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Journal de Théorie des Nombres de Bordeaux, tome 12, n° 1 (2000),
p. 69-79

http://www.numdam.org/item?id=JTNB_2000__12_1_69_0

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Topological Properties of Two-Dimensional Number Systems

par SHIGEKI AKIYAMA et JÖRG M. THUSWALDNER

RÉSUMÉ. Pour une matrice réelle M d'ordre 2 donnée, on peut définir la notion de représentation M -adique d'un élément de \mathbb{R}^2 . On note \mathcal{F} le domaine fondamental constitué des nombres de \mathbb{R}^2 dont le développement " M -adique" ne commence pas par 0. C'est l'analogie dans \mathbb{R}^2 des nombres q -adiques, où la matrice M joue le rôle de la base q . Kátai et Kőrnyci ont démontré que \mathcal{F} est compact, et que \mathbb{R}^2 s'écrit comme la réunion dénombrable de certains translatés de \mathcal{F} , l'intersection de 2 quelconques d'entre eux étant de mesure nulle. Dans cet article, nous construisons des points qui appartiennent simultanément à trois translatés de \mathcal{F} , et nous montrons que \mathcal{F} est connexe. Nous donnons aussi une propriété sur la structure des points intérieurs de \mathcal{F} .

ABSTRACT. In the two dimensional real vector space \mathbb{R}^2 one can define analogs of the well-known q -adic number systems. In these number systems a matrix M plays the role of the base number q . In the present paper we study the so-called fundamental domain \mathcal{F} of such number systems. This is the set of all elements of \mathbb{R}^2 having zero integer part in their " M -adic" representation. It was proved by Kátai and Kőrnyci, that \mathcal{F} is a compact set and certain translates of it form a tiling of the \mathbb{R}^2 . We construct points, where three different tiles of this tiling coincide. Furthermore, we prove the connectedness of \mathcal{F} and give a result on the structure of its inner points.

1. INTRODUCTION

In this paper we use the following notations: \mathbb{R} , \mathbb{Q} , \mathbb{Z} and \mathbb{N} denote the set of real numbers, rational numbers, integers and positive integers, respectively. If $x \in \mathbb{R}$ we will write $[x]$ for the largest integer less than or equal to x . λ will denote the 2-dimensional Lebesgue measure. Furthermore, we write ∂A for the boundary of the set A and $\text{int}(A)$ for its interior.

$\text{diag}(\lambda_1, \lambda_2)$ denotes a 2×2 diagonal matrix with diagonal elements λ_1 and λ_2 .

Let $q \geq 2$ be an integer. Then each positive integer n has a unique q -adic representation of the shape $n = \sum_{k=0}^H a_k q^k$ with $a_k \in \{0, 1, \dots, q-1\}$ ($0 \leq k \leq H$) and $a_H \neq 0$ for $H \neq 0$. These q -adic number systems have been generalized in various ways. In the present paper we deal with analogs of these number systems in the 2-dimensional real vector space, that emerge from number systems in quadratic number fields. The first major step in the investigation of number systems in number fields was done by Knuth [13], who studied number systems with negative bases as well as number systems in the ring of Gaussian integers. Meanwhile, Kátai, Kovács, Pethő and Szabó invented a general notion of number systems in rings of integers of number fields, the so-called *canonical number systems* (cf. for instance [10, 11, 12, 15]). We recall their definition.

Let K be a number field with ring of integers Z_K . For an algebraic integer $b \in Z_K$ define $\mathcal{N} = \{0, 1, \dots, |N(b)| - 1\}$, where $N(b)$ denotes the norm of b over \mathbb{Q} . The pair (b, \mathcal{N}) is called a *canonical number system* if any $\gamma \in Z_K$ admits a representation of the shape

$$\gamma = c_0 + c_1 b + \dots + c_H b^H,$$

where $c_k \in \mathcal{N}$ ($0 \leq k \leq H$) and $c_H \neq 0$ for $H \neq 0$.

These number systems resemble a natural generalization of q -adic number systems to number fields. Each of these number systems gives rise to a number system in the n -dimensional real vector space. Since we are only interested in the 2-dimensional case, we construct these number systems only for this case. Consider a canonical number system (b, \mathcal{N}) in a quadratic number field K with ring of integers Z_K . Let $p_b(x) = x^2 + Ax + B$ be the minimal polynomial of b . It is known, that for bases of canonical number systems $-1 \leq A \leq B \geq 2$ holds (cf. [10, 11, 12]). Now consider the embedding $\Phi : K \rightarrow \mathbb{R}^2$, $\alpha_1 + \alpha_2 b \mapsto (\alpha_1, \alpha_2)$, where $\alpha_1, \alpha_2 \in \mathbb{Q}$. Kovács [14] proved, that $\{1, b\}$ forms an integral basis of Z_K . Thus we have $\Phi(Z_K) = \mathbb{Z}^2$. Furthermore, note that $\Phi(bz) = M\Phi(z)$ with

$$M = \begin{pmatrix} 0 & -B \\ 1 & -A \end{pmatrix}.$$

Since the elements of \mathcal{N} are rational integers, for each $c \in \mathcal{N}$, $\Phi(c) = (c, 0)^T$. Summing up we see, that $(M, \Phi(\mathcal{N}))$ forms a number system in the two dimensional real vector space in the following sense (cf. also [8], where some properties of these number systems are studied). Each $g \in \mathbb{Z}^2$ has a unique representation of the form

$$g = d_0 + M d_1 + \dots + M^H d_H,$$

with $d_k \in \Phi(\mathcal{N})$ ($0 \leq k \leq H$) and $d_H \neq (0, 0)^T$ for $H \neq 0$. These number systems form the object of this paper. In particular, we want to study the so-called *fundamental domains* of these number systems. The *fundamental domain* of a number system $(M, \Phi(\mathcal{N}))$ is defined by

$$\mathcal{F} = \left\{ z \mid z = \sum_{j \geq 1} M^{-j} d_j, d_j \in \Phi(\mathcal{N}) \right\}.$$

Sloppily spoken, \mathcal{F} contains all elements of \mathbb{R}^2 , with integer part zero in their “ M -adic” representation. In Figure 1 the fundamental domain corresponding to the M -adic representations arising from the Gaussian integer $-1 + i$ is depicted. This so-called “twin dragon” was studied extensively by Knuth in his book [13].

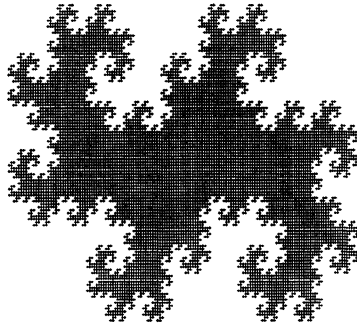


FIGURE 1. *The fundamental domain of a number system*

Fundamental domains of number systems have been studied in various papers. Kátai and Kórnyci [9] proved, that \mathcal{F} is a compact set that tessellates the plane in the following way.

$$(1) \quad \bigcup_{g \in \mathbb{Z}^2} (\mathcal{F} + g) = \mathbb{R}^2 \quad \text{where} \quad \lambda((\mathcal{F} + g_1) \cap (\mathcal{F} + g_2)) = 0$$

$$(g_1, g_2 \in \mathbb{Z}^2; g_1 \neq g_2).$$

Furthermore, we want to mention, that the boundary of \mathcal{F} has fractal dimension. Its Hausdorff and box counting dimension has been calculated by Gilbert [4], Ito [7], Müller-Thuswaldner-Tichy [16] and Thuswaldner [17]. In the present paper we are interested in topological properties of \mathcal{F} . Before we give a survey on our results we shall define some basic objects. Let S be the set of all translates of \mathcal{F} , that “touch” \mathcal{F} , i.e.

$$S := \{g \in \mathbb{Z}^2 \setminus (0, 0)^T \mid \mathcal{F} \cap (\mathcal{F} + g) \neq \emptyset\}.$$

Then by (1) the boundary of \mathcal{F} has the representation

$$(2) \quad \partial\mathcal{F} = \bigcup_{g \in \mathcal{S}} (\mathcal{F} \cap (\mathcal{F} + g)).$$

Hence, the boundary of \mathcal{F} is the set of all elements of \mathcal{F} , that are contained in $\mathcal{F} + g$ for a certain $g \neq (0, 0)^T$. Of course, $\partial\mathcal{F}$ may contain points, that belong to \mathcal{F} and two other different translates of \mathcal{F} . These points we call *vertices* of \mathcal{F} . Thus the set of vertices of \mathcal{F} is defined by

$$V := \{z \in \mathcal{F} \mid z \in (\mathcal{F} + g_1) \cap (\mathcal{F} + g_2), g_1, g_2 \in \mathbb{Z}^2; g_1 \neq g_2, g_1 \neq 0, g_2 \neq 0\}.$$

In Section 2 we study the set of vertices of \mathcal{F} . It turns out, that, apart from one exceptional case, \mathcal{F} has at least 6 vertices. In some cases we derive that V is an infinite or even uncountable set. In Section 3 we prove the connectedness of \mathcal{F} and show that each element of \mathcal{F} , which has a finite M -adic expansion, is an inner point of \mathcal{F} .

2. VERTICES OF THE FUNDAMENTAL DOMAIN \mathcal{F}

In this section we give some results on the set of vertices V of \mathcal{F} . For number systems emerging from Gaussian integers, similar results have been established with help of different methods in Gilbert [3]. We start with the definition of useful abbreviations. Let

$$(3) \quad g = M^{-H_1} d_{-H_1} + \dots + M^{H_2} d_{H_2}$$

be the M -adic representation of g . Note, that the digits d_j ($-H_1 \leq j \leq H_2$) are of the shape $d_j = (c_j, 0)^T \in \Phi(\mathcal{N})$. Thus for the expansion (3) we will write

$$g = c_{H_2} c_{H_2-1} \dots c_1 c_0 \cdot c_{-1} \dots c_{-H_1}.$$

If the string $c_1 \dots c_H$ occurs j times in an M -adic representation, then we write $[c_1 \dots c_H]_j$. If a representation is ultimately periodic, i.e. a string $c_1 \dots c_H$ occurs infinitely often, we write $[c_1 \dots c_H]_\infty$. First we show, that for $A > 0$ any fundamental domain \mathcal{F} contains at least 6 vertices.

Theorem 2.1. *Let $(M, \Phi(\mathcal{N}))$ be a number system in \mathbb{R}^2 , which is induced by the base b of a canonical number system. Let $p_b(x) = x^2 + Ax + B$ with $A > 0$ be the minimal polynomial of b . Then the set of vertices V of the fundamental domain \mathcal{F} of this number system contains the points*

$$\begin{aligned} P_1 &= 0.[0(A-1)(B-1)]_\infty, & P_2 &= 0.[(A-1)(B-1)0]_\infty, \\ P_3 &= 0.[0(B-1)(B-A)]_\infty, & P_4 &= 0.[(B-1)(B-A)0]_\infty, \\ P_5 &= 0.[(B-A)0(B-1)]_\infty, & P_6 &= 0.[(B-1)0(A-1)]_\infty. \end{aligned}$$

Depending on the cases $A = 1$, $1 < A < B$ and $A = B$, the points P_j ($1 \leq j \leq 6$) belong to the following translates $\mathcal{F} + w$ of \mathcal{F} .

	values of w for $1 < A < B$	values of w for $A = B$
P_1	0, 1, $1A$	0, 1, $1(A-1)10$
P_2	0, $1(A-1)$, $1A(B-1)$	0, $1(A-1)$, $1(A-1)10(A-1)$
P_3	0, $1A$, $1(A-1)$	0, $1(A-1)$, $1(A-1)10$
P_4	0, $1A(B-1)$, $1(A-1)(B-A)$	0, $1A(A-1)$, $1(A-1)0$
P_5	0, $1(A-1)(B-A+1)$, 1	0, $1(A-1)1$, 1
P_6	0, $1(A-1)(B-A)$, $1(A-1)(B-A+1)$	0, $1(A-1)1$, $1(A-1)0$

The case $A = 1$ is very similar to the case $1 < A < B$; just replace the representation $1(A-1)(B-A+1)$ by $11(B-1)0$ in the above table.

Remark 2.1. Note, that we have $0 < A \leq B \geq 2$. Hence the digits of the 6 points indicated in Theorem 2.1 are all admissible.

Proof of the theorem. We will prove that each of the 6 points P_1, \dots, P_6 is contained in three different translates of \mathcal{F} , as indicated in the statement of the theorem. First we consider the point P_1 . Write $\bar{x} = -x$. By using $b^2 + Ab + B = 0$, we see that

$$(4) \quad 0.1(A-1)(B-A)\bar{B} = 0.1[(A-1)(B-A)\overline{(B-1)}]_{\infty} = 0$$

are formal representations of zero. Adding the second representation for 0 given in (4) twice, we have

$$\begin{aligned} P_1 &= 0.[0(A-1)(B-1)]_{\infty} + 1.[(A-1)(B-A)\overline{(B-1)}]_{\infty} \\ &= 1.[(A-1)(B-1)0]_{\infty} \\ &= 1.[(A-1)(B-1)0]_{\infty} + 1(A-1).[(B-A)\overline{(B-1)}(A-1)]_{\infty} \\ &= 1A.[(B-1)0(A-1)]_{\infty}. \end{aligned}$$

For $A < B$ this yields

$$P_1 \in \mathcal{F} \cap (\mathcal{F} + 1) \cap (\mathcal{F} + 1A).$$

For $A = B$ the last expansion $1A.[(B-1)0(A-1)]_{\infty}$ is not admissible since $A > B - 1$. In order to settle this case we use the first representation of zero given in (4) to get $1A = 1B = 1B + 1(B-1)0\bar{B} = 1(A-1)10$. As a result, we have

$$P_1 \in \mathcal{F} \cap (\mathcal{F} + 1) \cap (\mathcal{F} + 1(A-1)10)$$

for $A = B$. Since $P_2 = MP_1$, we get the desired results also for P_2 . Now we treat

$$P_3 = 0.[0(B-1)(B-A)]_{\infty}.$$

In the same way as before, we get, using both representations of zero in (4)

$$\begin{aligned} P_3 &= 0.[0(B-1)(B-A)]_{\infty} + 1A.B - 0.1[(A-1)(B-A)\overline{(B-1)}]_{\infty} \\ &= 1A.[(B-1)(B-A)0]_{\infty} \\ &= 1A.[(B-1)(B-A)0]_{\infty} - 1.[(A-1)(B-A)\overline{(B-1)}]_{\infty} \\ &= 1(A-1).[(B-A)0(B-1)]_{\infty}, \end{aligned}$$

which implies

$$P_3 \in \mathcal{F} \cap (\mathcal{F} + 1A) \cap (\mathcal{F} + 1(A - 1))$$

for $A < B$ and

$$P_3 \in \mathcal{F} \cap (\mathcal{F} + 1(A - 1)10) \cap (\mathcal{F} + 1(A - 1))$$

for $A = B$. Since \mathcal{F} permits an involution $\varphi : x \rightarrow \sum_{j \geq 1} M^{-j}(B-1, 0)^T - x$, \mathcal{F} is symmetric with respect to the center $\frac{1}{2} \sum_{j \geq 1} M^{-j}(B-1, 0)^T$. For $w \in \mathbb{Z}^2$ this map sends each $\mathcal{F} + w$ to $\mathcal{F} - w$. Thus we have

$$\begin{aligned} \varphi(\mathcal{F} + 1) &= \mathcal{F} + 1A(B - 1), \\ \varphi(\mathcal{F} + 1(A - 1)) &= \begin{cases} \mathcal{F} + 1(A - 1)(B - A + 1) & \text{for } A > 1, \\ \mathcal{F} + 11(B - 1)0 & \text{for } A = 1, \end{cases} \\ \varphi(\mathcal{F} + 1A) &= \mathcal{F} + 1(A - 1)(B - A), \end{aligned}$$

for $A < B$ and

$$\begin{aligned} \varphi(\mathcal{F} + 1) &= \mathcal{F} + 1(A - 1)10(A - 1), \\ \varphi(\mathcal{F} + 1(A - 1)) &= \mathcal{F} + 1(A - 1)1, \\ \varphi(\mathcal{F} + 1(A - 1)10) &= \mathcal{F} + 1(A - 1)0, \end{aligned}$$

for $A = B$. Furthermore, it is easy to see, that $\varphi(P_1) = P_4$, $\varphi(P_2) = P_5$ and $\varphi(P_3) = P_6$. Thus also P_4 , P_5 and P_6 are vertices of \mathcal{F} that are contained in the translates of \mathcal{F} indicated in the statement of the theorem. \square

In the case $A = 0$ it is easy to see that \mathcal{F} is a square. It has exactly 4 vertices. These are the “usual” vertices of the square. Thus we only have to deal with the case $A = -1$. We will formulate the corresponding result as a corollary.

Corollary 2.1. *Let the same settings as in Theorem 2.1 be in force, but assume now, that $A = -1$. Then the following table gives 6 points P_j ($1 \leq j \leq 6$), that are contained in the set of vertices V of \mathcal{F} . Furthermore, we give the translates $\mathcal{F} + w$, to which P_j belongs.*

P_j	translates w , for which $P_j \in \mathcal{F} + w$
$0.[0(B-1)(B-1)(B-1)00]_\infty$	$0, 10(B-1), 10(B-1)(B-1)$
$0.[000(B-1)(B-1)(B-1)]_\infty$	$0, 1, 10$
$0.[00(B-1)(B-1)(B-1)0]_\infty$	$0, 10, 10(B-1)$
$0.[(B-1)000(B-1)(B-1)]_\infty$	$0, 1, 10(B-1)(B-1)1$
$0.[(B-1)(B-1)000(B-1)]_\infty$	$0, 10(B-1)(B-1), 10(B-1)(B-1)1$
$0.[(B-1)(B-1)(B-1)000]_\infty$	$0, 10(B-1)(B-1)0, 10(B-1)(B-1)1$

Proof. Let $M_1 = \begin{pmatrix} 0 & -B \\ 1 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & -B \\ 1 & 1 \end{pmatrix}$ be bases of number systems in \mathbb{R}^2 and let \mathcal{F}_1 and \mathcal{F}_2 be the fundamental domains corresponding

to M_1 and M_2 , respectively. We know the vertices of \mathcal{F}_1 from Theorem 2.1 and will construct the vertices of \mathcal{F}_2 from it. To this matter let $M_1 = G_1 \text{diag}(b_1, b_2) G_1^{-1}$. It is easy to see, that then $M_2 = G_2 \text{diag}(-b_1, -b_2) G_2^{-1}$ with $G_2 = \text{diag}(-1, 1) G_1$. Now suppose, that $\sum_{k \geq 1} M_1^{-k} a_j \in \mathcal{F}_1 \cap \mathcal{F}_1 + (v_1, v_2)^T \cap \mathcal{F}_1 + (w_1, w_2)^T$ with $v_1, v_2, w_1, w_2 \in \mathbb{Z}$ is a vertex of \mathcal{F}_1 . Using the fact, that $G_1^{-1} a_k = -G_2^{-1} a_k$ for $a_k \in \mathcal{M}$ and setting $d = 0.[0(B - 1)]_\infty$ we easily derive that

$$(5) \quad Q := \sum_{k \geq 1} (-1)^{k+1} M_2^{-k} a_k + d$$

$$\in \text{diag}(-1, 1)(\mathcal{F}_1 \cap \mathcal{F}_1 + (v_1, v_2)^T \cap \mathcal{F}_1 + (w_1, w_2)^T) + d.$$

Observe, that by the selection of d , Q has an admissible M_2 -adic representation with integer part zero. Thus $Q \in \mathcal{F}_2$. Since any element of \mathcal{F}_2 can be constructed from elements of \mathcal{F}_1 in the same way we conclude, that $\mathcal{F}_2 = \text{diag}(-1, 1)\mathcal{F}_1 + d$. But with that (5) reads $Q \in \mathcal{F}_2 \cap \mathcal{F}_2 + (-v_1, v_2)^T \cap \mathcal{F}_2 + (-w_1, w_2)^T$. Thus Q is a vertex of \mathcal{F}_2 . The representations in the table above, can now easily be obtained from the results for $A = 1$ in Theorem 2.1. □

The following corollary is an immediate consequence of Theorem 2.1 and Corollary 2.1.

Corollary 2.2. *For $1 < A < B$ we have*

$$S \supset \{1, 1A, 1(A - 1), 1A(B - 1), 1(A - 1)(B - A), 1(A - 1)(B - A + 1)\},$$

for $A = B$

$$S \supset \{1, 1(A - 1)10, 1(A - 1), 1(A - 1)10(A - 1), 1(A - 1)0, 1(A - 1)1\},$$

while for $A = 1$

$$S \supset \{1, 10, 10(B - 1), 10(B - 1)(B - 1), 10(B - 1)(B - 1)0, \\ 10(B - 1)(B - 1)1\}$$

holds.

Remark 2.2. Note, that “ \supset ” may be replaced by “ $=$ ” in Corollary 2.2 if $2A < B + 3$. This is shown for the Gaussian case in [16]. For arbitrary quadratic number fields this fact can be proved in a similar way.

Theorem 2.2. *Let the same settings as in Theorem 2.1 be in force. If $2A = B + 3$ then \mathcal{F} has infinitely many vertices.*

Proof. Set $K = B - A + 1 = \frac{B-1}{2}$. Then, using $b^2 + Ab + B = 0$, we get ($j \geq 0$)

$$(6) \quad 0 = \sum_{k=2}^{\infty} (-1)^k \left(M^{-k+2}(1,0)^T + M^{-k+1}(A,0)^T + M^{-k}(B,0)^T \right) \\ = 1.(A-1)[K\bar{K}]_{\infty}.$$

Here we set $\bar{x} = -x$, as before. We will show, that the points

$$(7) \quad Q_j = 1A.[(B-1)0(A-1)]_{2j}(B-1)0[K]_{\infty} \quad (j \in \mathbb{N})$$

are vertices of \mathcal{F} . Therefore we need the representation (6). With help of this representation we define the following representations of zero.

$$\begin{aligned} N_1 &:= 1(A-1).[K\bar{K}]_{\infty} = 0, \\ N_2 &:= 1.(A-1)[K\bar{K}]_{\infty} = 0, \\ X_j &:= 0.[0]_j 1AB = 0 \quad (j \geq 0). \end{aligned}$$

In the sequel we write kX_j ($k \in \mathbb{Z}$) if we want to multiply each digit of the representation X_j by k . Furthermore, addition and subtraction of representations is always meant digit-wise. After these definitions we define the following, more complicated representations of zero.

$$\begin{aligned} Z_1(j) &:= N_1 + \sum_{k=1}^j (X_{6k-1} - 2X_{6k-2} + 2X_{6k-3} - X_{6k-4}) + (1.AB) - 2(1A.B) \\ &= \overline{1A}.[(\overline{B-1})(A-1)(B-A)]_{2j}(\overline{B-1})(A-1)[K\bar{K}]_{\infty}, \\ Z_2(j) &:= N_2 + \sum_{k=1}^j (-X_{6k-3} + 2X_{6k-4} - 2X_{6k-5} + X_{6k-6}) - (1A.B) \\ &= \overline{1(A-1)}.[(\overline{B-A})(B-1)(\overline{A-1})(\overline{B-A})]_{2j-1}(B-1)(\overline{A-1})\bar{K}[K\bar{K}]_{\infty}. \end{aligned}$$

Finally, we observe, that for $j \in \mathbb{N}$

$$\begin{aligned} Q_j &= Q_j + Z_1(j) \\ &= 0.[0(A-1)(B-1)]_{2j}0(A-1)[(B-1)0]_{\infty} \\ &= Q_j + Z_2(j) \\ &= 1.(A-1)[(B-1)0(A-1)]_{2j-1}(B-1)0KK[0(B-1)]_{\infty}, \end{aligned}$$

and this implies $Q_j \in V$. It remains to show, that the elements Q_j , $j \geq 1$, are pairwise different. This follows from the following observation. Select $k \in \mathbb{N}$ arbitrary and let $j_1, j_2 \leq k$. Suppose, that Q_{j_1} and Q_{j_2} are represented by the representation (7) for $j = j_1$ and $j = j_2$, respectively. Then $Q_{j_1} = Q_{j_2}$ if and only if $M^{6k+2}Q_{j_1} = M^{6k+2}Q_{j_2}$. For $k \geq \max(j_1, j_2)$, $M^{6k+2}Q_{j_1}$ and $M^{6k+2}Q_{j_2}$ have the same digit string $[0(B-1)]_{\infty}$ after the comma. Hence, they can only be equal, if their integer parts are equal. But since $(M, \Phi(\mathcal{N}))$ is a number system, this can only be the case, if the digit strings of their integer parts are the same. This implies $j_1 = j_2$. So we have proved, that the points Q_j are pairwise different for $j \leq k$. Since k

can be selected arbitrary, the result follows. Thus we found infinitely many different vertices of \mathcal{F} . □

Theorem 2.3. *Let the same settings as in Theorem 2.1 be in force. If $2A > B + 3$ then \mathcal{F} has uncountably many vertices.*

Proof. Set $K = B - A + 1$ and $\xi = \lfloor (B - 1)/2 \rfloor$. As $\xi + K, \xi - K \in \mathcal{N}$, by using (6), we see that

$$\begin{aligned} 0.[\xi]_\infty &= 1(A - 1).[(\xi + K)(\xi - K)]_\infty \\ &= 1(A - 1)K.[(\xi - K)(\xi + K)]_\infty. \end{aligned}$$

Thus $0.[\xi]_\infty$ is a vertex of \mathcal{F} . Fix an integer k , such that all eigenvalues of M^k are greater than 2 (such an integer exists, since the eigenvalues of M are all greater than 1). This implies, that the representations $0.c_1[0]_k c_2[0]_k c_3[0]_k c_4 \dots$, $c_j \in \{0, 1\}$ ($j \geq 1$) represent pairwise different elements of \mathbb{R}^2 for different $\{0, 1\}$ sequences $\{c_j\}_{j \geq 1}$. Because $\xi + K < B - 1$, each of the uncountably many representations

$$0.[\xi]_\infty + 0.c_1[0]_k c_2[0]_k c_3[0]_k c_4 \dots \quad (c_j \in \{0, 1\}, j \geq 1)$$

corresponds to a vertex of \mathcal{F} . Since they are pairwise different, the theorem is proved. □

3. CONNECTEDNESS AND INNER POINTS OF THE FUNDAMENTAL DOMAIN \mathcal{F}

In this section we will show, that the fundamental domain \mathcal{F} is arcwise connected. To establish this result, we will apply a general theorem due to Hata (cf. [5, 6]) which assures arcwise connectedness for a large class of sets. The second result of this section is devoted to the structure of the inner points of \mathcal{F} . In particular, we prove, that each point with finite M -adic representation is an inner point of \mathcal{F} . In this section we will use the notation

$$\mathcal{F}_k := \left\{ z \mid z = \sum_{j=1}^k M^{-j} a_j, a_j \in \Phi(\mathcal{N}) \right\} \quad (k \in \mathbb{N}).$$

We start with the connectedness result.

Theorem 3.1. *Let $(M, \Phi(\mathcal{N}))$ be a number system in \mathbb{R}^2 , which is induced by the base b of a canonical number system in a quadratic number field. Then the fundamental domain \mathcal{F} of $(M, \Phi(\mathcal{N}))$ is arcwise connected.*

Proof. It is an easy consequence of the definition of \mathcal{F} , that

$$(8) \quad \mathcal{F} = \bigcup_{g \in \Phi(\mathcal{N})} M^{-1}(\mathcal{F} + g).$$

Furthermore, Theorem 2.1 implies that $\mathcal{F} \cap (\mathcal{F} + (1, 0)^T) \neq \emptyset$. Thus the sets contained in the union of (8) form a *chain* in the sense that $(\mathcal{F} + g) \cap (\mathcal{F} + (g + (1, 0)^T)) \neq \emptyset$ for $g \in \Phi(\mathcal{N}) \setminus (B - 1, 0)^T$. Thus \mathcal{F} fulfills the conditions being necessary for the application of a theorem of Hata, namely [5, Theorem 4.6]. This theorem yields the arcwise connectedness of \mathcal{F} . \square

Now we prove the result on the inner points of \mathcal{F} . Note, that the existence of inner points is an immediate consequence of [9, Theorem 1].

Theorem 3.2. *Let $(M, \Phi(\mathcal{N}))$ be a number system in \mathbb{R}^2 , which is induced by the base b of a canonical number system in a quadratic number field. Then for each $k \in \mathbb{N}$ we have*

$$\mathcal{F}_k \subset \text{int}(\mathcal{F}).$$

Proof. First we will show, that 0 is an inner point of \mathcal{F} . Suppose, that 0 is contained in the boundary of \mathcal{F} . Then by (2) there exists a representation of zero of the shape

$$(9) \quad 0 = c_{H_1} c_{H_1-1} \dots c_1 c_0 \cdot c_{-1} c_{-2} \dots$$

This representation implies $0 \in \mathcal{F} + c_{H_1} c_{H_1-1} \dots c_1 c_0$. If we multiply (9) by M^j for $j \in \mathbb{N}$ arbitrary, we conclude, that $0 \in \mathcal{F} + c_{H_1} c_{H_1-1} \dots c_1 c_0 c_{-1} \dots c_{-j}$ for each $j \in \mathbb{N}$. Hence, 0 is contained in infinitely many different translates of \mathcal{F} . But since \mathcal{F} is a compact set this is a contradiction to (1). Thus $0 \in \text{int}(\mathcal{F})$.

Now fix $k \in \mathbb{N}$ and $g \in \mathcal{F}_k$. Then $0 \in \text{int}(\mathcal{F})$ implies, that $g \in \text{int}(M^{-k}\mathcal{F} + g)$. The result now follows from the representation

$$\mathcal{F} = \bigcup_{g \in \mathcal{F}_k} (M^{-k}\mathcal{F} + g).$$

\square

There is a direct alternative proof of this theorem by using the methods of [1] and [2]. In these papers a similar result for the tiling generated by Pisot number systems is shown.

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