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Primitive Substitutive Numbers are Closed
Under Rational Multiplication

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Résumé. Soit \( M(r) \) l'ensemble des réels \( \alpha \) dont le développement en base \( r \) contient une queue qui est l'image d'un point fixe d'une substitution primitive par un morphisme de lettres. Nous démontrons que l'ensemble \( M(r) \) est stable par multiplication par les rationnels, mais non stable par addition.

Abstract. Let \( M(r) \) denote the set of real numbers \( \alpha \) whose base-\( r \) digit expansion is ultimately primitive substitutive, i.e., contains a tail which is the image (under a letter to letter morphism) of a fixed point of a primitive substitution. We show that the set \( M(r) \) is closed under multiplication by rational numbers, but not closed under addition.

1. Introduction

A sequence on a finite alphabet \( A \) is called \( q \)-automatic if it is the image (under a letter to letter morphism) of a fixed point of a substitution of constant length \( q \). Let \( M(q, r) \) denote the set of real numbers whose fractional part has a \( q \)-automatic base-\( r \) digit expansion. J.H. Loxton and A. van der Poorten stated that each number \( \alpha \in M(q, r) \) is either rational or transcendental [LoPo]. Unfortunately a gap has been reported in their proof.

Analogously, a sequence on a finite alphabet \( A \) is called primitive substitutive if it is the image (under a letter to letter morphism) of a fixed point of a primitive substitution. Let \( M(r) \) denote the set of real numbers whose base-\( r \) digit expansion is ultimately primitive substitutive, i.e., contains a tail which is primitive substitutive. It is believed that a number \( \alpha \) in \( M(r) \) is either rational or transcendental. This has been verified in a few special cases (see [AlZa], [FeMa], and [RiZa]).

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In [Le] S. Lehr showed that $M(q, r)$ is a $\mathbb{Q}$-vector space. We will show that $M(r)$ is closed under multiplication by $\mathbb{Q}$ but not closed under addition. Lehr's proof relies in part on a theorem of J.-P. Allouche and M. Mendès France:

**Theorem 1** (J.-P. Allouche, M. Mendès France, [AlMe]). Let $\ast$ be an associative binary operation on a finite set $A$ and let $\omega = \omega_1 \omega_2 \omega_3 \ldots$ be a $q$-automatic sequence in $A^\mathbb{N}$. Then the induced sequence of partial products

$$\omega_1, \omega_1 \ast \omega_2, \omega_1 \ast \omega_2 \ast \omega_3, \omega_1 \ast \omega_2 \ast \omega_3 \ast \omega_4, \ldots$$

is $q$-automatic.

We will use the following analogue of Theorem 1

**Theorem 2** (C. Holton, L.Q. Zamboni, [HoZa]). Let $\ast$ be a binary operation on a finite set $A$ and let $\omega = \omega_1 \omega_2 \omega_3 \ldots$ be an ultimately primitive substitutive sequence in $A^\mathbb{N}$. Then the induced sequence of partial products

$$\omega_1, \omega_1 \ast \omega_2, (\omega_1 \ast \omega_2) \ast \omega_3, ((\omega_1 \ast \omega_2) \ast \omega_3) \ast \omega_4, \ldots$$

is ultimately primitive substitutive.

2. MAIN THEOREM

**Theorem 1.** The set $M(r)$ is closed under multiplication by $\mathbb{Q}$ but not closed under addition.

*Proof.* We begin by observing that $\mathbb{Q} \subset M(r)$. In fact, the digit expansion of a rational number is ultimately periodic, and a periodic sequence is primitive substitutive. Let $\xi \in M(r)$. We show that for positive integers $n$ and $p$, both $n\xi$ and $\frac{\xi}{p}$ are in $M(r)$. In each case we can assume that $0 < \xi < 1$ and that $\xi \notin \mathbb{Q}$. Hence we can write $\xi = \sum_{k=1}^{\infty} \xi_k r^{-k}$ with $\xi_k \in A_r = \{0, 1, \ldots, r - 1\}$. The sequence $\{\xi_k\}$ is then ultimately primitive substitutive but not ultimately periodic. We begin by showing that $y = \frac{\xi}{p} \in M(r)$. We write $y = \sum_{k=1}^{\infty} y_k r^{-k}$ with $y_k \in A_r$. Then following [Le] we have

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1This is a special case of a more general result of F.M. Dekking in [De]
Consider the sequence $\{(\xi_k, r)\}$ in the alphabet $A_{pr} \times A_{pr}$. Since the sequence $\{\xi_k\}$ is ultimately primitive substitutive, the same is true of the sequence $\{(\xi_k, r)\}$. Let $\ast$ denote the associative binary operation on $A_{pr} \times A_{pr}$ given by $^2$

$$(a, \alpha) \ast (b, \beta) = (a\beta + b \mod pr, \alpha\beta \mod pr).$$

For each $k \geq 1$ we set

$$x_k = (\xi_1, r) \ast (\xi_2, r) \ast \ldots \ast (\xi_k, r) = \left(\sum_{i=1}^{k} \xi_i r^{k-i} \mod pr, r^k \mod pr\right).$$

$^2$There is a typographical error in the definition of the binary operation $\ast$ given in [Le]. It should be the same as $\ast$. 

$$y_k = \left\lfloor \frac{\xi \cdot r^k}{p} \right\rfloor \mod r$$

$$= \frac{r^k}{p} \sum_{i=1}^{\infty} \xi_i r^{-i} \mod r$$

$$= \frac{1}{p} \sum_{i=1}^{\infty} \xi_i r^{k-i} \mod r$$

$$= \frac{1}{p} \sum_{i=1}^{k} \xi_i r^{k-i} + \frac{1}{p} \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \mod r$$

Since

$$\frac{1}{p} \sum_{i=1}^{k} \xi_i r^{k-i} = \frac{m}{p}$$

for some natural number $m$, and

$$\frac{1}{p} \sum_{i=k+1}^{\infty} \xi_i r^{k-i} < \frac{1}{p}$$

we obtain

$$y_k = \left\lfloor \frac{1}{p} \sum_{i=1}^{k} \xi_i r^{k-i} \right\rfloor \mod r = \left\lfloor \frac{\left(\sum_{i=1}^{k} \xi_i r^{k-i}\right) \mod pr}{p} \right\rfloor \mod r.$$
By Theorem 2 the sequence \( \{x_k\} \) is ultimately primitive substitutive, and hence so is the sequence \( \{y_k\} \) as required.

We next show that \( n \xi \in M(r) \).

**Lemma 2.** Let \( \{\xi_k\} \) be as above. There is a positive integer \( M = M(n) \) such that for each \( k \geq 0 \)

\[
\left[ n \sum_{i=1}^{\infty} \frac{\xi_{k+i}}{r^i} \right] = \left[ n \left( \frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+M}}{r^M} \right) \right].
\]

**Proof.** Fix a positive integer \( l \) so that \( r^l \geq n \), and set \( (a)_f = a - \lfloor a \rfloor \) for each \( a \in \mathbb{R} \). Then

\[
\left[ n \sum_{i=1}^{\infty} \frac{\xi_{k+i}}{r^i} \right] = \left[ n \left( \frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+l}}{r^l} \right) \right] + \left[ S_k + n \sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i} \right]
\]

where \( S_k = \left( n \left( \frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+l}}{r^l} \right) \right)_f \).

Note that for \( k \geq 0 \) the quantity \( [S_k + n \sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i}] \) is either 0 or 1.

Let \( S = \{S_k \mid k \geq 1\} \). Then \( \text{Card}(S) \leq r^l \). For each \( s \in S \) there exist words \( V_s \) and \( U_s \) (in the alphabet \( A_r \)) such that the base-\( r \) digit expansion of \( \frac{1}{n}(1 - s) \in \mathbb{Q} \) is given by \( V_sU_s^\omega \). Since \( \xi \notin \mathbb{Q} \), for each \( s \in S \) there is a positive integer \( m_s \) so that the sequence \( \{\xi_k\} \) does not contain the subword \( U_s^{m_s} \). Set \( M_s = |V_s| + m_s|U_s| \) and \( M' = \max \{M_s \mid s \in S\} \). Then for each \( k \geq 0 \) we have

\[
[S_k + n \sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i}] = 1
\]

if and only if

\[
\sum_{i=l+1}^{\infty} \frac{\xi_{k+i}}{r^i} > \frac{1}{n}(1 - S_k)
\]

if and only if

\[
\frac{\xi_{k+l+1}}{r^{l+1}} + \frac{\xi_{k+l+2}}{r^{l+2}} + \ldots + \frac{\xi_{k+l+M'}}{r^{l+M'}} > \frac{1}{n}(1 - S_k).
\]

Thus \( M = l + M' \) satisfies the conclusion of Lemma 2. \( \square \)

We return to the proof of Theorem 1. Let \( z = n \xi \). Then we can write \( (z)_f = \sum_{k=1}^{\infty} z_k r^{-k} \). It suffices to show that the sequence \( \{z_k\} \) is ultimately...
primitive substitutive. Let $M$ be as in Lemma 2. Then for $k \geq 0$ we have

$$z_k = \left\lfloor n \xi r^k \right\rfloor \mod r$$

$$= \left\lfloor n r^k \sum_{i=1}^{\infty} \xi_i r^{-i} \right\rfloor \mod r$$

$$= \left\lfloor n \sum_{i=1}^{k-1} \xi_i r^{k-i} + n \xi_k + n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \mod r$$

$$= \left\lfloor n \xi_k + n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \mod r$$

$$= \left\lfloor n \xi_k \right\rfloor \mod r + \left\lfloor n \sum_{i=k+1}^{\infty} \xi_i r^{k-i} \right\rfloor \mod r$$

$$= \left\lfloor n \xi_k \right\rfloor \mod r + n \left( \frac{\xi_{k+1}}{r} + \frac{\xi_{k+2}}{r^2} + \ldots + \frac{\xi_{k+M}}{r^M} \right) \mod r.$$
Since $|\tau^n(121)| = |\tau^n(211)|$ it follows that $a_j \neq b_j$ (and hence $c_j = 3$) for infinitely many values of $j$. Thus no tail of the decimal expansion of $\alpha + \beta$ is a minimal sequence. In particular $\{c_i\}$ is not ultimately primitive substitutive.

\section*{REFERENCES}


