Takao Komatsu

On inhomogeneous diophantine approximation with some quasi-periodic expressions, II

Journal de Théorie des Nombres de Bordeaux, tome 11, n° 2 (1999), p. 331-344

<http://www.numdam.org/item?id=JTNB_1999__11_2_331_0>
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Résumé. On s'intéresse aux valeurs de

\[ M(\theta, \phi) = \liminf_{|q| \to \infty} |q||q\theta - \phi| \]

lorsque \( \theta \) est un réel ayant un développement en fraction continue quasi-périodique.

Abstract. We consider the values concerning

\[ M(\theta, \phi) = \liminf_{|q| \to \infty} |q||q\theta - \phi| \]

where the continued fraction expansion of \( \theta \) has a quasi-periodic form. In particular, we treat the cases so that each quasi-periodic form includes no constant. Furthermore, we give some general conditions satisfying \( M(\theta, \phi) = 0 \).

1. Introduction

Let \( \theta \) be irrational and \( \phi \) real. We suppose throughout that \( q\theta - \phi \) is never integral for any integer \( q \). Define the value of the function

\[ M(\theta, \phi) = \liminf_{|q| \to \infty} |q||q\theta - \phi| , \]

which is called inhomogeneous approximation constant for the pair \( \theta, \phi \). It is convenient to introduce the functions

\[ M_+(\theta, \phi) = \liminf_{q \to +\infty} q||q\theta - \phi| \]

and

\[ M_-(\theta, \phi) = \liminf_{q \to -\infty} q||q\theta + \phi| = \liminf_{q \to -\infty} |q||q\theta - \phi| . \]

Then \( M(\theta, \phi) = \min(M_+(\theta, \phi), M_-(\theta, \phi)) \). Several authors have treated \( M(\theta, \phi) \) or \( M_+(\theta, \phi) \) by using their own algorithms (See [1], [2], [4], [5], [11] e.g.), but it has been difficult to find the exact values of \( M(\theta, \phi) \) for

Manuscrit reçu le 2 avril 1998.
the concrete pair of $\theta$ and $\phi$. For example, Cusick, Rockett and Szüss ([2]) obtain
\[
\mathcal{M}\left(\theta, \frac{1}{2}\right) = \frac{1}{4\sqrt{5}} \quad \text{and} \quad \mathcal{M}\left(\theta, \frac{1}{\sqrt{5}}\right) = \frac{1}{5\sqrt{5}}
\]
when $\theta = (1 + \sqrt{5})/2 = [1; 1, 1, \ldots]$. And author ([5]) obtains
\[
\mathcal{M}\left(\theta, \frac{1}{a}\right) = \frac{1}{a^2\sqrt{a^2 + 4}},
\]
\[
\mathcal{M}\left(\theta, \frac{1}{2a}\right) = \frac{1}{4a^2\sqrt{a^2 + 4}},
\]
\[
\mathcal{M}\left(\theta, \frac{1}{a^2 + 4}\right) = \frac{1}{(a^2 + 4)\sqrt{a^2 + 4}} \quad \text{and}
\]
\[
\mathcal{M}\left(\theta, \frac{1}{2}\right) = \frac{1}{4\sqrt{a^2 + 4}} \quad (a \text{ is odd } \geq 3)
\]
when $\theta = (\sqrt{a^2 + 4} - a)/2 = [0; a, a, \ldots]$. However, it is not easy to apply these methods to find the value $\mathcal{M}(\theta, \phi)$ about the different types of $\theta$.

In [6] author establishes the relationship between $\mathcal{M}(\theta, \phi)$ and the algorithm of Nishioka, Shiokawa and Tamura. If we use this result, we can evaluate the exact value of $\mathcal{M}(\theta, \phi)$ for any pair of $\theta$ and $\phi$ at least when $\theta$ is a positive real root of the quadratic equation and $\phi \in \mathbb{Q}(\theta)$. For example,
\[
\mathcal{M}\left(\theta, \frac{1}{2}\right) = \begin{cases} 
\frac{\min(a, b)}{4\sqrt{D}} & \text{if both } a \text{ and } b \text{ are odd}, \\
\frac{a}{4\sqrt{D}} & \text{otherwise},
\end{cases}
\]
\[
\mathcal{M}\left(\theta, \frac{1}{\sqrt{D}}\right) = \frac{a}{D\sqrt{D}} \quad \text{and}
\]
\[
\mathcal{M}\left(\theta, \frac{1}{a}\right) = \frac{1}{a\sqrt{D}} \quad (a \geq 2) \quad (D = ab(ab + 4))
\]
are given when $\theta = (\sqrt{ab(ab + 4)} - ab)/(2a) = [0; a, b, a, b, \ldots]$. Furthermore, in [7] author is so successful applying the Nishioka-Shiokawa-Tamura algorithm that the exact value of $\mathcal{M}(\theta, \phi)$ can be calculated even if $\theta$ is a Hurwitzian number, namely its continued fraction expansion has a quasi-periodic form. And it is the first time to find a concrete pair of $\theta$ and $\phi$ so that $\mathcal{M}(\theta, \phi) = 0$. For example, for a positive integer $s$
\[
\mathcal{M}\left(e^{\frac{1}{s}}, \frac{1}{3}\right) = \begin{cases} 
0 & \text{if } s \equiv 2 \pmod{3}, \\
\frac{1}{18} & \text{otherwise}
\end{cases}
\]
is given.

In this paper we consider the cases so that each quasi-periodic form includes no constant, and conditions satisfying $\mathcal{M}(\theta, \phi) = 0$. 
2. NST ALGORITHM

We first introduce the NST algorithm ([9]). \( \theta = [a_0; a_1, a_2, \ldots] \) denotes the continued fraction expansion of \( \theta \), where

\[
\theta = a_0 + \theta_0, \quad a_0 = [\theta], \\
1/\theta_{n-1} = a_n + \theta_n, \quad a_n = [1/\theta_{n-1}] \quad (n = 1, 2, \ldots).
\]

The k-th convergent \( p_k/q_k = [a_0; a_1, \ldots, a_k] \) of \( \theta \) is then given by the recurrence relations

\[
p_k = a_np_{k-1} + p_{k-2} \quad (k = 0, 1, \ldots), \quad p_2 = 0, \quad p_1 = 1, \\
q_k = a_kq_{k-1} + q_{k-2} \quad (k = 0, 1, \ldots), \quad q_2 = 1, \quad q_1 = 0.
\]

Denote \( \phi = \theta[b_0; b_1, b_2, \ldots, ] \) be the expansion of \( \phi \) in terms of the sequence \{\( \theta_0, \theta_1, \ldots \)\}, where

\[
\phi = b_0 - \phi_0, \quad b_0 = [\phi], \\
\phi_{n-1}/\theta_{n-1} = b_n - \phi_n, \quad b_n = [\phi_{n-1}/\theta_{n-1}] \quad (n = 1, 2, \ldots).
\]

Then, \( \phi \) is represented by

\[
\phi = b_0 - b_1\theta_0 + b_2\theta_0\theta_1 - \cdots + (-1)^k b_k\theta_0\theta_1\cdots\theta_{k-1} - (-1)^k\theta_0\theta_1\cdots\theta_{k-1}\phi_k
\]

\[
= b_0 - \sum_{k=0}^{\infty} (-1)^k b_{k+1}\theta_0\theta_1\cdots\theta_k = b_0 - \sum_{k=0}^{\infty} b_{k+1}D_k,
\]

where \( D_k = q_k\theta - p_k = (-1)^k\theta_0\theta_1\cdots\theta_k \). Now, the following theorem is established in [6].

Theorem 1.

\[
\mathcal{M}_-(\theta, \phi) = \liminf_{n \to +\infty} \min(B_n\theta + \phi, B_n^*\theta + \phi),
\]

where \( B_n = \sum_{k=1}^{n} b_kq_{k-1} \) and \( B_n^* = B_n - q_{n-1} \).

Remark. It is also known in [6] that \( \|B_n\theta + \phi\| = \phi_n|D_{n-1}| \) and \( \|B_n^*\theta + \phi\| = (1 - \phi_n)|D_{n-1}| \). Together with \( \mathcal{M}_+(\theta, \phi) = \mathcal{M}_-(\theta, 1 - \phi) \), one can obtain the value \( \mathcal{M}(\theta, \phi) \).

3. THE CASE \( \mathcal{M}(\theta, \phi) = 0 \)

Continued fraction expansions of the form

\[
[c_0; c_1, \ldots, c_n, \overline{Q_1(k), \ldots, Q_p(k)}]_{k=1}^{\infty}
\]

are called Hurwitzian if \( c_0 \) is an integer, \( c_1, \ldots, c_n \) are positive integers, \( Q_1(k), \ldots, Q_p(k) \) are polynomials with rational coefficients which takes positive integral values for \( k = 1, 2, \ldots \) and at least one of the polynomials
is not constant. $Q_1(k), \ldots, Q_p(k)$ are said to form a quasi-period. The expansions
\[
e = \left[ 2; \frac{1}{2k}, 1 \right]_{k=1}^{\infty} \quad \text{and} \quad e^{1/s} = \left[ 1; \frac{(2k-1)s-1}{2k}, 1, 1 \right]_{k=1}^{\infty},
\]
where $s$ is a positive integer with $s \geq 2$, are well-known examples (See [3], [8], [10] e.g.). In [7] for a positive integer $s$ we have $\mathcal{M}(e^{1/s}, (e^{1/s} - 1)/2) = 0$, $\mathcal{M}(e^{1/s}, 1/2) = 1/8$ and $\mathcal{M}(e^{1/s}, 1/3) = 0$ if $s \equiv 2 \pmod{3}$; $1/18$ otherwise.

Then, what is the condition such that $\mathcal{M}(\theta, \phi) = 0$ holds? It seems that a non-constant polynomial in a quasi-periodic part influences whether $\mathcal{M}(\theta, \phi) = 0$ or not. So, we consider the cases each quasi-periodic form includes no constant.

\[
\frac{e^{1/s} - 1}{e^{1/s} + 1} = \left[ 0; \frac{(4k - 2)s}{2k+1} \right]_{k=1}^{\infty},
\]
where $s$ is a positive integer, or

\[
\frac{e^{2/s} - 1}{e^{2/s} + 1} = \left[ 0; \frac{(2k - 1)s}{2k+1} \right]_{k=1}^{\infty},
\]
where $s$ is an odd positive integer with $s \geq 3$, is one of the well-known examples (See [10] e.g.).

In any of two expansions of $\theta$ above $a_k$ is increasing and $a_k \to \infty (k \to \infty)$. So, one may be apt to conjecture that $\mathcal{M}(\theta, \phi) = 0$ for almost all of $\phi$. But, there is a case satisfying $\mathcal{M}(\theta, \phi) \neq 0$.

**Theorem 2.**

\[
\mathcal{M} \left( \frac{e^{1/s} - 1}{e^{1/s} + 1}, \frac{e^{1/s}}{e^{1/s} + 1} \right) = \frac{1}{4}.
\]

**Proof.** First, note that in the expansion of $\theta = (e^{1/s} - 1)/(e^{1/s} + 1)$
\[
a_n = (4n - 2)s \to \infty \quad (n = 1, 2, \ldots \to \infty),
\]
yielding
\[
\theta_{n-1} = \frac{1}{a_n + \theta_n} \to 0 \quad (n = 1, 2, \ldots \to \infty).
\]
It is convenient to see that
\[
q_n|D_{n-1}| = \frac{1}{1 + \theta_n q_{n-1}/q_n} \to 1 \quad (n = 1, 2, \ldots \to \infty)
\]
and
\[
q_{n-1}|D_{n-1}| = \frac{1}{(4n-2)s + \theta_n + q_{n-2}/q_{n-1}} \to 0 \quad (n = 1, 2, \ldots \to \infty).
\]
$\phi = (\theta + 1)/2 = e^{1/s}/(e^{1/s} + 1)$ is expanded as
\[
\phi = \left[ 1; \frac{(2k - 1)s}{2k}, \frac{a_k}{2} \right]_{k=1}^{\infty} = \left[ 1; \frac{a_k}{2} \right]_{k=1}^{\infty}
\]
and
\[ \phi_n = \frac{1 - \theta_n}{2} \to \frac{1}{2} \quad (n = 0, 1, 2, \ldots \to \infty). \]

For \( n = 1, 2, \ldots \)
\[ B_n = \sum_{i=1}^{n} \frac{a_i}{2} q_{i-1} = \frac{q_n + q_{n-1} - 1}{2}. \]

Hence,
\[ B_n \| B_n \theta + \phi \| = B_n \phi_n |D_{n-1}| \]
\[ = \frac{1}{2} (q_n |D_{n-1}| + q_{n-1} |D_{n-1}| - |D_{n-1}|) \phi_n \]
\[ \to \frac{1}{2} (1 + 0 - 0) \cdot \frac{1}{2} = \frac{1}{4} \quad (n \to \infty) \]

and
\[ B_n^* \| B_n^* \theta + \phi \| = (B_n - q_{n-1})(1 - \phi_n) |D_{n-1}| \]
\[ = \frac{1}{2} (q_n |D_{n-1}| - q_{n-1} |D_{n-1}| - |D_{n-1}|) (1 - \phi_n) \]
\[ \to \frac{1}{2} (1 - 0 - 0) \left( 1 - \frac{1}{2} \right) = \frac{1}{4} \quad (n \to \infty), \]
yielding that \( M_- (\theta, \phi) = 1/4. \)

Next, \( 1 - \phi = (1 - \theta)/2 = 1/(e^{1/s} + 1) \) is expanded as
\[ 1 - \phi = \delta 1; s + 1, (2k - 1)s \sum_{k=2}^{\infty} = \delta 1; a_1/2 + 1, \overline{a_k/2} \sum_{k=2}^{\infty} \]
and
\[ \phi_0 = \frac{1 + \theta_0}{2}, \quad \phi_n = \frac{1 - \theta_n}{2} \to \frac{1}{2} \quad (n = 1, 2, \ldots \to \infty). \]

For \( n = 1, 2, \ldots \)
\[ B_n = 1 + \sum_{i=1}^{n} \frac{a_i}{2} q_{i-1} = \frac{q_n + q_{n-1} + 1}{2}. \]

In a similar manner, by
\[ B_n \| B_n \theta - \phi \| \to \frac{1}{4} \quad \text{and} \quad B_n^* \| B_n^* \theta - \phi \| \to \frac{1}{4} \quad (n \to \infty) \]
one has \( M_+ (\theta, \phi) = 1/4. \) Therefore, \( M(\theta, \phi) = M_\pm (\theta, \phi) = 1/4. \)

Contrary to this result, there is, of course, a case satisfying \( M(\theta, \phi) = 0. \)

**Theorem 3.**
\[ M \left( \frac{e^{1/s} - 1}{e^{1/s} + 1}, \frac{1}{2} \right) = 0. \]
Remark. It is interesting to see that in [7]
\[ \mathcal{M} \left( e^{1/s}, \frac{e^{1/s} - 1}{2} \right) = 0 \quad \text{and} \quad \mathcal{M} \left( e^{1/s}, \frac{1}{2} \right) = \frac{1}{8} \neq 0 \]
in comparison with Theorem 2 above and this Theorem.

Proof. \( \phi = 1/2 \) is expanded as
\[
\frac{1}{2} = \sum_{k=1}^{\infty} \left[ (8k - 2)s, (4k + 1)s \right] \quad \text{and} \quad \phi_0 = 1/2, \quad \text{for } n = 1, 2, \ldots \]
and \( \phi_{2n-1} = 1 - \frac{1}{2} \theta_{2n-1} \to 1, \quad \phi_{2n} = \frac{1}{2} - \theta_{2n} \to \frac{1}{2} \quad (n \to \infty). \)

Since for \( n = 1, 2, \ldots \)
\[
B_{2n-1} = \frac{1}{2} + 1 + \sum_{i=1}^{n-1} \left( a_{2i}q_{2i-1} + \frac{1}{2}a_{2i+1}q_{2i} \right) = \frac{1}{2} q_{2n-1} + q_{2n-2},
\]
one finds that
\[
B_{2n-1} \| B_{2n-1} \theta + \phi \| = B_{2n-1} \phi_{2n-1} |D_{2n-2}|
\]
\[
= \left( \frac{1}{2} q_{2n-1} |D_{2n-2}| + q_{2n-2} |D_{2n-2}| \right) \phi_{2n-1}
\]
\[
\to \left( \frac{1}{2} \cdot 1 + 0 \right) \cdot 1 = \frac{1}{2},
\]
\[
B_{2n-1}^* \| B_{2n-1}^* \theta + \phi \| = (B_{2n-1} - q_{2n-2})(1 - \phi_{2n-1}) |D_{2n-2}|
\]
\[
= \frac{1}{2} q_{2n-1} |D_{2n-2}| (1 - \phi_{2n-1}) \to \frac{1}{2} \cdot 1 \cdot (1 - 1) = 0,
\]
\[
B_{2n} \| B_{2n} \theta + \phi \| = (B_{2n-1} + b_{2n}q_{2n-1}) \phi_{2n} |D_{2n-1}|
\]
\[
= \left( q_{2n} |D_{2n-1}| + \frac{1}{2} q_{2n-1} |D_{2n-1}| \right) \phi_{2n}
\]
\[
\to \left( 1 + \frac{1}{2} \cdot 0 \right) \cdot \frac{1}{2} = \frac{1}{2},
\]
\[
B_{2n}^* \| B_{2n}^* \theta + \phi \| = (B_{2n} - q_{2n-1})(1 - \phi_{2n}) |D_{2n-1}|
\]
\[
= \left( q_{2n} |D_{2n-1}| - \frac{1}{2} q_{2n-1} |D_{2n-1}| \right) (1 - \phi_{2n})
\]
\[
\to \left( 1 - \frac{1}{2} \cdot 0 \right) \left( 1 - \frac{1}{2} \right) = \frac{1}{2}
\]
as \( n \) tends to infinity. Therefore, we have \( \mathcal{M}(\theta, 1/2) = \mathcal{M}_{\pm}(\theta, 1/2) = 0. \) □
We shall show one more case satisfying $\mathcal{M}(\theta, \phi) = 0$.

**Theorem 4.**

$$
\mathcal{M}\left(\frac{e^{1/s} - 1}{e^{1/s} + 1}, \frac{1}{3}\right) = 0.
$$

**Proof.** When $s \equiv 0 \pmod{3}$, $\phi = 1/3$ is expanded as

$$
\frac{1}{3} = \underbrace{1; \frac{2}{3}a_1 + 1, a_{2k}, \frac{2}{3}a_{2k+1}}_{k=1}^{\infty}
$$

and $\phi_0 = 2/3$, for $n = 1, 2, \ldots$

$$
\phi_{2n-1} = 1 - \frac{2}{3}\theta_{2n-1} \to 1, \quad \phi_{2n} = \frac{2}{3} - \theta_{2n} \to \frac{2}{3} \quad (n \to \infty).
$$

Since for $n = 1, 2, \ldots$

$$
B_{2n-1} = \frac{2}{3}a_1 + 1 + \sum_{i=1}^{n-1} \left(a_{2i}q_{2i-1} + \frac{2}{3}a_{2i+1}q_{2i}\right) = \frac{2}{3}q_{2n-1} + q_{2n-2},
$$

one finds that

$$
B_{2n-1}^* B_{2n-1}^* \theta + \phi \| = (B_{2n-1} - q_{2n-2})(1 - \phi_{2n-1})|D_{2n-2}|
$$

$$
= \frac{2}{3}q_{2n-1}|D_{2n-2}|(1 - \phi_{2n-1}) \to \frac{2}{3} \cdot 1 \cdot (1 - 1) = 0,
$$

as $n$ tends to infinity. Hence, we have $\mathcal{M}_-(\theta, 1/3) = 0$.

$1 - \phi = 2/3$ is expanded as

$$
\frac{2}{3} = \underbrace{1; \frac{1}{3}a_1 + 1, a_{2k}, \frac{1}{3}a_{2k+1}}_{k=1}^{\infty}
$$

and $\phi_0 = 1/3$, for $n = 1, 2, \ldots$

$$
\phi_{2n-1} = 1 - \frac{1}{3}\theta_{2n-1} \to 1, \quad \phi_{2n} = \frac{1}{3} - \theta_{2n} \to \frac{1}{3} \quad (n \to \infty).
$$

Since for $n = 1, 2, \ldots$

$$
B_{2n-1} = \frac{1}{3}a_1 + 1 + \sum_{i=1}^{n-1} \left(a_{2i}q_{2i-1} + \frac{1}{3}a_{2i+1}q_{2i}\right) = \frac{1}{3}q_{2n-1} + q_{2n-2},
$$

one finds that

$$
B_{2n-1}^* B_{2n-1}^* \theta - \phi \| = (B_{2n-1} - q_{2n-2})(1 - \phi_{2n-1})|D_{2n-2}|
$$

$$
= \frac{1}{3}q_{2n-1}|D_{2n-2}|(1 - \phi_{2n-1}) \to \frac{1}{3} \cdot 1 \cdot (1 - 1) = 0,
$$

as $n$ tends to infinity. Hence, we have $\mathcal{M}_+(\theta, 1/3) = 0$.

Therefore, $\mathcal{M}(\theta, 1/3) = 0$.  

When $s \equiv 1 \pmod{3}$, $\phi = 1/3$ is expanded as

$$
\frac{1}{3} = \frac{1}{3}(a_1 + 1), \quad \frac{2}{3} a_{6k-4}, \quad \frac{2}{3} a_{6k-3} + \frac{1}{3}, \quad \frac{2}{3} a_{6k-2}, \quad \frac{2}{3} a_{6k-1}, \quad a_{6k}, \quad \frac{2}{3} a_{6k+1} - \frac{1}{3}_{k=1}^{\infty}
$$

and $\phi_0 = 2/3$, for $n = 1, 2, \ldots$

$$
\phi_{6n-5} = \frac{2}{3} (1 - \theta_{6n-5}) \rightarrow \frac{2}{3}, \quad \phi_{6n-4} = \frac{2}{3} (1 - \theta_{6n-4}) \rightarrow \frac{2}{3},
$$

$$
\phi_{6n-3} = 1 - \frac{2}{3} \theta_{6n-3} \rightarrow 1, \quad \phi_{6n-2} = \frac{2}{3} - \theta_{6n-2} \rightarrow \frac{2}{3},
$$

$$
\phi_{6n-1} = 1 - \frac{2}{3} \theta_{6n-1} \rightarrow 1, \quad \phi_{6n} = \frac{2}{3} - \theta_{6n} \rightarrow \frac{2}{3}
$$

as $n$ tends to infinity. Since for $n = 1, 2, \ldots$

$$
B_{6n-5} = \frac{2}{3} (a_1 + 1) + \sum_{i=1}^{n-1} \left( \frac{2}{3} a_{6i-4} q_{6i-5} + \frac{2}{3} a_{6i-3} + \frac{1}{3} q_{6i-4}
\right.

\left. + a_{6i-2} q_{6i-3} + \frac{2}{3} a_{6i-1} q_{6i-2} + a_{6i} q_{6i-1} + \frac{2}{3} a_{6i+1} - \frac{1}{3} q_{6i} \right)

= \frac{2}{3} (q_{6n-5} + q_{6n-6}),
$$

one finds that

$$
B_{6n-3}^* B_{6n-3}^* \theta + \phi = (B_{6n-3} - q_{6n-4})(1 - \phi_{6n-3})|D_{6n-6}|
$$

$$
= \frac{2}{3} q_{6n-3} |D_{6n-4}| (1 - \phi_{6n-3}) \rightarrow \frac{2}{3} \cdot 1 \cdot (1 - 1) = 0,
$$

$$
B_{6n-1}^* B_{6n-1}^* \theta + \phi = \frac{2}{3} q_{6n-1} |D_{6n-2}| (1 - \phi_{6n-1}) \rightarrow \frac{2}{3} \cdot 1 \cdot (1 - 1) = 0,
$$

as $n$ tends to infinity. Hence, we have $M_-(\theta, 1/3) = 0$.

$1 - \phi = 2/3$ is expanded as

$$
\frac{2}{3} = \frac{1}{3} (a_1 + 1), \quad \frac{1}{3} a_{6k-4}, \quad \frac{a_{6k-3} + \frac{2}{3}}{3}, \quad \frac{a_{6k-2}}{3}, \quad \frac{a_{6k-1}}{3}, \quad a_{6k}, \quad \frac{a_{6k+1} - \frac{2}{3}}{3}_{k=1}^{\infty}
$$

and $\phi_0 = 1/3$, for $n = 1, 2, \ldots$

$$
\phi_{6n-5} = \frac{1}{3} (1 - \theta_{6n-5}) \rightarrow \frac{1}{3}, \quad \phi_{6n-4} = \frac{1}{3} (1 - \theta_{6n-4}) \rightarrow \frac{1}{3},
$$

$$
\phi_{6n-3} = 1 - \frac{1}{3} \theta_{6n-3} \rightarrow 1, \quad \phi_{6n-2} = \frac{1}{3} - \theta_{6n-2} \rightarrow \frac{1}{3},
$$

$$
\phi_{6n-1} = 1 - \frac{1}{3} \theta_{6n-1} \rightarrow 1, \quad \phi_{6n} = \frac{1}{3} - \theta_{6n} \rightarrow \frac{1}{3}
$$
as $n$ tends to infinity. Since for $n = 1, 2, \ldots$

$$B_{6n-5} = \frac{1}{3} (q_{6n-5} + q_{6n-6}),$$

one finds that

$$B_{6n-3}^* B_{6n-3}^\ast \theta - \phi = \frac{1}{3} q_{6n-3} |D_{6n-4}| (1 - \phi_{6n-3}) \to \frac{1}{3} \cdot 1 \cdot (1 - 1) = 0,$$

$$B_{6n-1}^* B_{6n-1}^\ast \theta - \phi = \frac{1}{3} q_{6n-1} |D_{6n-2}| (1 - \phi_{6n-1}) \to \frac{1}{3} \cdot 1 \cdot (1 - 1) = 0,$$

as $n$ tends to infinity. Hence, we have $M_+(\theta, 1/3) = 0$.

Therefore, $M(\theta, 1/3) = 0$.

When $s \equiv 2 \pmod{3}$, $\phi = 1/3$ is expanded as

$$\frac{1}{3} \phi = d \left[ 1; \frac{2a_1 + 1}{3}, \frac{2}{3} \frac{a_{6k-4}}{a_{6k-3} + 1}, a_{6k-2}, \frac{2}{3} a_{6k-1}, a_{6k}, \frac{2}{3} (a_{6k+1} - 1) \right]_{k=1}^\infty$$

and $\phi_0 = 2/3$, for $n = 1, 2, \ldots$

$$\phi_{6n-5} = \frac{1}{3} (1 - 2\theta_{6n-5}) \to \frac{1}{3}, \quad \phi_{6n-4} = \frac{1}{3} (2 - \theta_{6n-4}) \to \frac{2}{3},$$

$$\phi_{6n-3} = 1 - \frac{2}{3} \theta_{6n-3} \to 1, \quad \phi_{6n-2} = \frac{2}{3} - \theta_{6n-2} \to \frac{2}{3},$$

$$\phi_{6n-1} = 1 - \frac{2}{3} \theta_{6n-1} \to 1, \quad \phi_{6n} = \frac{2}{3} - \theta_{6n} \to \frac{2}{3}$$

as $n$ tends to infinity. Since for $n = 1, 2, \ldots$

$$B_{6n-5} = \frac{2a_1 + 1}{3} + \sum_{i=1}^{n-1} \left( \frac{1}{3} a_{6i-4} q_{6i-5} + \frac{2}{3} (a_{6i-3} + 1) q_{6i-4} + a_{6i-2} q_{6i-3} + \frac{2}{3} a_{6i-1} q_{6i-2} + a_{6i} q_{6i-1} + \frac{2}{3} (a_{6i+1} - 1) q_{6i} \right)$$

$$= \frac{1}{3} (2q_{6n-5} + q_{6n-6}),$$

one finds that

$$B_{6n-3}^* B_{6n-3}^\ast \theta + \phi = \frac{2}{3} q_{6n-3} |D_{6n-4}| (1 - \phi_{6n-3}) \to \frac{2}{3} \cdot 1 \cdot (1 - 1) = 0,$$

$$B_{6n-1}^* B_{6n-1}^\ast \theta + \phi = \frac{2}{3} q_{6n-1} |D_{6n-2}| (1 - \phi_{6n-1}) \to \frac{2}{3} \cdot 1 \cdot (1 - 1) = 0,$$

as $n$ tends to infinity. Hence, we have $M_-(\theta, 1/3) = 0$.

$1 - \phi = 2/3$ is expanded as

$$\frac{2}{3} = d \left[ 1; \frac{a_1 + 2}{3}, \frac{2}{3} a_{6k-4}, \frac{a_{6k-3} + 1}{3}, a_{6k-2}, \frac{1}{3} a_{6k-1}, a_{6k}, \frac{a_{6k+1} - 1}{3} \right]_{k=1}^\infty$$
and $\phi_0 = 1/3$, for $n = 1, 2, \ldots$

$$\phi_{6n-5} = \frac{1}{3}(2 - \theta_{6n-5}) \rightarrow \frac{2}{3}, \quad \phi_{6n-4} = \frac{1}{3}(1 - 2\theta_{6n-4}) \rightarrow \frac{1}{3},$$

$$\phi_{6n-3} = 1 - \frac{1}{3}\theta_{6n-3} \rightarrow 1, \quad \phi_{6n-2} = \frac{1}{3} - \theta_{6n-2} \rightarrow \frac{1}{3},$$

$$\phi_{6n-1} = 1 - \frac{1}{3}\theta_{6n-1} \rightarrow 1, \quad \phi_{6n} = \frac{1}{3} - \theta_{6n} \rightarrow \frac{1}{3}$$

as $n$ tends to infinity. Since for $n = 1, 2, \ldots$

$$B_{6n-5} = \frac{1}{3}(q_{6n-5} + 2q_{6n-6}),$$

one finds that

$$B_{6n-3}^* || B_{6n-3}^* - \phi || = \frac{1}{3}q_{6n-3}|D_{6n-4}|(1 - \phi_{6n-3}) \rightarrow \frac{1}{3} \cdot 1 \cdot (1 - 1) = 0,$$

$$B_{6n-1}^* || B_{6n-1}^* - \phi || = \frac{1}{3}q_{6n-1}|D_{6n-2}|(1 - \phi_{6n-1}) \rightarrow \frac{1}{3} \cdot 1 \cdot (1 - 1) = 0,$$

as $n$ tends to infinity. Hence, we have $M_+(\theta, 1/3) = 0$. Therefore, $M(\theta, 1/3) = 0$. $\square$

4. The cases $M((e^{2/s} - 1)/(e^{2/s} + 1), \phi) = 0$

Let us calculate $M(\theta, \phi)$ when

$$\theta = \frac{e^{2/s} - 1}{e^{2/s} + 1} = [0; \frac{(2k - 1)s}{k = 1}^{\infty}],$$

where $s$ is an odd positive integer with $s \geq 3$. The situations are a little bit different from the previous results. Notice that $a_n = (2n - 1)s \rightarrow \infty$, so $\theta_{n-1} = 1/(a_n + \theta_n) \rightarrow 0$ ($n = 1, 2, \ldots \rightarrow \infty$). $\lim_{n \rightarrow \infty} q_n|D_{n-1}| = 1$ and $\lim_{n \rightarrow \infty} q_{n-1}|D_{n-1}| = 0$ hold for this $\theta$ too. The first result is quite different from Theorem 2.

Theorem 5.

$$M\left(\frac{e^{2/s} - 1}{e^{2/s} + 1}, \frac{e^{2/s}}{e^{2/s} + 1}\right) = 0.$$

Proof. $\phi = (\theta + 1)/2 = e^{2/s}/(e^{2/s} + 1)$ is expanded as

$$\phi = d[1; \frac{(6k-5)s+1}{2}, \frac{(6k-3)s}{2}, \frac{(6k-1)s-1}{2}]_{k=1}^{\infty},$$

$$= d[1; \frac{a_{3k-2}+1}{2}, a_{3k-1}, \frac{a_{3k-1}}{2}]_{k=1}^{\infty}.$$
and
\[ \phi_{3n} = \frac{1 - \theta_{3n}}{2} \to \frac{1}{2} \quad (n = 0, 1, 2, \ldots \to \infty), \]
\[ \phi_{3n-2} = 1 - \frac{1}{2} \theta_{3n-2} \to 1, \quad \phi_{3n-1} = \frac{1}{2} - \theta_{3n} \to \frac{1}{2} \quad (n = 1, 2, \ldots \to \infty). \]

Since for \( n = 1, 2, \ldots \)
\[ B_{3n} = \sum_{i=1}^{n} \left( \frac{a_{3i-2} + 1}{2} q_{3i-3} + a_{3i-1} q_{3i-2} + \frac{a_{3i-1} - 1}{2} q_{3i-1} \right) \]
\[ = \frac{1}{2} (q_{3n} + q_{3n-1} - 1). \]

one finds that
\[ B_{3n+1}^* B_{3n+1}^* \theta + \phi \| = (B_{3n+1} - q_{3n})(1 - \phi_{3n+1})|D_{3n}| \]
\[ = \frac{1}{2} (q_{3n+1} |D_{3n}| - |D_{3n}|)(1 - \phi_{3n+1}) \]
\[ \to \frac{1}{2} (1 - 0)(1 - 1) = 0 \quad (n \to \infty), \]
yielding \( M_-(\theta, \phi) = 0. \)

Next, \( 1 - \phi = (1 - \theta)/2 = 1/(e^{2/s} + 1) \) is expanded as
\[ 1 - \phi = \left[ 1; \frac{a_1 + 3}{2}, \frac{a_{3k-1}}{2}, \frac{a_{3k}}{2}, \frac{a_{3k+1} + 1}{2} \right]_{k=2}^{\infty} \]
and
\[ \phi_0 = \frac{1 + \theta_0}{2}, \quad \phi_{3n-2} = 1 - \frac{1}{2} \theta_{3n-2} \to 1, \]
\[ \phi_{3n-1} = \frac{1}{2} - \theta_{3n-1}, \quad \phi_{3n} = \frac{1}{2}(1 - \theta_{3n}) \to \frac{1}{2} \quad (n = 1, 2, \ldots \to \infty). \]

Since for \( n = 1, 2, \ldots \)
\[ B_{3n-2} = \frac{1}{2} q_{3n-2} + q_{3n-3} + \frac{1}{2}. \]

one finds that
\[ B_{3n-2}^* B_{3n-2}^* \theta - \phi \| = \frac{1}{2} (q_{3n-2} |D_{3n-3}| + |D_{3n-3}|)(1 - \phi_{3n-2}) \]
\[ \to \frac{1}{2} (1 + 0)(1 - 1) = 0 \quad (n \to \infty), \]
yielding \( M_+(\theta, \phi) = 0. \) Therefore, \( M(\theta, \phi) = M_+(\theta, \phi) = 0. \)

\( \square \)

**Theorem 6.**
\[ M_0 \left( \frac{e^{2/s} - 1}{e^{2/s} + 1}, \frac{1}{2} \right) = 0. \]
Proof. $\phi = 1/2$ is expanded as
$$\frac{1}{2} = \{ 1; \frac{a_1+1}{2}, \frac{a_{3k-1}+1}{2}, a_{3k}, \frac{a_{3k+1}-1}{2} \}_k^{\infty}$$
and
$$\phi_0 = \frac{1}{2}, \quad \phi_{3n-2} = \frac{1}{2}(1 - \theta_{3n-2}) \rightarrow \frac{1}{2},$$
$$\phi_{3n-1} = 1 - \frac{1}{2}\theta_{3n-1} \rightarrow 1, \quad \phi_{3n} = \frac{1}{2} - \theta_{3n} \rightarrow \frac{1}{2} \quad (n = 1, 2, \ldots \rightarrow \infty).$$
Since for $n = 1, 2, \ldots$
$$B_{3n-2} = \frac{a_1+1}{2} + \sum_{i=1}^{n-1} \left( \frac{a_{3i-1}+1}{2}q_{3i-2} + a_{3i}q_{3i-1} + \frac{a_{3i+1}-1}{2}q_{3i} \right),$$
one finds that
$$B_{3n-1}^* B_{3n-2}^* \theta + \phi = (B_{3n-1} - q_{3n-2})(1 - \phi_{3n-1})|D_{3n-2}|$$
$$= \frac{1}{2}q_{3n-1}|D_{3n-2}|(1 - \phi_{3n-1}) \rightarrow \frac{1}{2} \cdot 1 \cdot (1 - 1) = 0$$
as $n$ tends to infinity. Therefore, we have $M(\theta, 1/2) = M_{\pm}(\theta, 1/2) = 0$. \hfill \Box

Theorem 7.
$$M \left( \frac{e^{2/3} - 1}{e^{2/3} + 1}, \frac{1}{3} \right) = 0.$$  

Proof. When $s \equiv 3, s \equiv 5, s \equiv 1 \pmod{6}$, the situation is completely the same as the case of
$$\theta = \frac{e^{1/3} - 1}{e^{1/3} + 1}$$
with $s \equiv 0, s \equiv 1, s \equiv 2 \pmod{3}$, respectively. \hfill \Box

5. Some conditions satisfying $M(\theta, \phi) = 0$

We have already seen several examples so that $M(\theta, \phi) = 0$ holds. Then, what is the condition of $M(\theta, \phi) = 0$? Of course, the following is clear.

Theorem 8. If $\phi_n \rightarrow 0$ or $\phi_n \rightarrow 1 \quad (n \rightarrow \infty)$ for infinitely many positive integers $n$, then $M(\theta, \phi) = 0$.

Proof. First, we shall show that $\theta_n - 1 < B_n|D_{n-1}| < 4$ for any positive integer $n$. Since
$$B_n = \sum_{i=1}^{n} b_i q_{i-1} \leq \sum_{i=1}^{n} (a_i + 1)q_{i-1} = q_n + 2q_{n-1} + (q_{n-2} + \cdots + q_1) < 4q_n,$$
we obtain
\[ B_n |D_{n-1}| < \frac{4q_n}{q_n + \theta_n q_{n-1}} < 4. \]

On the other hand,
\[ B_n |D_{n-1}| \geq \frac{\sum_{i=1}^{n} q_{i-1}}{q_n + \theta_n q_{n-1}} > \frac{1}{a_n + \theta_n} = \theta_{n-1}. \]

If \( \phi_n \to 0 \quad (n \to \infty) \), then
\[ B_n \| B_n \theta + \phi \| = B_n |D_{n-1}| \phi_n \to 0 \quad (n \to \infty). \]

If \( \phi_n \to 1 \quad (n \to \infty) \), then
\[ B_n^* \| B_n^* \theta + \phi \| = B_n^* |D_{n-1}| (1 - \phi_n) \to 0 \quad (n \to \infty). \]

\[ \square \]

**Corollary.** When \( b_n = 1 \), \( \phi_{n-1} \to 0 \) if and only if \( \phi_n \to 1 \quad (n \to \infty) \).

This is very generous. So, we state the following.

**Theorem 9.** If \( |a_n - b_n| \leq c \) and \( a_n \to \infty \quad (n \to \infty) \) for infinitely many positive integers \( n \), then \( \mathcal{M}(\theta, \phi) = 0 \). Here, \( c \) is a constant not depending upon \( n \).

**Remark.** In fact, \( a_n = b_n \to \infty \quad (n \to \infty) \) holds in all previous theorems above implying \( \mathcal{M}(\theta, \phi) = 0 \).

**Proof.** If \( |a_n - b_n| \leq c \), then \( \frac{1}{\theta_{n-1}} - \frac{\phi_{n-1}}{\theta_{n-1}} < c + 2 \) or \( 0 < 1 - \phi_{n-1} < (c + 2)\theta_{n-1} \). And if \( \lim_{n \to \infty} a_n = \infty \), then
\[ \theta_{n-1} = \frac{1}{a_n + \theta_n} \to 0 \quad (n \to \infty). \]

Thus, \( 1 - \phi_{n-1} \to 0 \quad (n \to \infty) \) entails that
\[ B_{n-1}^* \| B_{n-1}^* \theta + \phi \| = B_{n-1}^* |D_{n-2}| (1 - \phi_{n-1}) \to 0 \quad (n \to \infty). \]

\[ \square \]

**References**


Takao KOMATSU
Faculty of Education
Mie University
Mie, 514-8507
Japan
E-mail: komatsu@edu.mie-u.ac.jp