrt{(-23 + 3\sqrt{137})/2})$ and its class number is one. Thus, $K_1$ is an imaginary $D_4$-extension of $\mathbb{Q}$ with class number one having an unramified $A_5$-extension. This field is only example we know of an imaginary normal extension of $\mathbb{Q}$ with class number one having an unramified nonsolvable Galois extension. (On the other hand, there exist many (probably infinitely many) real quadratic number fields with class number one having an unramified $A_n$-extension for each $n \geq 5$ [55, 60].)
APPENDIX 1. CALCULATION OF CLASS NUMBERS OF $S_4$-EXTENSIONS OF $\mathbb{Q}$

Let $M$ be an $S_4$-extension of $\mathbb{Q}$. We consider here how to calculate its class number. If we know the class numbers of a cubic subfield and sextic subfields containing it, we get the class number $h(M)$ of $M$ up to a power of two by using class number relation for $V_4$-extension repeatedly. Let $L$ be the unique normal subfield of $M$ which is an $S_3$-extension of $\mathbb{Q}$. Then $M/L$ is a $V_4$-extension and three intermediate fields are conjugate. Therefore, if we know the class numbers of $L$ and any one of the intermediate fields of $M/L$, then we get $h(M)$ up to a 2-power. Let $E$ be a cubic subfield of $L$. Then $M/E$ is a $D_4$-extension and the $V_4$-subextension $N$ of $M/E$ is an intermediate field of $M/L$. Thus, if the class numbers of three intermediate fields of $N/E$, which are sextic, are known, we get $h(N)$ up to a 2-power.

Now, we give the sextic fields more explicitly by starting with a quartic subfield of $M$. Let $F$ be a quartic subfield of $M$ and $f(X) = X^4 + a_1X^3 + a_2X^2 + a_3X + a_4 \in \mathbb{Z}[X]$ its generating polynomial, that is, $F = \mathbb{Q}(\beta)$, where $\beta$ is a root of $f(X)$. Then the resolvent cubic $f_r(X) = X^3 - a_2X^2 + (a_1a_3 - 4a_4)X - a_4(a_1^2 - 4a_2) - a_3^2$ of $f(X)$ is irreducible over $\mathbb{Q}$ and its roots are $\alpha = \beta_1\beta_2 + \beta_3\beta_4$, $\beta_1\beta_3 + \beta_2\beta_4$, $\beta_1\beta_4 + \beta_2\beta_3$, where $\beta_1, \beta_2, \beta_3$, and $\beta_4$ are the roots of $f(X)$. Let $E = \mathbb{Q}(\alpha)$ and $N$ the $V_4$-subextension of $M/E$. We want to know the three intermediate fields of $N/E$ and their class numbers. One is $L = KE = \mathbb{Q}(\sqrt{d})$, which is the splitting field of $f_r(X)$, where $K$ is the unique quadratic subfield of $M$ and $d$ its discriminant. Other one is $\mathbb{Q}(\beta_1\beta_2) = \mathbb{Q}(\beta_1\beta_2 - \beta_3\beta_4)$. Note that $(\beta_1\beta_2 - \beta_3\beta_4)^2 = (\beta_1\beta_2 + \beta_3\beta_4)^2 - 4\beta_1\beta_2\beta_3\beta_4 = \alpha^2 - 4a_4$ and we can write this field as $E(\sqrt{d(\alpha^2 - 4a_4)})$. Thus, the other one is $E(\sqrt{d(\alpha^2 - 4a_4)}) = \mathbb{Q}(\sqrt{d(\alpha_1^2 - 4a_3)}) = \mathbb{Q}(\sqrt{d(\beta_1\beta_2 - \beta_3\beta_4)})$. The minimal polynomial $g(X)$ of $\beta_1\beta_2$ is

$$X^6 - a_2X^5 + (a_1a_3 - a_4)X^4 + (-a_1^2a_4 + 2a_2a_4 - a_3^2)X^3 + (a_1a_3a_4 - a_4^2)X^2 - a_2a_4^2X + a_4^3$$

and its discriminant is $d_f^2a_4^6(-2a_1^2a_3^2a_4 + a_1^4a_4^2 + a_4^4)$, where $d_f$ is the discriminant of $f(X)$. The minimal polynomial $u(X)$ of $\sqrt{d(\beta_1\beta_2 - \beta_3\beta_4)}$ is

$$X^6 - (a_2^2 - 2a_1a_3 - 4a_4)dX^4 + (-2a_1^2a_2a_4 + a_1^2a_3^2 + 8a_1a_3a_4 - 2a_2a_3^2)d^2X^2 - (-2a_1^2a_3a_4 + a_1^2a_4^2 + a_3^2)d^3$$

and its discriminant is $64d_f^{11}(a_1a_2a_3 - a_1^2a_4^2 - a_3^2)(-2a_1^2a_3a_4 + a_1^2a_4^2 + a_3^2)$. 

We describe ramification. H. J. Godwin [15] proved that the discriminant of $\mathbb{Q}(\beta_1\beta_2)$ equals $d_Ed_F$ (see also [24]). In order that the extension $M/K$ is unramified at all finite primes, $d_F = d$ is necessary and sufficient (see [26]). In that case, we have also $d_E = d$ and therefore $d_{\mathbb{Q}(\beta_1\beta_2)} = d^2 = d_E^2$. This implies that both $N/\mathbb{Q}(\sqrt{d(\alpha^2 - 4a_4)})$ and $\mathbb{Q}(\beta_1\beta_2)/E$ are unramified at all finite primes.

Now we explain how we get $h(K_2)$ for $K = \mathbb{Q}(\sqrt{-643})$. It is a classical result that $K_2$ for $K = \mathbb{Q}(\sqrt{-643})$ is an $S_4$-extension of $\mathbb{Q}$. So, put $M = K_2$ and we use the notations above. Then we can take $f(X) = X^4 + X^3 + 2X + 1$ and $d_f = d = -643$. (From the irreducibility and the squarefreeness of $d_f$, we can deduce the fact that the splitting field of $f(X)$ is an $S_4$-extension of $\mathbb{Q}$ and is unramified over the unique quadratic subfield $K = \mathbb{Q}(\sqrt{-643})$. (See [56].)) Then $f_r(X) = X^3 - 2X - 5$ and we can check $h(E) = 2$ by computer. Since $h(K) = 3$, by using the class number relation in Lemma 3, we get $h(L) = 4$. (More precisely, by Lemma 5, $\text{Cl}(L) \cong V_4$.) Now $g(X) = X^6 + X^4 - 5X^3 + X^2 + 1, u(X) = X^6 - 8\cdot 643X^4 + 20\cdot 643^2X^2 + 9\cdot 643^3$ and KANT gives $h(\mathbb{Q}(\beta_1\beta_2)) = 1$. Hence in this case, the field $\mathbb{Q}(\beta_1\beta_2)$ is the Hilbert class field of $E$ and $N/\mathbb{Q}(\sqrt{d(\alpha^2 - 4a_4)})$ is unramified. The function 'polred' of pari-gp reduces $u(X)$ to $X^6 - 3X^5 + 11X^4 - 9X^3 + 23X^2 + 8X + 57$ and KANT gives $\text{Cl}(\mathbb{Q}(\sqrt{d(\alpha^2 - 4a_4)})) \cong C_8$. We also note that the field $\mathbb{Q}(\sqrt{d(\alpha^2 - 4a_4)})$ is totally imaginary and therefore the narrow class group of it concides with its class group. Then the class number relation for $V_4$-extension shows that $h(N)$ is a power of 2, and since $N$ is the unramified quadratic extension of $\mathbb{Q}(\sqrt{d(\alpha^2 - 4a_4)})$, we get $\text{Cl}(N) \cong C_4$. Using the class number relation for $V_4$-extension again, $h(M)$ is a power of 2, and since $M$ is the unramified quartic extension of $\mathbb{Q}(\sqrt{d(\alpha^2 - 4a_4)})$, we get $h(M) = 2$. By the same method, we get $h(K_2) = 2$ also for $K = \mathbb{Q}(\sqrt{-283})$ and $\mathbb{Q}(\sqrt{-331})$. Moreover, if we take $M$ as the normal closure of the third (resp. fourth, and eighth) quartic number field of type $S_4$, which is an unramified $A_4$-extension of $\mathbb{Q}(\sqrt{-491})$ (resp. $\mathbb{Q}(\sqrt{-563})$, and $\mathbb{Q}(\sqrt{-751})$), we get $h(M) = 6$ (resp. $h(M) = 6$, and $h(M) = 10$).

**APPENDIX 2. IMAGINARY QUADRATIC NUMBER FIELDS $K$ WITH $K_2 = (K_g)_1 \neq K_1$**

We give here a note on the following.

**PROBLEM 1.** Characterize the imaginary quadratic number fields $K$ with

\[ (*) \quad K_2 = (K_g)_1 \neq K_1. \]
Let $K$ be an imaginary quadratic number field. The inequality $(K_g)_1 \neq K_1$ is equivalent to $h(K_g) > h(K)/[K_g : K]$ and we can easily calculate $h(K), [K_g : K],$ and $h(K_g)$. Since $K_g = K$ if and only if $h(K)$ is odd, for $(K_g)_1 \neq K_1$, $h(K)$ must be even and therefore the discriminant $d$ of $K$ must have two distinct prime factors.

On the other hand, it is hard to calculate $h(K_1)$ in general. However, by considering the $p$-class field tower, we can get some necessary condition for $K_2 = (K_g)_1$. Let $p$ be an odd prime. If the class number $h(K^{(p)}_1)$ of the Hilbert $p$-class field $K^{(p)}_1$ of $K$ is divisible by $p$, then we have

$$(K_g)_1 = (K_g)_1 K^{(p)}_1 \subseteq (K_g)_1 K^{(p)}_2 \subseteq K_2,$$

because the odd part of the class group of $K_g$ is the direct product of the odd parts of the class groups of the quadratic subfields of $K_g$. If the $p$-rank of the $p$-class group of $K$ is larger than one for some odd prime $p$, then $h(K^{(p)}_1)$ is divisible by $p$ [37, Theorem 2] and therefore in this case $K_2 \nsubseteq (K_g)_1$. For example, let $K = \mathbb{Q}(-3896)$. Then we have $\text{Cl}(K) \cong C_{12} \times C_3, K_g = \mathbb{Q}(\sqrt{2}, \sqrt{-487})$, and $\text{Cl}(K_g) \cong C_{42} \times C_3$. Therefore $K_2 \nsubseteq K^{(3)}_1(K_g)_1 = (K_g)_1$. Thus, for $K_2 = (K_g)_1$, the $p$-class field tower must terminate with $K^{(p)}_1$ for all odd primes, in other words, the odd part of the class group of $K$ must be cyclic.

Problem 1 seems to be very difficult. Thus, it is reasonable to consider the following before treating Problem 1.

PROBLEM 2. Characterize the imaginary quadratic number fields $K$ with

$$K^{(2)}_2 = (K_g)^{(2)}_1 \neq K^{(2)}_1.$$

Put $\Lambda = \text{Gal}(K^{(2)}_2/K)$. Then $\text{Gal}(K^{(2)}_2/K^{(2)}_1) = \Lambda'$, $\text{Gal}(K^{(2)}_2/K_g) = \Phi(\Lambda)$, and $\text{Gal}(K^{(2)}_2/(K_g)_1) = \Phi(\Lambda)'$. Problem 2 consists in characterizing $K$ such that

$$\Lambda' \neq \Phi(\Lambda)' = (\Lambda'' = \{1\}).$$

If $\text{Cl}^{(2)}(K)$ is elementary abelian, then $K_g = K^{(2)}_1$ and therefore if moreover $h(K_g)$ is divisible by 2, then $(**)$ trivially holds. We can characterize such $K$: $\text{Cl}^{(2)}(K)$ is elementary abelian if and only if $d$ does not have a factorization of the second kind (in the terminology of Rédei-Reichardt). F. Lemmermeyer [29] characterized $K$ for which $\Lambda$ is abelian. By known
results on the 2-class field tower, if the 2-class group of $K^{(2)}_1$ is nontrivial and cyclic, (***) holds. (Refer to §5.) In fact, for $\Lambda = M_0^r$, $M_0^r$, $\Gamma_{m,t}$ or $C_{m,t}$, (***) holds. Therefore the remaining case is where $Cl^{(2)}(K)$ is not elementary abelian and $Cl^{(2)}(K^{(2)}_1)$ is noncyclic. The author knows no example of $K$ with $K^{(2)}_2 \cong (K^{(2)}_1)_2 \ncong K^{(2)}_1$.

Finally, we note that there exist 2-groups $\Lambda$ satisfying $\Lambda' \neq \Phi(\Lambda)' \neq \Lambda'' = \{1\}$. The groups $64\Gamma_{22}a_1, 64\Gamma_{22}a_2, 64\Gamma_{23}a_1, 64\Gamma_{23}a_2, 64\Gamma_{23}a_3$, and $64\Gamma_{23}a_4$ are so. This is the complete list of the nonabelian 2-groups of orders $\leq 64$ having nonabelian Frattini subgroup. (See [17].)

REFERENCES

7. D. A. Buell, Small class number and extreme values of $L$-functions of quadratic fields, Math. Comp. 31 (1977), no. 139, 786–796; MR 55 #12684.
12. F. Diaz y Diaz, Tables minorant la racine $n$-ième du discriminant d'un corps de degré $n$, Publications Mathématiques d'Orsay 80, 6., Université de Paris-Sud, Département de Mathématique, Orsay, 1980; MR 82i:12007.


32. ______, Private communication, 1996.
43. R. Schoof, Private communication, 1996.

52. C. Wagner, *Class number 5, 6 and 7*, Math. Comp. 65 (1996), no. 214, 785–800; MR 96g:11135.


54. Y. Yamamoto, *Divisibility by 16 of class numbers of quadratic fields whose 2-class groups are cyclic*, Osaka J. Math. 21 (1984), no. 1, 1–22; MR 85g:11092.


57. _____, *The determination of the imaginary abelian number fields with class number one*, Math. Comp. 62 (1994), no. 206, 899-921; MR 94g:11096.

58. _____, *The maximal unramified extensions of the imaginary quadratic number fields with class number two*, J. Number Theory 60 (1996), no. 2, 42–50; MR 97g:11119.

59. _____, *Determination of the non-CM imaginary normal octic number fields with class number one*, submitted for publication.

60. _____, *Real quadratic number fields with class number one having an unramified $A_n$-extension*, in preparation.


Ken YAMAMURA
Department of Mathematics
National Defence Academy
Hashirimizu Yokosuka 239
JAPAN
e-mail : yamamura@cc.nda.ac.jp