NIKOLAI G. MOSHCHEVITIN

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Continued Fractions, Multidimensional Diophantine Approximations and Applications

par NIKOLAI G. MOSHCHEVITIN

RÉSUMÉ. Cet article rassemble des résultats généraux d'approximation diophantienne, sur les meilleures approximations et leurs applications à la théorie de répartition uniforme.

ABSTRACT. This paper is a brief review of some general Diophantine results, best approximations and their applications to the theory of uniform distribution.

1. DIOPHANTINE APPROXIMATIONS.

1.1. One-dimensional approximations.

1.1.1. *Lagrange spectrum.* Let α be an irrational number. Dirichlet's theorem states that there are infinitely many positive integers q such that

$$\|q\alpha\| < \frac{1}{q}$$

holds, where $\|\cdot\|$ denotes the distance to the nearest integer. Hurwitz obtained a more precise result: for any irrational number α , the inequality

$$\|q\alpha\| < \frac{1}{\sqrt{5}q}$$

has infinitely many solutions in q . Moreover, there is a countable set of numbers α for which this inequality is an exact one, that is, for any positive ε there are only finitely positive integers q such that the inequality

$$\|q\alpha\| < \left(\frac{1}{\sqrt{5}} - \varepsilon\right) \frac{1}{q}$$

holds.

We define the *Lagrange spectrum* to be the set of the real numbers λ for which there exists $\alpha = \alpha(\lambda)$ such that the inequality

$$\|q\alpha\| < \lambda \frac{1}{q}$$

has infinitely many solutions, and for any positive ε the inequality

$$||q\alpha|| < (\lambda - \varepsilon) \frac{1}{q}$$

has only a finite number of solutions. It is well-known that Lagrange spectrum has a discrete part

$$\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{8}}, \dots,$$

and the minimal λ for which there are uncountably many $\alpha = \alpha(\lambda)$ is $\lambda = 1/3$. Also it is well-known that Lagrange spectrum contains an interval $[0, \lambda^*]$.

Moreover, for any decreasing function ψ satisfying $\psi(y) = o(y^{-1})$, as y tends to infinity, there is an uncountable set of real numbers α such that the inequality

$$||q\alpha|| < \psi(q)$$

has infinitely many solutions, but for any $\varepsilon > 0$, the stronger inequality

$$||q\alpha|| < (1 - \varepsilon)\psi(q)$$

has only a finite number of solutions.

One can find the above results in [5]. All of them can be obtained from the continued fraction expansion [14].

1.1.2. *Best approximations and continued fractions.* Any real number α may be written as

$$\alpha = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \dots}}}$$

where $b_0 \in \mathbb{Z}$ and, for $j > 0$, b_j are a nonnegative integers. For convenience, we use the notation

$$\alpha = [b_0; b_1, b_2, b_3, \dots].$$

This representation is infinite and unique when α is irrational. If α is rational, we have $\alpha = [b_0; b_1, b_2, b_3, \dots, b_t]$, and this representation is unique if we impose the condition $b_t \neq 0, 1$.

Convergents to α of the order ν are defined as

$$\frac{p_\nu}{q_\nu} = [b_0; b_1, b_2, b_3, \dots, b_\nu]$$

A simple theorem states that these fraction and only these form *the best approximations*, that is the relation

$$||q_\nu \alpha|| = \min_{q < q_\nu} ||q\alpha||$$

holds for the denominators q_ν and only for them (see [14]). We now give two other easy facts.

Theorem 1. *We have*

$$\|q_\nu \alpha\| \asymp (q_{\nu+1})^{-1}, \quad (\text{in order of approximation}).$$

Proposition 2. *We have*

$$\Delta_\nu = \begin{vmatrix} p_\nu & q_\nu \\ p_{\nu+1} & q_{\nu+1} \end{vmatrix} = \pm 1.$$

1.1.3. *Klein polygons.* We now consider the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Let $(q, a) \in \mathbb{Z}^2$ be a primitive point ($\gcd(q, a) = 1$) and $q, a > 0$. We define the two angles φ_+ and φ_- by

$$\begin{aligned} \varphi_+ &= \left\{ Z = (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq \frac{a}{q}x \right\}, \\ \varphi_- &= \left\{ Z = (x, y) \in \mathbb{R}^2 : y \geq 0, y \leq \frac{a}{q}x \right\}, \end{aligned}$$

Klein polygons $\mathcal{K}_+(a, q)$ and $\mathcal{K}_-(a, q)$ are defined to be respectively the following borders

$$\partial(\text{conv}(\varphi_+ \cap (\mathbb{Z}^2 \setminus \{0\})))$$

and

$$\partial(\text{conv}(\varphi_- \cap (\mathbb{Z}^2 \setminus \{0\})))$$

which consist of finite (nontrivial) intervals.

We now define $\Delta(a, q)$ to be the domain:

$$\begin{aligned} \Delta(a, q) &= \left\{ Z = (x, y) \in \mathbb{R}^2 : x > 0, y > 0, \right. \\ &\quad \left. Z \notin \text{conv}(\varphi_+ \cap (\mathbb{Z}^2 \setminus \{0\})), \quad Z \notin \text{conv}(\varphi_- \cap (\mathbb{Z}^2 \setminus \{0\})) \right\}. \end{aligned}$$

We have

Theorem 3 ([7, 9]). 1. *The vertices of $\mathcal{K}_-(a, q)$, (different from (q, a)) are integer points of the form $(q_{2\mu}, p_{2\mu})$, where $(p_{2\mu}/q_{2\mu})$ is the 2μ -th convergent to a/q .*

2. *The vertices of $\mathcal{K}_+(a, q)$ (different from (q, a)) are integer points of the form $(q_{2\nu+1}, p_{2\nu+1})$, where $p_{2\nu+1}/q_{2\nu+1}$ is the $(2\nu+1)$ -th convergent to a/q .*

3. *If $(u, v) \in (\mathcal{K}_+(a, q) \cup \mathcal{K}_-(a, q)) \cap \mathbb{Z}^2$ is an integer point then v/u is a convergent to a/q or one of the intermediate fractions $(wp_\nu + p_{\nu-1})/(wq_\nu + q_{\nu-1})$, $1 \leq w \leq b_{\nu+1}$.*

4. $\Delta(a, q) \cap \mathbb{Z}^2 = \emptyset$.

One can easily verify the same results for infinite continued fractions (i.e. for irrational numbers).

Recently, several papers [1, 21, 45, 46, 27] devoted to multidimensional generalization of Klein polygons have appeared. Unfortunately one must notice that there is something incorrect in papers [45, 46].

1.1.4. *Representation of rationals.* The rationals a/q with bounded partial quotients b_j are of great interest (see [22, 23, 24, 11]).

Let $N(k, q)$ be the number of integers A , $1 \leq A < q$, $\gcd(A, q) = 1$ such that any component b_i of the continued fraction expansion

$$\frac{A}{q} = [0; b_1, \dots, b_{n(A)}]$$

is bounded by k : $b_i \leq k$, $i = 1, \dots, n(A)$. It is known ([22, 4, 52]) that if $k > \gamma \log q$ with γ sufficiently large, then $N(k, q) \geq 1$. Moreover we can show that for *almost all* positive integers q and A with $1 \leq A < q$, all partial quotients are bounded by $O(\log q)$.

By the way we may recall a famous and still open conjecture which asserts that for any $q \geq 1$, we have $N(6, q) \geq 1$. However it is known that the conjecture holds when $q = 2^\alpha$ or $q = 3^\alpha$ ([39]).

Sergei Konyagin (see [17]), by means of Farey fractions, proved the following upper bound for $N(k, q)$:

Theorem 4. *For any $\gamma < 1$ and for any $k \geq k(\gamma)$ we have*

$$N(k, q) \ll \varphi(q) q^{-\frac{\gamma}{k \log k}},$$

where φ denotes the Euler function.

We define the sequence $A_1 < A_2 < \dots < A_d$ to be an *almost arithmetic progression* if

$$\exists w > 1 : w \leq A_{j+1} - A_j \leq 3w, \quad j = 1, \dots, d-1.$$

In [32], the author shows that numbers with bounded partial quotients cannot appear very regularly: they cannot form long almost arithmetic progressions. The following theorem improves the result from [32].

Theorem 5. *For $d \geq 3$, let $A_0, \dots, A_d$ be positive integers. Suppose*

- (i) $0 < A_0 < \dots < A_d$ form an almost arithmetic progression;
- (ii) $\gcd(A_i, q) = 1$, $i = 0, \dots, d$.

Let $A_\nu/q = [b_{\nu,1}, \dots, b_{\nu,s(\nu)}]$. Then there exist ν_0 and μ_0 such that

$$0 \leq \nu_0 \leq d, \quad 1 \leq \mu_0 \leq s(\nu_0)$$

and

$$b_{\nu_0, \mu_0} \gg d^{1/2}.$$

Theorem 5 is proved by means of Klein polygons. The same result is true for real-valued (not integer) almost arithmetic progressions and in the last case S. Konyagin showed that the result for real-valued progressions is exact in order.

1.2. Simultaneous approximations. Let $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive and real-valued function. For given $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$, a positive integer p is said to be a ψ -approximation of α , if

$$\max_{j=1, \dots, s} \|p\alpha_j\| = \max_{j=1, \dots, s} \min_{a \in \mathbb{Z}} |p\alpha_j - a| \leq \psi(p).$$

1.2.1. Dirichlet and Liouville's theorems. Dirichlet's theorem states that for any $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$, where $1, \alpha_1, \dots, \alpha_s$ linearly independent over $\mathbb{Z}$, there are infinitely many ψ -approximations of α with $\psi(y) = y^{-1/s}$.

On the other hand, Liouville's theorem ([2], ch.5) shows that for any $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that $1, \alpha_1, \dots, \alpha_s$ form a basis of a real algebraic field of degree $s + 1$, there exists $C(\alpha)$ such that

$$\max_{j=1, \dots, s} \|p\alpha_j\| \geq C(\alpha)p^{-1/s}, \quad \forall p \in \mathbb{N}.$$

One can see that there are only *countably many* algebraic $\alpha = (\alpha_1, \dots, \alpha_s)$.

1.2.2. Theorem by Cassels and Davenport and the result by Jarnik. In [3, 6] the following result is obtained.

Theorem 6. *There exists a constant C_s for which there exists an uncountable set of elements $\alpha \in \mathbb{R}^s$ which do not have any ψ -approximation where $\psi(y) = C_s y^{-1/s}$.*

V. Jarnik [12, 13] proved another result:

Theorem 7. *Let ψ and λ be positive real-valued functions such that $\psi(y)y^{1/s}$ decreases as $y \rightarrow \infty$ and $\lambda(y) \rightarrow 0$ as $y \rightarrow \infty$. Then there exists an uncountable set of elements $\alpha \in \mathbb{R}^s$ for which there are infinitely many ψ -approximations, but only finitely many $\psi\lambda$ -approximations.*

A review of other results can be found in [43, 10, 2].

1.2.3. Exact results in terms of the order of approximation. Generalizing the work [3] by means of chains of parallelepipeds [28, 50, 7, 8] we improve Jarnik's result.

Theorem 8. *For $y \geq 1$, let ψ and ω such that Let $\psi(y) = y^{-1/s}\omega(y)$, where $\omega(y)$ decreases as $y \rightarrow \infty$ and*

$$\omega(1) \leq 2^{-(s+1)(s+2)}(s!)^{-1/s}.$$

Then there exists a vector $\alpha = (\alpha_1, \dots, \alpha_s)$ which has infinitely many ψ -approximations but not any $2^{-(s+3)}\psi$ -approximation.

Theorem 9. *Let ω and ψ be as in Theorem 8 and suppose that*

$$\omega(1) \leq 2^{-(s+1)(s+3)}(s!)^{-1/s}.$$

Then there exists an uncountable set of vectors $\alpha = (\alpha_1, \dots, \alpha_s)$, each of them having infinitely many ψ -approximations but not any $2^{-(s+3)}\psi$ -approximation.

It follows that in Cassels Theorem 6 we may put

$$C_s = 2^{-(s+2)(s+3)}(s!)^{-1/s}.$$

We say that $\alpha = (\alpha_1, \dots, \alpha_s)$ satisfies the ψ -condition if α has infinitely many ψ -approximations but not any $c\psi$ -approximation for some $c = c(\alpha)$

Theorem 10. *Let ψ be defined by $\psi(y) = y^{-1/s}\omega(y)$ where ω is decreasing positive function. Then in any Jordan s -dimensional domain Ω with $\text{Vol } \Omega > 0$, there exists an uncountable set of $\alpha \in \mathbb{R}^s$ satisfying the ψ -condition.*

Theorems 8, 9, 10 are discussed in [33].

1.2.4. Successive best approximations. Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$. We define a *best simultaneous approximation* (b.a) of α to be any integer point $\zeta = (p, a_1, \dots, a_s) \in \mathbb{Z}^{s+1}$ such that $\forall q, \forall (b_1, \dots, b_s) \in \mathbb{Z}^s, 1 \leq q \leq p, (q, b_1, \dots, b_s) \neq (p, a_1, \dots, a_s)$, we have

$$D(\zeta) = \max_{j=1, \dots, s} |p\alpha_j - a_j| < \max_{j=1, \dots, s} |q\alpha_j - b_j|.$$

Let $\alpha_j \notin \mathbb{Q}, j = 1, \dots, s$. Then all b.a. of α form infinite sequences

$$\zeta^\nu = (p^\nu, a_1^\nu, \dots, a_s^\nu), \quad \nu = 1, 2, \dots$$

where $p^1 < p^2 < \dots < p^\nu < p^{\nu+1} < \dots$ and

$$D(\zeta^1) > D(\zeta^2) > \dots > D(\zeta^\nu) > D(\zeta^{\nu+1}) > \dots$$

Let

$$M_\nu[\alpha] = \begin{pmatrix} p^\nu & a_1^\nu & \dots & a_s^\nu \\ \dots & \dots & \dots & \dots \\ p^{\nu+s} & a_1^{\nu+s} & \dots & a_s^{\nu+s} \end{pmatrix}.$$

For $\alpha = (\alpha_1, \dots, \alpha_s)$ satisfying $\alpha_j \notin \mathbb{Q}, j = 1, \dots, s$, we define $R(\alpha) \in [2, s+1]$ to be the integer

$$R(\alpha) = \min \{n : \text{there exist a lattice } \Lambda \subseteq \mathbb{Z}^{s+1} \text{ with } \dim \Lambda = n \text{ and a natural } \nu_0 \text{ such that } \zeta^\nu \in \Lambda, \forall \nu > \nu_0\}.$$

Proposition 11. *Let $s = 1$. Then for any $\nu \geq 1$ we have $\det M_\nu[\alpha] = \pm 1$ ($\text{rank } M_\nu[\alpha] = 2, \forall \nu$).*

Proposition 12. *For any $s \geq 1$ we have $R(\alpha) = \dim_{\mathbb{Z}} (\alpha_1, \dots, \alpha_s, 1)$.*

Proposition 13. *Let $s = 2$ and α_1, α_2 such that $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Z}$. Then for infinitely many ν we have*

$$\text{rank } M_\nu[\alpha] = 3 = \dim_{\mathbb{Z}}(\alpha_1, \alpha_2, 1).$$

Proposition 11 – 13 can be easily verified. The following result is proved in [36].

Theorem 14. *Let $s \geq 3$. There exists an uncountable set of elements $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that*

(i) $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Z}$,

(thus $\dim_{\mathbb{Z}}(\alpha_1, \dots, \alpha_s, 1) = s + 1$),

and

(ii) $\text{rank } M_\nu[\alpha] \leq 3, \forall \nu \geq 1$,

(Hence for all $\nu \geq 1$ we have $\det M_\nu[\alpha] = 0$).

Theorem 14 represents a counterexample to the conjecture of J.S. Lagarias [26]. It shows that the successive b.a. have no such an asymptotic property as a reader can see in Proposition 12. The idea of the proof was suggested to the author by N.P. Dolbilin.

1.3. Linear forms. Again, let $\alpha_1, \dots, \alpha_s$ be real numbers such that $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Z}$, and put $\alpha = (\alpha_1, \dots, \alpha_s)$.

For $m = (m_0, m_1, \dots, m_s) \in \mathbb{Z}^{s+1} \setminus \{0\}$ we define

$$\zeta(m) = m_0 + m_1\alpha_1 + \dots + m_s\alpha_s, \quad M = \max_{j=0,1,\dots,s} |m_j|.$$

A vector $m \in \mathbb{Z}^{s+1} \setminus \{0\}$ is a *best approximation of α in sense of linear form* if

$$\zeta(m) = \min_{\substack{n \in \mathbb{Z}^{s+1} \setminus \{0\} \\ \max_j |n_j| \leq M}} |\zeta(n)|.$$

All best approximations form sequences

$$\zeta_1 > \zeta_2 > \dots > \zeta_\nu > \zeta_{\nu+1} > \dots,$$

$$M_1 < M_2 < \dots < M_\nu < M_{\nu+1} < \dots$$

where $m_\nu = (m_{0,\nu}, \dots, m_{s,\nu})$ is the vector of the ν -th b.a., $\zeta_\nu = \zeta(m_\nu)$ and $M_\nu = \max_j |m_{j,\nu}|$.

By Minkowski's Theorem we have $\zeta_\nu M_{\nu+1}^s \leq 1$.

1.3.1. Singular systems. The theorem on the order of approximations from §1.1.2 does not admit multidimensional generalization in the sense of linear form.

Theorem 15 (see [29, 35]). *Let s be an integer ≥ 1 and ψ a function such that $\psi(y)$ decreases to zero when y tends to infinity. Then there exists an uncountable set of elements $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that*

(i) $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Z}$,

and

(ii) the sequence of the best approximations of α satisfies

$$\zeta_\nu \leq \psi(M_{\nu+s-1}).$$

In the case $s = 1$ this theorem means that there are real numbers with any given order of the best approximations. In higher dimensions it gives something more.

Khinchin [15] defined a vector $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ to be a ψ -singular system if for any $T > 0$ the system

$$\|m_1\alpha_1 + \dots + m_s\alpha_s\| < \psi(T), \quad M = \max_{1 \leq j \leq s} |m_j| < T$$

has a nontrivial solution $(m_1, m_2, \dots, m_s) \in \mathbb{Z}^s$.

Proposition 16. *System is ψ -singular $\iff \zeta_\nu < \psi(M_{\nu+1})$, $\forall \nu$.*

1.3.2. *Successive best approximations for linear form.* Here we define Δ_ν^s to be the determinant of the successive best approximations

$$\Delta_\nu^s = \begin{vmatrix} m_{0,\nu} & m_{1,\nu} & \dots & m_{s,\nu} \\ \dots & \dots & \dots & \dots \\ m_{0,\nu+s} & m_{1,\nu+s} & \dots & m_{s,\nu+s} \end{vmatrix}.$$

The proposition below follows from Minkowski theorem on convex body. It seems to me that it is a well-known fact, but I could not find the corresponding reference.

Proposition 17. *Let $s = 2$. Then for infinitely many ν we have $\Delta_\nu^2 \neq 0$.*

The theorem below was proved by the author in [35] by means of singular systems.

Theorem 18. *Lets $s \geq 3$. Then there exists a uncountable set of vectors $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{R}^s$ such that*

(i) $1, \alpha_1, \dots, \alpha_s$ are linearly independent over $\mathbb{Z}$,

and

(ii) there exists a linear subspace $\mathcal{L}_\alpha \subset \mathbb{R}^{s+1}$, $\dim \mathcal{L}_\alpha = 3$ satisfying the condition

$$m_\nu \in \mathcal{L}_\alpha, \quad \forall \nu > \nu_0.$$

We see that for $s \geq 3$ almost all best approximations may asymptotically lie in a three-dimensional plane but they cannot lie in two-dimensional plane. Of course these examples are degenerated in sense of measure. For almost all vectors $\alpha \in \mathbb{R}^s$ (in the sense of Lebesgue) best approximations are asymptotically $(s + 1)$ -dimensional.

2. INTEGRALS FROM QUASIPERIODIC FUNCTIONS

In the text below we discuss applications of diophantine results to certain problems of uniform distribution of irrational rotations on torus. A full review of the methods and results of the theory of uniform distribution is given in [25].

2.1. Uniform distribution. Let $\mathbf{T}$ be the one-dimensional torus and $f : \mathbf{T}^s \rightarrow \mathbb{R}$ defined by the series

$$f(x_1, \dots, x_s) = \sum_{\substack{m \in \mathbb{Z}^s \\ m \neq 0}} f_m \exp(2\pi i(m_1 x_1 + \dots + m_s x_s)).$$

We also define the integral

$$I(T, \varphi) = I_{f, \omega}(T, \varphi) = \int_0^T f(\omega_1 t + \varphi_1, \dots, \omega_s t + \varphi_s) dt.$$

where $\omega_1, \dots, \omega_s \in \mathbb{R}$ are linearly independent over $\mathbb{Z}$, and $\varphi = (\varphi_1, \dots, \varphi_s) \in \mathbb{R}^s$.

H. Weyl [51] proved that if f is a continuous function, then for any φ we have $I(T, \varphi) = o(T)$, $T \rightarrow \infty$. This equality holds uniformly in φ if we suppose moreover that f is smooth.

V.V. Kozlov conjectured that the integral $I(T, \varphi)$ is *recurrent* that is the following condition holds:

$$(*) \quad \forall \varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |I(T^*, \varphi)| < \varepsilon.$$

This conjecture is true when f is any trigonometric polynomial, and in this case $(*)$ holds uniformly in φ . This implies that for any trigonometric polynomial f of finite degree, J^∞ defined by

$$J^\infty(T) = J_{f, \omega}^\infty(T) = \sup_{\varphi \in \mathbb{R}^s} |I(T, \varphi)|,$$

is itself recurrent, that is

$$(\$) \quad \forall \varepsilon > 0, \quad \forall T, \quad \exists T^* > T : |J^\infty(T^*)| < \varepsilon.$$

2.2. Case $s = 2$. In the two-dimensional case, the conjecture above was proved by V.V. Kozlov himself for functions $f \in C^2(\mathbf{T}^2)$ in [18] (see also [19]). It is also easy to see that when f is a smooth function, then $(*)$ holds uniformly in φ , that is $(\$)$ is true. E.A. Sidorov [44] obtained a similar result for “absolutely” continuous f .

2.3. The general result. The author [34] proved the conjecture in the general case:

Theorem 19. *Suppose that*

$$f(x_1, \dots, x_s) = \sum_{\substack{m \in \mathbb{Z}^s \\ m \neq 0}} f_m \exp(2\pi i(m_1 x_1 + \dots + m_s x_s))$$

belongs to the class $C^d(\mathbf{T}^s)$, where $d > Cs^{s^3}$ and $\omega_1, \dots, \omega_s$ are linearly independent over $\mathbb{Z}$. Then for any φ , the integral $I(T, \varphi)$ satisfies ().*

The proof is based on consideration of best approximations in the sense of linear form (see §1.3.2.).

2.4. Metric results. It is known [49, 48] that for almost all (in the sense of Lebesgue) vectors $\omega = (\omega_1, \dots, \omega_s) \in \mathbb{R}^s$, if f is smooth enough, then the integral $I(T, \varphi)$ is bounded when $T \rightarrow \infty$ uniformly in φ . Hence the integral $I(T, \varphi)$ satisfies (*), uniformly in φ . But even in the case $s \geq 3$, this result is not universal.

Let Φ be a decreasing function and assume that the series $\sum_{m \in \mathbb{Z}} \Phi(m)$ converges. We define a periodic function $\Theta : \mathbf{T}^s \rightarrow \mathbb{R}$ to be of the type Φ if, the coefficients $\Theta_{m_1, \dots, m_s}$ in the expansion

$$\Theta(x_1, \dots, x_s) = \sum \Theta_{m_1, \dots, m_s} e^{2\pi i m_1 x_1 + \dots + m_s x_s},$$

satisfy

$$|\Theta_{m_1, \dots, m_s}| \leq \Phi(M), \quad \text{where } M = \max_j |m_j|.$$

We consider

$$J^\infty(T) = J_{f, \omega}^\infty(T) = \max_{\varphi \in \mathbf{T}^s} |I_{f, \omega}(T, \varphi)|;$$

$$J^2(T) = J_{f, \omega}^2(T) = \left(\int_{\mathbf{T}^s} |I_{f, \omega}(T, \varphi)|^2 d\varphi \right)^{1/2}.$$

The result below is proved in [29].

Theorem 20. *Let $s \geq 3$. Then for any function Φ which decreases to zero as $y \rightarrow \infty$ and for any function ψ with $\psi(y) = o(1)$ as $y \rightarrow \infty$, there exist $\omega_1, \omega_2, \dots, \omega_s$ which are linearly independent over $\mathbb{Z}$, and a function f of type Φ such that $\int_{\mathbf{T}^s} f(x) dx = 0$ and*

$$J^l(T) \gg T\psi(T) \quad \forall T, \quad l = 2, \infty.$$

We will reformulate Theorem 20 in the following way.

Theorem 21. *Let $s \geq 3$, and $f : \mathbf{T}^s \rightarrow \mathbb{R}$ be smooth with zero mean value. Assume that in the expansion*

$$f(x_1, \dots, x_s) = \sum_{(m_1, \dots, m_s) \neq 0} f_{m_1, \dots, m_s} e^{2\pi i(m_1 x_1 + \dots + m_s x_s)}$$

the coefficients $f_{m_1, \dots, m_s}$, where $(m_1, \dots, m_s) \neq 0$, are all different from zero.

Then there exist $\omega_1, \omega_2, \dots, \omega_s$ which are linearly independent over $\mathbb{Z}$ such that

$$J^l(t) \gg t\psi(t) \quad \forall t; \quad l = 2, \infty.$$

An improvement of the latter result was obtained recently by E.V. Kolo-meikina [20].

One can see that the behaviour of integrals J_l in two-dimensional case radically differs from the case $s \geq 3$.

2.5. Odd functions. Sergei Konyagin's result. Recently, S. Konyagin [16] obtained the following result.

Theorem 22. *The Kozlov's conjecture is true (that is $(*)$ holds) for arbitrary $s \geq 1$ and any function f satisfying the condition*

$$f(-x_1, \dots, -x_s) = f(x_1, \dots, x_s), \quad f \in C^\tau(\mathbf{T}^s), \quad \tau \asymp s2^s.$$

2.6. The smoothness. In [42],[41] it is shown that we need some kind of smoothness conditions on f to insure that $(*)$ is true : indeed in the two-dimensional case ($s = 2$), there exists a function $f : \mathbf{T}^2 \rightarrow \mathbb{R}$ (with zero mean value) of the class $C \setminus C^1(\mathbf{T}^2)$ such that $I(T, 0)$ tends to infinity when $T \rightarrow \infty$ (with the choice $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$). On the other hand, in [44] it is shown that when $s = 2$, a sufficient condition on f for having $(*)$, is f to be absolutely continuous.

Developping an idea of D.V. Treshchev, the author, in [31], generalized Poincaré's example. He proved that for any real $\omega_1, \dots, \omega_s$ which form a basis of a real algebraic field, there exists a function $f \in C^{s-2}(T^s) \setminus C^{s-1}(T^s)$ such that $I(., 0)$ does not satisfy the property $(*)$ with $\varphi = 0$.

One may find some results on algebraic numbers in [40] and [38]. Recently, S.V. Konyagin [16] proved that for some Liouville transcendental numbers, there exists $f \in C^d(\mathbf{T}^s)$ with $d \asymp 2^s/s$ such that $(*)$ is not satisfied.

Some early results are reviewed in [37].

2.7. Vector-functions: counterexample in dimension $s = 3$. Let $f^j : \mathbf{T}^s \rightarrow \mathbb{R}$, $j = 1, 2$ be defined by

$$f^j(x_1, \dots, x_s) = \sum_{\substack{k \in \mathbb{Z}^s \\ k \neq 0}} f_k^j \exp(2\pi i(k_1 x_1 + \dots + k_s x_s)).$$

For $\omega_1, \dots, \omega_s \in \mathbb{R}$ be linearly independent over $\mathbb{Z}$, we put

$$I^j(T) = \int_0^T f^j(\omega_1 t, \dots, \omega_s t) dt, \quad j = 1, 2.$$

The analogue of property (*) for the vector-integral $I = (I^1, I^2) : \mathbb{R} \rightarrow \mathbb{R}^2$ becomes

$$(\%) \quad \forall \varepsilon > 0 \quad \forall T \quad \exists T^* > T : |I^1(T^*)| + |I^2(T^*)| < \varepsilon.$$

Proposition 23. *In the case when $s = 2$ and f is a smooth vector-functions, then (%) holds.*

Proposition 24. *The analogue of Theorem 22 holds for vector-function, that is (%) is satisfied for any odd smooth vector-function f .*

Let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a positive function such that $\sum_{k \in \mathbb{Z}^s} \Phi(\max_{1 \leq j \leq s} |k_j|)$ converges. A vector-function $f = (f^1, f^2) : \mathbb{T}^s \rightarrow \mathbb{R}^2$ is defined to be a function of type Φ if we have

$$|f_k^j| \leq \Phi(\max_{1 \leq j \leq s} |k_j|) \quad \forall k, \quad j = 1, 2.$$

Recently, the author [34] constructed the following example.

Theorem 25. *For any given positive function Φ , there exist a vector-function $f = (f^1, f^2) : \mathbb{T}^3 \rightarrow \mathbb{R}^2$ of the type Φ with zero mean value ($\int_{\mathbb{T}^3} f^j(x) dx = 0, j = 1, 2$) and numbers $\omega_1, \omega_2, \omega_3$, which are linearly independent over $\mathbb{Z}$ such that*

$$|I^1(T)| + |I^2(T)| \rightarrow \infty, \quad \text{as } T \rightarrow +\infty.$$

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Nikolai G. MOSHCHEVITIN
 Department of Theory of Numbers
 Faculty of Mathematics & mechanics
 Moscow State University
 Vorobiovy Gory
 119899, Moscow, Russia
E-mail : MOSH@mv.math.msu.su