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The mean values of logarithms of algebraic integers

par ARTŪRAS DUBICKAS

RÉSUMÉ. Soit $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ l'ensemble des conjugués d'un entier algébrique α de degré d , n'étant pas une racine de l'unité. Dans cet article on propose de minorer

$$M_p(\alpha) = \sqrt[p]{\frac{1}{d} \sum_{i=1}^d |\log |\alpha_i||^p}$$

où $p > 1$.

ABSTRACT. Let α be an algebraic integer of degree d with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$. In the paper we give a lower bound for the mean value

$$M_p(\alpha) = \sqrt[p]{\frac{1}{d} \sum_{i=1}^d |\log |\alpha_i||^p}$$

when α is not a root of unity and $p > 1$.

1. INTRODUCTION.

Let α be an algebraic number of degree $d \geq 2$ with

$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_0 = a_d (x - \alpha_1) (x - \alpha_2) \dots (x - \alpha_d)$$

as its minimal polynomial over \mathbb{Z} and a_d positive. Following Mahler, the Mahler measure of α is defined by

$$M(\alpha) = a_d \prod_{i=1}^d \max(1, |\alpha_i|).$$

The house of an algebraic number is the maximum of the modulus of its conjugates:

$$\overline{|\alpha|} = \max \{|\alpha_1|, |\alpha_2|, \dots, |\alpha_d|\}.$$

Put also

$$d(\alpha) = \max \{ \overline{|\alpha|}, \overline{|\alpha^{-1}|} \} = \max \{ |\alpha_1|, \dots, |\alpha_d|, 1/|\alpha_1|, \dots, 1/|\alpha_d| \}$$

for the "symmetric deviation" of conjugates from the unit circle. Denote for $p > 0$

$$M_p(\alpha) = \sqrt[p]{\frac{1}{d} \sum_{i=1}^d |\log |\alpha_i||^p}.$$

Our main concern here is the lower bound for this mean value when α is an algebraic integer ($a_d = 1$) which is not a root of unity.

In 1933, D.H. Lehmer [8] asked whether it is true that for every positive ε there exists an algebraic number α for which $1 < M(\alpha) < 1 + \varepsilon$. In its strong form Lehmer's problem has been reformulated as whether it is true that if α is not a root of unity then $M(\alpha) \geq \alpha_0 = 1.1762808\dots$ where α_0 is the root of the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

In 1971, C.J. Smyth [16] proved that if α is a non-reciprocal algebraic integer then $M(\alpha) \geq \theta = 1.32471\dots$ where θ is the real root of the polynomial $x^3 - x - 1$. This result reduces Lehmer's problem to the case of reciprocal algebraic integers (those with minimal polynomial satisfying the identity $P(x) \equiv x^d P(1/x)$). P.E. Blanksby and H.L. Montgomery [2] used Fourier analysis to prove that $M(\alpha) > 1 + 1/52d \log(6d)$. In 1978, C.L. Stewart [18] proved that $M(\alpha) > 1 + 1/10^4 d \log d$. Although this result is weaker than the previous one, the method used has become very important and led to further improvements. Recently M. Mignotte and M. Waldschmidt [12] obtained Stewart's result via the interpolation determinant.

In 1979, E. Dobrowolski [4] obtained a remarkable improvement of these results showing that for each $\varepsilon > 0$, there exists an effective $d(\varepsilon)$ such that for $d > d(\varepsilon)$

$$(1.1) \quad M(\alpha) > 1 + (1 - \varepsilon) \left(\frac{\log \log d}{\log d} \right)^3.$$

D.C. Cantor and E.G. Straus [3] in 1982 introduced the interpolation determinant to simplify Dobrowolski's proof and to replace the constant $1 - \varepsilon$ by $2 - \varepsilon$. Finally, R. Louboutin [9] was able to improve this constant to $9/4 - \varepsilon$. M. Meyer [11] obtained Louboutin's result using a version of Siegel's lemma due to Bombieri and Vaaler. Recently P. Voutier [19] showed that inequality (1) holds for all $d \geq 2$ with the weaker constant $1/4$ instead of $1 - \varepsilon$.

In 1965, A. Schinzel and H. Zassenhaus [13] conjectured that there exists an absolute positive constant γ such that $|\alpha| > 1 + \gamma/d$ whenever α is not a root of unity. The best known result on this problem is due to the author [5]:

we have

$$(1.2) \quad |\overline{\alpha}| > 1 + \left(\frac{64}{\pi^2} - \varepsilon\right) \frac{1}{d} \left(\frac{\log \log d}{\log d}\right)^3$$

where $d > d_1(\varepsilon)$. In fact, both inequalities (1), (2) and the respective conjectures can be considered in terms of the lower bound for $M_p(\alpha)$. Indeed, notice that

$$M_1(\alpha) = \frac{2 \log M(\alpha) - \log |a_0|}{d}.$$

Therefore, for $|a_0| \geq 2$,

$$M_1(\alpha) = \frac{2 \log |a_0| - \log |a_0|}{d} \geq \frac{\log 2}{d}.$$

If $|a_0| = 1$, then

$$M_1(\alpha) = \frac{2 \log M(\alpha)}{d}.$$

Louboutin's result can be written as follows

$$(1.3) \quad dM_1(\alpha) > \left(\frac{9}{2} - \varepsilon\right) \left(\frac{\log \log d}{\log d}\right)^3.$$

Taking $p = \infty$, we can write the inequality (2) in the following form

$$(1.4) \quad dM_\infty(\alpha) = d \log d(\alpha) \geq d \log |\overline{\alpha}| > \left(\frac{64}{\pi^2} - \varepsilon\right) \left(\frac{\log \log d}{\log d}\right)^3.$$

The function $p \rightarrow M_p(\alpha)$ is nondecreasing. Hence the inequality $dM_p(\alpha) \geq c_p$ where $1 < p < \infty$ and $c_p > 0$ lies between the conjecture of Lehmer $p = 1$ and the "symmetric" form of the conjecture of Schinzel and Zassenhaus $p = \infty$ (see also [1] for a problem which lies between these two conjectures). We have noticed above that the conjectural value for c_1 is $2 \log \alpha_0$. It would be of interest to find out whether it is true that $d(\alpha) \geq \sqrt[d]{2}$. The equality holds for the polynomial $x^d - 2$. We conjecture that the answer to the above question is affirmative, so that $c_\infty = \log 2$. In this paper, we take up the interpolation determinant again (see [3],[5],[9],[10], [19]) and fill the gap between inequalities (3) and (4) (Theorem 2). One can also consider the mean value of conjugates of an algebraic integer

$$m_p(\alpha) = \sqrt[p]{\frac{1}{d} \sum_{i=1}^d |\alpha_i|^p}$$

and the mean value of the differences

$$t_p(\alpha) = \sqrt[p]{\frac{2}{d(d-1)} \sum_{i \leq j} |\alpha_i - \alpha_j|^p}.$$

The lower bound for $m_1(\alpha)$ where α is a totally positive integer was considered by I. Schur [14], C.L. Siegel [15], C.J. Smyth [17]. In 1988, M. Langevin [7] solved Favard's problem proving that $t_\infty(\alpha) := \max_{i,j} |\alpha_i - \alpha_j| > 2 - \varepsilon$ for an algebraic integer of a sufficiently large degree. The author [6] proved that $t_2(\alpha) > \sqrt[4]{\varepsilon} - \varepsilon$. The problem of finding an upper bound for $t_{-\infty}(\alpha) := 1 / \min_{i \neq j} |\alpha_i - \alpha_j|$ is known as a separation problem. In this article, we apply the lower bound for $M_2(\alpha)$ to estimate $m_p(\alpha)$ from below (Theorem 3).

2. STATEMENT OF THE RESULTS.

The notations are the following. Let $G(x)$ be a real valued function in $[0; 1]$ such that $G(0) = 1$, $G(1) = 0$. Let also the derivative of $G(x)$ be continuous and negative in the interval $(0; 1)$. Put

$$(2.5) \quad I = \int_0^1 G(x) dx,$$

$$(2.6) \quad J = \int_0^1 (G(x))^2 dx,$$

$$(2.7) \quad L = \int_0^1 (G'(x))^2 dx.$$

Put also for brevity

$$\delta(d) = \left(\frac{\log \log d}{\log d} \right)^3.$$

Let α be a reciprocal algebraic integer, i.e. $d = 2m$, $m \in \mathbb{N}$, $\alpha_{2m} = 1/\alpha_1$, $\alpha_{2m-1} = 1/\alpha_2, \dots, \alpha_{m+1} = 1/\alpha_m$ where $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_m| \geq 1$. Suppose also that α is not a root of unity. With these hypotheses, our main result is the following:

Theorem 1. *For every $\varepsilon > 0$ there exists $d_0(\varepsilon)$ such that we have*

$$(2.8) \quad \sum_{j=1}^{d/2} \left(I - \frac{2j}{d} J \right) \log |\alpha_j| > \frac{1 - \varepsilon}{L} \delta(d)$$

whenever $d > d_0(\varepsilon)$.

The constant $d_0(\varepsilon)$ and the constants $d_1(\varepsilon), d_2(\varepsilon), d_3(\varepsilon), d_4, d_5(p)$ used below are effective. Taking $G(x) = (1 - x)^2$, we get $I = 1/3$, $J = 1/5$, $L = 4/3$. Hence the following inequality holds:

Corollary 1. *For every $\varepsilon > 0$ there exists $d_1(\varepsilon)$ such that*

$$\sum_{j=1}^{d/2} \left(1 - \frac{6j}{5d}\right) \log |\alpha_j| > \left(\frac{9}{4} - \varepsilon\right) \delta(d)$$

whenever $d > d_1(\varepsilon)$.

This inequality obviously implies Louboutin’s result. On the other hand, taking $G(x) = 1 - \sin(\pi x/2)$, we have $I = 1 - 2/\pi$, $J = 3/2 - 4/\pi$, $L = \pi^2/8$. Hence

$$\sum_{j=1}^{d/2} \left(1 - \frac{2}{\pi} - \left(3 - \frac{8}{\pi}\right) \frac{j}{d}\right) \log |\alpha_j| > \left(\frac{8}{\pi^2} - \varepsilon\right) \delta(d).$$

We can replace in the inequality above $\log |\alpha_j|$ by $|\alpha_j| - 1$, and so Theorem 1 yields the following Corollary.

Corollary 2. *For every $\varepsilon > 0$ there exists $d_2(\varepsilon)$ such that for $d > d_2(\varepsilon)$ we have*

$$\sum_{j=1}^{d/2} \tau_j |\alpha_j| > 1 + \left(\frac{64}{\pi^2} - \varepsilon\right) \frac{\delta(d)}{d},$$

where

$$\tau_j = \left(1 - \frac{2}{\pi}\right) \frac{8}{d} - \left(3 - \frac{8}{\pi}\right) \frac{8j - 4}{d^2}.$$

Corollary 2 implies the inequality (2), since $\sum_{j=1}^{d/2} \tau_j = 1$. The following theorem fills the gap between (3) and (4).

Theorem 2. *Let $1 < p < \infty$ and $\varepsilon > 0$. Then there is $d_3(\varepsilon)$ such that for $d > d_3(\varepsilon)$ we have*

$$dM_p(\alpha) > (b_p - \varepsilon) \delta(d),$$

where the constant b_p is given by

$$(2.9) \quad b_p = \frac{2}{L} \left(\frac{(2p - 1)J}{(p - 1)(I^{(2p-1)/(p-1)} - (I - J)^{(2p-1)/(p-1)})} \right)^{1-1/p}.$$

We are not solving the problem of computing the maximum in (9) for a fixed p from the interval $(1; \infty)$. However, notice that if $G(x) = (1 - x)^{1.7}$ and $p = 2$ then by (5)-(7) and (9) we get $b_2 > 6.2679$.

Corollary 3. *There is $d_4 > 0$ such that for $d > d_4$ we have*

$$dM_2(\alpha) > 6.2679\delta(d).$$

Theorem 3. *If α is an algebraic integer which is not a root of unity, then for every $p > 0$ there exists $d_5(p)$ such that for $d > d_5(p)$ we have*

$$\left(m_p(\alpha)\right)^p > 1 + 19.64\left(p\delta(d)/d\right)^2.$$

In particular,

$$m_1(\alpha) = \frac{|\alpha_1| + \dots + |\alpha_d|}{d} > 1 + 19.64\left(\frac{\delta(d)}{d}\right)^2.$$

Proof of Theorem 1. Let $f(x)$ be a continuous non-negative function in $[0; 1]$ such that $\int_0^1 f(x)dx = 1$, and let $G(x) = \int_x^1 f(y)dy$. Put

$$\begin{aligned} s &= \left\lceil \frac{L}{2} \left(\frac{\log d}{\log \log d} \right)^2 \right\rceil, \\ k_0 &= \left\lceil \frac{s^2 \log s}{\log d} \right\rceil, \\ k_r &= \left\lceil s f\left(\frac{r}{s}\right) \right\rceil, \quad 1 \leq r \leq s. \end{aligned}$$

Define

$$\begin{aligned} h_0(z) &= h(z) = \left(1, z, z^2, \dots, z^{N-1}\right)^t, \\ h_k(z) &= \frac{z^k d^k h(z)}{k! d^k z} = \left(0, \dots, \binom{N-2}{k} z^{N-2}, \binom{N-1}{k} z^{N-1}\right)^t. \end{aligned}$$

Consider the determinant

$$D = \det \left\| h_{u_r}(\alpha_j^{p_r}) \right\|,$$

where the matrix consists of $N = (k_0 + k_1 + \dots + k_s)d$ columns, $u_r = 0, 1, \dots, k_r - 1$, $j = 1, 2, \dots, d$. Here p_r is the r -th prime number ($p_0 = 1, p_1 = 2, p_2 = 3, \dots$). Recall that α is reciprocal and $\alpha_{2m} = 1/\alpha_1, \dots, \alpha_{m+1} = 1/\alpha_m$. Then see ([3], [5], [9], [10], [19]) the determinant D is given by

$$D = \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v}\right)^{k_u k_v} \left(\alpha_i^{-p_u} - \alpha_j^{-p_v}\right)^{k_u k_v} \prod \left(\alpha_i^{p_u} - \alpha_j^{-p_v}\right)^{k_u k_v}$$

where the first product is taken over $i, j = 1, 2, \dots, m$ and $0 \leq u \leq v \leq s$ (if $u = v$, then $i < j$). The second product is taken over all $i, j = 1, 2, \dots, m$; $u, v = 0, 1, 2, \dots, s$. Let us denote these products by P_1 and P_2 respectively.

We first consider P_1 . We have:

$$\begin{aligned}
 P_1 &= \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v} \right)^{2k_u k_v} \prod \alpha_i^{-p_u k_u k_v} \alpha_j^{-p_v k_u k_v} \\
 &= \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v} \right)^{2k_u k_v} \prod_{i,j;u < v} \alpha_i^{-p_u k_u k_v} \prod_{i,j;u < v} \alpha_j^{-p_v k_u k_v} \\
 &\quad \times \prod_{i < j; u} (\alpha_i \alpha_j)^{-p_u k_u^2} \\
 &= \pm M(\alpha)^{-m \left(\sum_{u < v} p_u k_u k_v + \sum_{u > v} p_u k_u k_v \right) - (m-1) \sum p_u k_u^2} \\
 &\quad \times \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v} \right)^{2k_u k_v} \\
 &= \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v} \right)^{2k_u k_v} M(\alpha)^{-m \sum p_u k_u \sum k_v + \sum p_u k_u^2}.
 \end{aligned}$$

Next, we have for the product P_2

$$\begin{aligned}
 P_2 &= \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right)^{k_u k_v} \prod \alpha_i^{p_u k_u k_v} \\
 &= \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right)^{k_u k_v} M(\alpha)^{m \sum p_u k_u \sum k_v}
 \end{aligned}$$

Combining these results we find

$$D = \pm \prod \left(\alpha_i^{p_u} - \alpha_j^{p_v} \right)^{2k_u k_v} \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right)^{k_u k_v} M(\alpha)^{\sum p_u k_u^2}.$$

Now from each term $\alpha_i^{p_u} - \alpha_j^{p_v}$ in the first product we take

1. $\alpha_i^{p_u}$, if $u = v, i < j$;
2. $\alpha_j^{p_v}$, if $u < v, j \leq i$;
3. $\alpha_i^{p_u} \alpha_j^{p_v - p_u}$, if $u < v, i < j$.

This is the key point of our argument. Write the determinant D as follows

$$\begin{aligned}
 D &= \pm \prod \alpha_i^{2p_u k_u^2} \left(1 - (\alpha_j / \alpha_i)^{p_u} \right)^{2k_u^2} \prod \alpha_j^{2p_v k_u k_v} \left(\alpha_i^{p_u} \alpha_j^{-p_v} - 1 \right)^{2k_u k_v} \\
 &\quad \times \prod \alpha_i^{2p_u k_u k_v} \alpha_j^{2(p_v - p_u) k_u k_v} \left(\alpha_j^{p_u - p_v} - (\alpha_j / \alpha_i)^{p_u} \right)^{2k_u k_v} \\
 &\quad \times \prod \left(1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right)^{k_u k_v} M(\alpha)^{\sum p_u k_u^2}
 \end{aligned}$$

Denote $y_1 = \alpha_2/\alpha_1, y_2 = \alpha_3/\alpha_2, \dots, y_{m-1} = \alpha_m/\alpha_{m-1}, y_m = 1/\alpha_m$. Then D can be expressed in the form

$$\begin{aligned} D &= \pm M(\alpha) p_u k_u^2 \prod \alpha_i^{2p_u k_u^2} \prod \alpha_j^{2p_v k_u k_v} \prod \alpha_i^{2p_u k_u k_v} \alpha_j^{2(p_v - p_u) k_u k_v} \\ &\quad \times p(y_1, y_2, \dots, y_m) \\ &= \prod_{j=1}^m \alpha_j^{s_j} \times p(y_1, y_2, \dots, y_m) \end{aligned}$$

where $p(y_1, \dots, y_m)$ is a polynomial in y_1, y_2, \dots, y_m . The power s_j is given by

$$\begin{aligned} s_j &= \sum p_u k_u^2 + 2(m - j) \sum p_u k_u^2 + 2(m - j + 1) \sum_{u < v} p_v k_u k_v \\ &\quad + 2(m - j) \sum_{u < v} p_u k_u k_v + 2(j - 1) \sum_{u < v} (p_v - p_u) k_u k_v \\ &= (2m - 2j + 1) \sum p_u k_u^2 + 2m \sum_{u < v} p_v k_u k_v + (2m - 4j + 2) \sum_{u < v} p_u k_u k_v \\ &= (2m - 2j + 1) \sum p_u k_u^2 + 2m \sum_{u < v} p_v k_u k_v \\ &\quad + (2m - 4j + 2) \left(\sum p_u k_u \sum k_v - \sum_{v < u} p_u k_u^2 - \sum_{v < u} p_u k_u k_v \right) \\ &= (d - 4j + 2) \sum p_u k_u \sum k_v + (4j - 2) \sum_{v < u} p_u k_u k_v + (2j - 1) \sum p_u k_u^2 \\ &= (d - 4j + 2) \sum p_u k_u \sum k_v + (4j - 2) \sum_{v \leq u} p_u k_u k_v - (2j - 1) \sum p_u k_u^2 \end{aligned}$$

Using the maximum modulus principle and the inequalities $|y_j| \leq 1, j = 1, 2, \dots, m$, we have

$$\left| p(y_1, y_2, \dots, y_m) \right| \leq \left| p(y_1^0, y_2^0, \dots, y_m^0) \right|,$$

where $|y_1^0| = |y_2^0| = \dots = |y_m^0| = 1$. Now by Hadamard's inequality we find (see [5])

$$\log |D| \leq \frac{1}{2} d \log \left(d \sum_{v=0}^s k_v \right) \sum_{v=0}^s k_v^2 + \sum_{j=1}^{d/2} s_j \log |\alpha_j|.$$

On the other hand (see [9]),

$$\log |D| \geq k_0 d \sum_{v=1}^s k_v \log p_v.$$

For d tending to infinity the following asymptotic formulas hold:

$$\begin{aligned} \sum_{v=1}^s k_v \log p_v &\sim \sum_v s f\left(\frac{v}{s}\right) \log v \sim s^2 \log s \int_0^1 f(x) dx \sim s^2 \log s \sim \\ &\sim \frac{L^2 (\log d)^4}{2 (\log \log d)^3}, \\ k_0 &\sim \frac{L^2 (\log d)^3}{2 (\log \log d)}, \\ \sum_{v=0}^s k_v^2 &\sim k_0^2 + \sum_{v=1}^s s^2 f^2\left(\frac{v}{s}\right) \sim k_0^2 + s^3 \int_0^1 f^2(x) dx \sim \frac{3}{8} L^4 \left(\frac{\log d}{\log \log d}\right)^6. \end{aligned}$$

Similarly,

$$s_j \sim (d - 4j) s^5 \log s \int_0^1 f(x)x dx + 4j s^5 \log s \int_0^1 f(x)x \left(\int_0^x f(y) dy \right) dx.$$

Since

$$\int_0^1 f(x)x dx = - \int_0^1 G'(x)x dx = \int_0^1 G(x) dx = I$$

and

$$\begin{aligned} \int_0^1 f(x)x \left(\int_0^x f(y) dy \right) dx &= \int_0^1 f(x)x(1 - G(x)) dx \\ &= I - \int_0^1 f(x)x G(x) dx = I + \int_0^1 G'(x) G(x) x dx \\ &= I + \frac{1}{2} \int_0^1 (G^2(x))' x dx \\ &= I - \frac{1}{2} \int_0^1 G^2(x) dx = I - \frac{1}{2} J, \end{aligned}$$

we have

$$\begin{aligned} s_j &\sim s^5 \log s \left((d - 4j)I + 4j \left(I - \frac{12}{J} \right) \right) \\ &\sim (dI - 2jJ) \frac{L^5 (\log d)^{10}}{16 (\log \log d)^9}. \end{aligned}$$

For a sufficiently large d we have

$$\begin{aligned} & \sum_{j=1}^{d/2} (dI - 2jJ) \log |\alpha_j| \\ & > (1 - \varepsilon) \frac{16(\log \log d)^9}{L^5(\log d)^{10}} \left(\frac{dL^4(\log d)^7}{4(\log \log d)^6} - \frac{3dL^4(\log d)^7}{16(\log \log d)^6} \right) \\ & = (1 - \varepsilon) \frac{d}{L} \left(\frac{\log \log d}{\log d} \right)^3. \end{aligned}$$

This inequality implies (8). \square

Proof of Theorem 2. If α is not reciprocal, then by Smyth's result [16] $dM_1(\alpha) \geq 2 \log \theta$, and the theorem follows from $M_p(\alpha) \geq M_1(\alpha)$. Let α be reciprocal. Then by (8) and by Hölder's inequality we have

$$\begin{aligned} 1 - \frac{\varepsilon}{L} \delta(d) & < \sum_{j=1}^{d/2} \left(I - \frac{2j}{d} J \right) \log |\alpha_j| \\ & \leq \left(\sum_{j=1}^{d/2} (\log |\alpha_j|)^p \right)^{1/p} \left(\sum_{j=1}^{d/2} \left(I - \frac{2j}{d} J \right)^q \right)^{1/q} \end{aligned}$$

where $1/p + 1/q = 1$.

Note first that for a reciprocal α

$$\left(\sum_{j=1}^{d/2} (\log |\alpha_j|)^p \right)^{1/p} = (d/2)^{1/p} M_p(\alpha).$$

For d tending to infinity we have

$$\begin{aligned} \sum_{j=1}^{d/2} \left(I - \frac{2j}{d} J \right)^q & \sim \frac{d}{2} \int_0^1 (I - Jx)^q dx \\ & = \frac{d(I^{q+1} - (I - J)^{q+1})}{2J(q+1)}. \end{aligned}$$

Hence

$$1 - \frac{\varepsilon_1}{L} \delta(d) < \frac{d}{2} M_p(\alpha) \left(\frac{I^{q+1} - (I - J)^{q+1}}{J(q+1)} \right)^{1/q},$$

and Theorem 2, where the constant b_p is given by (9), follows. \square

Proof of Theorem 3. We have

$$\begin{aligned}
 (m_p(\alpha))^p &= \frac{1}{d} \sum_{i=1}^d |\alpha_i|^p \\
 &= \frac{1}{d} \sum_{i=1}^d \exp(p \log |\alpha_i|) \\
 &= \frac{1}{d} \sum_{i=1}^d \sum_{j=0}^{\infty} \frac{(p \log |\alpha_i|)^j}{j!} \\
 &= \frac{1}{d} \sum_{j=0}^{\infty} \frac{p^j}{j!} \sum_{i=1}^d (\log |\alpha_i|)^j.
 \end{aligned}$$

If α is reciprocal, then the inner sum equals $d(M_j(\alpha))^j$ for even j and zero for odd j . Hence

$$(m_p(\alpha))^p = 1 + \sum_{k=1}^{\infty} \frac{p^{2k}}{(2k)!} (M_{2k}(\alpha))^{2k} > 1 + \frac{p^2}{2} (M_2(\alpha))^2.$$

Utilizing Corollary 3 we have

$$(M_2(\alpha))^2 > 39.28 \left(\frac{\delta(d)}{d} \right)^2,$$

if d is large enough and the statement of Theorem 3 follows.

Suppose now that α is not reciprocal. If

$$|a_0| = \prod_{i=1}^d |\alpha_i| \geq 2$$

then

$$\begin{aligned}
 (m_p(\alpha))^p &= \frac{|\alpha_1|^p + \dots + |\alpha_d|^p}{d} \geq \prod_{i=1}^d |\alpha_i|^{p/d} \geq 2^{p/d} > 1 + \frac{p \log 2}{d} \\
 &> 1 + 19.64 \left(\frac{p\delta(d)}{d} \right)^2
 \end{aligned}$$

for $d > d_5(p)$. Hence it is sufficient to consider the case when $|a_0| = 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be the conjugates of α lying strictly outside the unit circle. Put

$$\Lambda = \prod_{i=1}^r |\alpha_i|.$$

Then

$$\begin{aligned} (m_p(\alpha))^p &= \frac{|\alpha_1|^p + \dots + |\alpha_d|^p}{d} \\ &\geq \frac{r}{d} (|\alpha_1| \dots |\alpha_r|)^{p/r} + \frac{d-r}{d} (|\alpha_{r+1}| \dots |\alpha_d|)^{p/(d-r)} \\ &= \frac{r}{d} \Lambda^{p/r} + \frac{d-r}{d} \Lambda^{-p/(d-r)}. \end{aligned}$$

We shall show now that the last expression is greater than

$$1 + \frac{(\log \theta)^2}{2} \left(\frac{p}{d}\right)^2$$

where $\theta = 1.32471\dots$. Indeed, if

$$h(\Lambda) = \frac{r}{d} \Lambda^{p/r} + \frac{d-r}{d} \Lambda^{-p/(d-r)}$$

then

$$\begin{aligned} h'(\Lambda) &= \frac{p}{d} \Lambda^{p/r-1} - \frac{p}{d} \Lambda^{-p/(d-r)-1} \\ &= \frac{p}{\Lambda d} \left(\Lambda^{p/r} - \Lambda^{-p/(d-r)} \right). \end{aligned}$$

Therefore, the function $h(\Lambda)$ is increasing in the interval $(1; \infty)$ and by Smyth's theorem

$$h(\Lambda) \geq h(\theta) = \frac{r}{d} \theta^{p/r} + \frac{d-r}{d} \theta^{-p/(d-r)}.$$

Put for brevity $p = zd$ and $r = yd$. We are going to prove that

$$g(z) = y\theta^{z/y} + (1-y)\theta^{-z/(1-y)} - 1 - \frac{(\log \theta)^2}{2} z^2 > 0$$

for $z > 0$ and $0 < y < 1$. Indeed, $g(0) = 0$ and

$$\begin{aligned} g'(z) &= \theta^{z/y} \log \theta - \theta^{-z/(1-y)} \log \theta - (\log \theta)^2 z \\ &> \theta^{z/y} \log \theta - \log \theta - (\log \theta)^2 z \\ &> \left(1 + \frac{z \log \theta}{y}\right) \log \theta - \log \theta - (\log \theta)^2 z \\ &= z \left(\frac{1}{y} - 1\right) (\log \theta)^2 > 0. \end{aligned}$$

Therefore, with our hypotheses

$$(m_p(\alpha))^p > 1 + \frac{(\log \theta)^2}{2} \left(\frac{p}{d}\right)^2 > 1 + 19.64 \left(\frac{p\delta(d)}{d}\right)^2$$

for $d > d_5(p)$. This completes the proof of Theorem 3. □

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