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Congruences modulo ℓ between ε factors for cuspidal representations of \( GL(2) \)


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**Congruences modulo \( \ell \) between \( \varepsilon \) factors for cuspidal representations of \( GL(2) \)**

par MARIE-FRANCE VIGNÉRAS

**Pour Jacques Martinet**

**Abstract.** Let \( \ell \neq p \) be two different prime numbers, let \( F \) be a local non archimedean field of residual characteristic \( p \), and let \( \overline{\mathbb{Q}}_\ell, \mathbb{Z}_\ell, \overline{\mathbb{F}}_\ell \) be an algebraic closure of the field of \( \ell \)-adic numbers \( \mathbb{Q}_\ell \), the ring of integers of \( \overline{\mathbb{Q}}_\ell \), the residual field of \( \mathbb{Z}_\ell \). We proved the existence and the unicity of a Langlands local correspondence over \( \overline{\mathbb{F}}_\ell \) for all \( n \geq 2 \), compatible with the reduction modulo \( \ell \) in [V5], without using \( L \) and \( \varepsilon \) factors of pairs.

We conjecture that the Langlands local correspondence over \( \overline{\mathbb{Q}}_\ell \) respects congruences modulo \( \ell \) between \( L \) and \( \varepsilon \) factors of pairs, and that the Langlands local correspondence over \( \overline{\mathbb{F}}_\ell \) is characterized by identities between new \( L \) and \( \varepsilon \) factors. The aim of this short paper is prove this when \( n = 2 \).

**Introduction**

The Langlands local correspondence is the unique bijection between all irreductible \( \overline{\mathbb{Q}}_\ell \)-representations of \( GL(n, F) \) and certain \( \ell \)-adic representations of an absolute Weil group \( W_F \) of dimension \( n \), for all integers \( n \geq 1 \),

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which is induced by the reciprocity law of local class field theory
\[ W_F^{ab} \simeq F^* \]
when \( n = 1 \) (\( W_F^{ab} \) is the biggest abelian Hausdorff quotient of \( W_F \)), and which respects \( L \) and \( \varepsilon \) factors of pairs \([LRS],[HT],[H2]\).

Let \( \psi : F \to \mathbb{Z}_l^* \) be a non trivial character. We denote by \( \text{Cusp}_R \text{GL}(n,F) \) the set of isomorphism classes of irreducible cuspidal \( R \)-representations of \( \text{GL}(n,F) \). When \( \pi \in \text{Cusp}_Q \text{GL}(n,F) \), Henniart \([H1]\) showed that \( \pi \) is characterized by the epsilon factors of pairs \( \varepsilon(\pi,\sigma) \) for all \( \sigma \in \text{Cusp}_Q \text{GL}(m,F) \) and for all \( m \leq n - 1 \) (note that \( L(\pi,\sigma) = 1 \)), using the theory of Jacquet, Piatetski-Shapiro, and Shalika \([JPS1]\).

Does this remain true for cuspidal irreducible \( \overline{F}_l \)-representations of \( \text{GL}(n,F) \)? We need first to define the epsilon factors of pairs.

Let \( \pi \in \text{Cusp}_Q \text{GL}(n,F) \). It is known that the constants of the epsilon factors of pairs \( \varepsilon(\pi,\sigma) \) belong to \( \mathbb{Z}_l^* \) for all \( \sigma \in \text{Cusp}_Q \text{GL}(m,F) \) and for all \( m \leq n - 1 \), and that the conductor does not change by reduction modulo \( l \) (this is proved by Deligne \([D]\) for the irreducible representations of the Weil group, and by the local Langlands correspondence over \( \mathbb{Q}_l \) is true for cuspidal representations).

Now let \( \pi \in \text{Cusp}_F \text{GL}(n,F) \). Then \( \pi \) lifts to \( \text{Cusp}_Q \text{GL}(n,F) \) \([V1,III.5.10]\). By reduction modulo \( l \), one can define epsilon factors of pairs \( \varepsilon(\pi,\sigma) \) for all \( \sigma \in \text{Cusp}_F \text{GL}(m,F) \) and for all \( m \leq n - 1 \). Let \( q \) be the order of the residual field of \( F \). We expect that \( \pi \) is characterized by the epsilon factors \( \varepsilon(\pi,\sigma) \) for all \( \sigma \), when the multiplicative order of \( q \) modulo \( l \) is \( > n - 1 \); otherwise, \( \pi \) should be characterized by less naive but natural epsilon factors. The same should be true when \( \pi \) is replaced by an \( \overline{F}_l \)-irreducible representation of the Weil group \( W_F \).

The existence \([V4]\) of an integral Kirillov model for \( \pi \in \text{Cusp}_Q \text{GL}(n,F) \) seems to be an adequate tool to solve the problem. The description of the representation \( \pi \) on the Kirillov model is given by the central character \( \omega_\pi \) and by the action of the symmetric group \( S_n \) (the Weyl group of \( GL(n,F) \)). The action of \( S_n \) is related with the \( \varepsilon(\pi,\sigma) \) for all \( \sigma \) as above \([GK, see the end of paragraph 7]\). When \( n = 2 \) Jacquet and Langlands \([JL]\) described the action of \( S_2 \) on the Kirillov model in terms of \( \varepsilon(\pi,\chi) = \varepsilon(\pi \otimes \chi) \) for all \( \overline{Q}_l \)-characters \( \chi \) of \( F^* \), using the Fourier transform on \( F^* \).

In the case \( n = 2 \) and only in this case, we will prove that two integral \( \pi, \pi' \in \text{Cusp}_Q \text{GL}(2,F) \) have the same reduction modulo \( l \) if and only if their central characters have the same reduction modulo \( l \) and the factors \( \varepsilon(\pi \otimes \chi), \varepsilon(\pi' \otimes \chi) \) have the same reduction modulo \( l \) for integral \( \overline{Q}_l \)-characters \( \chi \) of \( F^* \) when \( l \) does not divide \( q - 1 \). When \( l \) divides \( q - 1 \) this remains true with new epsilon factors taking into account the natural
congruences modulo ℓ satisfied by the ε(π ⊗ χ) for all χ. By reduction modulo ℓ, we get that the local Langlands \( \overline{F}_\ell \)-correspondence for \( n = 2 \) is characterized by the equality on \( L \) and new \( ε \) factors of pairs. The field \( \overline{F}_\ell \) can be replaced by any algebraically closed field \( R \) of characteristic \( ℓ \).

The case \( n = 3 \) could be treated probably, but the general case \( n \geq 4 \) remains an open and interesting question.

1. Integral Kirillov model

The definition of the \( L \) and \( ε \) factors of pairs [JPS1] uses the Whittaker model, or what is equivalent the Kirillov model. We showed [V4] that these models are compatible with the reduction modulo \( ℓ \).

We denote by \( O_F \) the ring of integers of \( F \). Let \( R \) be an algebraically closed field of characteristic \( \neq p \), and let \( ψ : F \to R^* \) be a character such that \( O_F \) is the biggest ideal on which \( ψ \) is trivial. We extend \( ψ \) to a \( R \)-character of the group \( N \) of strictly upper triangular matrices of \( G = GL(n, F) \) by \( ψ(n) = ψ(\sum n_{i,i+1}) \) for \( n = (n_{i,j}) \in N \). The mirabolic subgroup \( P \) of \( G \) is the semi-direct product of the group \( GL(n - 1, F) \) embedded in \( GL(n, F) \) by

\[
g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}
\]

and of the group \( F^{n-1} \) embedded in \( GL(n, F) \) by

\[
x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.
\]

The representation \( τ_R := \text{ind}_{P,N} ψ \) of the mirabolic subgroup \( P \) (compact induction) is called mirabolic. It is irreducible (this is a corollary of [V4 prop.1]), but it is not admissible when \( n \geq 2 \).

**Lemma.** \( \text{End}_{RP} τ_R \simeq R \).

**Proof.** This is a general fact: the representation \( τ_R \) is absolutely irreducible [V1, I.6.10], hence \( \text{End}_{RP} τ_R \simeq R \). From the Schur's lemma [V1, I.6.9] \( \text{End}_{RP} τ_R \simeq R \) when the cardinal of \( R \) is strictly bigger than \( \text{dim}_R τ_R \) (countable dimension). There exists an algebraically closed field \( R' \) which contains \( R \) and of uncountable cardinal. Two \( RP \)-endomorphisms of \( τ_R \) which are proportional over \( R' \) are proportional over \( R \).

**Theorem.** An irreducible \( R \)-representation \( π \) of \( G \) is cuspidal if and only if extends the mirabolic representation \( τ_R \).

**Proof.** This results from [BZ] and [V1]. Suppose that \( π \) is cuspidal. Then \( π|_P \) is the mirabolic representation: when \( R = \overline{Q}_\ell \simeq C \) see [BZ, 5.13 & 5.20], when \( R = \overline{F}_\ell \), \( π \) lifts to \( \overline{Q}_\ell \) [V1, III.5.10] where it is true then reduce. Conversely, suppose \( π|_P = τ_R \) and \( R = \overline{Q}_\ell \) or \( \overline{F}_\ell \). Then \( π \) is cuspidal [V1,
The case of a general $R$ is deduced from this two cases by the next lemma.

Let $G$ be the group of rational points of a reductive connected group over $F$. We denote by $\text{Irr}_R G$ the set of isomorphism classes of irreducible $R$-representations of $G$.

**Lemma.** (1) A non zero homomorphism of algebraically closed fields $f : R \to R'$ gives a natural injective map $\pi \mapsto f_*(\pi) : \text{Irr}_R G \to \text{Irr}_{R'} G$ which respects cuspidality.

(2) Let $\pi' \in \text{Cusp}_{R'} G$. Then there exists an unramified character $\chi$ of $G$ such that $\pi' \otimes \chi = f_*(\pi)$ with $\pi \in \text{Cusp}_R G$.

**Proof.** This results from [VI].

(1) $f_*$ respects irreducibility [VI, II.4.5], and commutes with the parabolic restriction. Hence it respects cuspidality. The linear independence of characters [V1, I.6.13] shows that if $\pi, \pi' \in \text{Irr}_R G$ are not isomorphic then $f_* \pi, f_* \pi'$ are not isomorphic.

(2) Let $Z$ be the center of $G$. The group of rational characters $X(Z)$ is a subgroup of finite index in the group $X(G)$. This implies that there exists an unramified character $\chi$ of $G$ such that the quotient $Z/Z_0$ of $Z$ by the kernel $Z_0$ of the central character $\omega$ of $\pi' \otimes \chi$ is profinite. Hence the values of $\omega$ are roots of unity. We deduce that $\pi' \otimes \chi$ has a model on $R$ [V1, II.4.9].

Let $\pi \in \text{Cusp}_R GL(n, F)$ of central character $\omega$. The realisation of $\pi$ on the mirabolic representation $\tau_R$ is called the Kirillov model $K(\pi)$ of $\pi$. It is sometimes useful to use the Whittaker model instead of the Kirillov model. By adjonction and the theorem $\text{Hom}_{RG}(\pi, \text{Ind}_{G,N} \psi) \simeq R$ (the unicity of the Whittaker model); the Whittaker model $W(\pi)$ is the unique realisation of $\pi$ in $\text{Ind}_{G,N} \psi$. By definition

$$W(g) = (\pi(g)W)(1)$$

for all $g \in G$ and for all Whittaker functions $W \in W(\pi)$. We denote by $\Gamma(j)$ the subgroup of matrices $k \in GL(n, O_F)$ of the form

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \in GL(n-1, O_F), d \in O_F^*, c \in \mathcal{P}_F^0 O_F$$

for any integer $j > 0$. The smallest $j > 0$ such that $\pi$ contains a non-zero vector transforming under $\Gamma(j)$ according to the one dimensional character

$$\omega_j(k) = \omega(d)$$

for $k \in \Gamma(j)$ as above, is called the conductor of $\pi$ and denoted $f$. 

...
Theorem. Let \( \pi \in \text{Cusp}_R \text{GL}(n, F) \) of central character \( \omega = \omega \) and conductor \( f \).

1. The restriction from \( G \) to \( P \) induces a \( G \)-equivariant isomorphism

\[
W \rightarrow W|_P : W(\pi) \simeq K(\pi)
\]

from the Whittaker model to the Kirillov model.

2. Let \( \pi' \in \text{Cusp}_R \text{GL}(n, F) \). There is a natural isomorphism \( W \rightarrow W' : W(\pi) \rightarrow W(\pi') \) of \( R \)-vector spaces defined by the condition \( W|_P = W'|_P \).

3. There is unique function \( W_\pi \in W(\pi) \) such that

\[
W_\pi|_{GL(n-1,F)} = 1_{GL(n-1,O_F)}.
\]

The function \( W_\pi \) is called the new vector of \( \pi \) and generates the space of vectors of \( \pi \) transforming under \( \Gamma(f) \) according to \( \omega_f \).

4. \( W(\pi) \) is contained in the compactly induced representation \( \text{Ind}_{G,NZ}^G \psi \otimes \omega_\pi \).

Proof. (1) There exists \( W \in W(\pi) \) with \( W(1) \neq 0 \), and \( f : W \rightarrow W_P \) is a non zero \( P \)-equivariant map from \( \pi \) to \( \text{Ind}_N^P \psi \). The map \( f \) is injective of image \( \text{ind}_N^P \psi \), because \( \text{End}_R \tau_R \simeq R \). We get also (2).

(3) The space of \( \tau_R \) is isomorphic by restriction to \( G' = \text{GL}(n-1, F) \), to the space of \( \text{ind}_{N',G'}^G \psi \) where \( N' = N \cap G' \). As \( \psi \) is trivial on \( O_F \), the characteristic function of \( \text{GL}(n-1, O_F) \) belongs to \( \text{ind}_{N'}^G \psi \). For the conductor [JPS2].

(4) Let \( W \in W(\pi) \). The function \( x \rightarrow W(xg) \) on the parabolic standard subgroup \( PZ \) is locally constant of compact support modulo \( NZ \) for all \( g \in G \). As \( G = \text{PZGL}(n, O_F) \), the function \( W \) is of compact support modulo \( NZ \).

Let \( \pi \in \text{Irr}_G \). Let \( E/Q_{\ell} \) be an extension contained in a finite extension of the maximal unramified extension of \( Q_{\ell} \). Example: the extension \( E/Q_{\ell} \) generated by the values of \( \psi \). The ring of integers \( O_E \) is principal. An \( O_E \)-free module \( L \) with an action of \( G \) such that \( L \) is a finite type \( O_E G \)-module and such that \( \overline{Q}_\ell \otimes_{O_E} L \simeq \pi \) is called an \( O_E \)-integral structure of \( \pi \). If such an \( L \) exists, \( \pi \) is called integral, the representation \( r_\ell L = L \otimes_{O_E} \overline{F}_\ell \) is of finite length. One calls \( \overline{Z}_\ell \otimes_{O_E} L \) an integral structure of \( \pi \). When \( L, L' \) are two integral structures of \( \pi \), then the semi-simplifications of \( r_\ell L, r_\ell L' \) are isomorphic (see [V1, II.5.11.b] when \( E/Q_{\ell} \) is finite, and [Vig4, proof of theorem 2, page 416] in general). When \( \pi \in \text{Cusp}_G \) is integral, \( r_\ell L = L \otimes_{O_E} \overline{F}_\ell \) is irreducible; the isomorphism class \( r_\ell \pi \) of \( r_\ell L \) is called the reduction of \( \pi \); any irreducible cuspidal \( \overline{F}_\ell \)-representation of \( G \) is the reduction of an integral irreducible cuspidal \( \overline{Q}_\ell \)-representation of \( G \). For all these facts see [V1, III.5.10].
A function with values in $\overline{Q}_\ell$ is called integral, when its values belong to $\overline{Z}_\ell$. We denote by $K(\pi, \overline{Z}_\ell)$, resp. $W(\pi, \overline{Z}_\ell)$, the set of integral functions in the Kirillov model, resp. Whittaker model, of $\pi \in \text{Cusp}_{\overline{Q}_\ell} G$. Let $\Lambda$ be the maximal ideal of $\overline{Z}_\ell$. The reduction modulo $\ell$ of an integral function $f$ is the function $\pi f$ with values in $\overline{Z}_\ell/\Lambda \simeq \overline{F}_\ell$ deduced from $f$.

**Theorem.** (A) Let $\pi \in \text{Cusp}_{\overline{Q}_\ell} G$ with central character $\omega_\pi$. Then the following properties are equivalent:

(A.1) $\omega_\pi$ is integral.

(A.2) $\pi$ is integral.

(A.3) $K(\pi, \overline{Z}_\ell)$ is a $\overline{Z}_\ell$-structure of $\pi$, called the integral Kirillov model.

(A.4) $W(\pi, \overline{Z}_\ell)$ is a $\overline{Z}_\ell$-structure of $\pi$, called the integral Whittaker model.

(B) When $\pi$ is integral, we have

(B.1) The restriction to $P$ from $W(\pi, \overline{Z}_\ell)$ to $K(\pi, \overline{Z}_\ell)$ is an isomorphism.

(B.2) The integral Kirillov model is $\overline{Z}_\ell P$-generated by any function $f$ with $f(1) = 1$. The integral Whittaker model $W(\pi, \overline{Z}_\ell)$ is $\overline{Z}_\ell G$ generated by the new vector.

(B.3) $\overline{F}_\ell \otimes_{\overline{Z}_\ell} K(\pi, \overline{Z}_\ell) = K(\pi, \overline{Z}_\ell)$ is the Kirillov model, and $\overline{F}_\ell \otimes_{\overline{Z}_\ell} W(\pi, \overline{Z}_\ell) = W(\pi, \overline{Z}_\ell)$ is the Whittaker model of $\pi f$.

**Proof.** The equivalence of (A1) (A2) [VI, 11.4.12]; for the rest [V4 th.2] and the last theorem.  

**Corollary.** Let $\pi, \pi' \in \text{Cusp}_{\overline{Q}_\ell} G$ integral, with central character $\omega_\pi, \omega_{\pi'}$. Then $r_\ell \pi = r_\ell \pi'$ if and only if

\[
r_\ell \omega_\pi = r_\ell \omega_{\pi}, \quad r_\ell \pi(w)(f) = r_\ell \pi'(w)(f)
\]

for all $w \in S_n$, and for all $f$ in the integral Kirillov model.

**Proof.** Use (B.3) and $\text{End}_{\overline{F}_\ell} \tau_{\overline{F}_\ell} \simeq \overline{F}_\ell$. 

**Questions.** Can one define an integral Kirillov or Whittaker model for $\pi \in \text{Irr}_{\overline{Q}_\ell} G$ integral and not cuspidal? What is the action of $S_n$ in the Kirillov model?

**2. The case $n = 2$**

We can go further in the case $n = 2$. Let $\pi \in \text{Cusp}_{\overline{Q}_\ell} G$ where $G = GL(2, F)$. The restriction of $GL(2, F)$ to $GL(1, F) = F^*$ gives an isomorphism from $K(\pi)$ to the space $C^\infty_c(F^*, \overline{Q}_\ell)$ of locally constant functions $F^* \to \overline{Q}_\ell$ with compact support, which respects the natural $\overline{Z}_\ell$-structures $K(\pi, \overline{Z}_\ell) \simeq C^\infty_c(F^*, \overline{Z}_\ell)$. The unique non trivial element of $S_2$ is represented by

\[
w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
The action of $\pi(w)$ on the Kirillov model was described by Jacquet and Langlands [JL, Prop. 2.10 p. 46], using Fourier transform for complex representations.

We choose a $\mathbb{Q}_\ell$-Haar measure $dx$ on $F^*$. The Fourier transform of $f \in C_c^\infty(F^*, \overline{Q}_\ell)$ with respect to $dx$ is

$$\hat{f}(\chi) := \int_{F^*} f(x)\chi(x)dx$$

for any character $\chi : F^* \to \overline{Q}_\ell$.

We choose a uniformizing parameter $p_F$ of $F$. A function $f \in C_c^\infty(F^*, \overline{Q}_\ell)$ is determined by the set of functions $f_n \in C_c^\infty(O_F^*, \overline{Q}_\ell)$ defined by $f_n(x) := f(p_F^{-n}x)$ for all $n \in \mathbb{Z}$. The functions $f_n$ depend on the choice of $p_F$. Extension by zero allows to consider $C_c^\infty(O_F^*, \overline{Q}_\ell)$ as a subspace of $C_c^\infty(F^*, \overline{Q}_\ell)$, because $O_F^*$ is open in $F^*$. We have

$$\hat{f}(\chi) = \sum_n \hat{f}_n(\chi)(p_F^{-n}).$$

For a given character $\chi$, the sum is finite. The functions $\hat{f}_n(\chi)$ depend only on the restriction of $\chi$ to $O_F^*$. Set $\hat{O}_F^* := \text{Hom}(O_F^*, \overline{Q}_\ell)$. One introduces the formal series

$$f(x, X) := \sum_{n \in \mathbb{Z}} f_n(x)X^n, \quad \hat{f}(\chi, X) := \sum_{n \in \mathbb{Z}} \hat{f}_n(\chi)X^n$$

for all $x \in O_F^*$ and for all $\chi \in \hat{O}_F^*$.

Jacquet and Langlands [JL Prop. 2.10 page 46] proved that the action of $\pi(w)$ on the Kirillov model is given by:

$$(\pi(w)f)_n \hat{}(\chi) = c(\pi \otimes \chi^{-1}) \hat{f}_m(\chi^{-1}w_\pi^{-1})$$

for all $\chi \in \hat{O}_F^*$, all integers $n \in \mathbb{Z}$, where $m = -n - f(\pi \otimes \chi^{-1})$, for some constant $c(?) \in \overline{Q}_\ell$ and some integer $f(?) \in \mathbb{Z}$. The formula and $c(\pi \otimes \chi^{-1})$ are independent of the choice of $dx$. The formula is equivalent to

$$(\pi(w)f)^\epsilon(\chi, X) = \epsilon(\pi \otimes \chi^{-1})\hat{f}(\chi^{-1}w_\pi^{-1}, X^{-1})$$

for all $\overline{Q}_\ell$-characters $\chi$ of $O_F^*$, where the epsilon factor is

$$\epsilon(\pi \otimes \chi^{-1}) = c(\pi \otimes \chi^{-1})X^{f(\pi \otimes \chi^{-1})}.$$
We suppose that $dx$ is a $\mathbb{Z}_\ell$-Haar measure on $F^*$ which is not divisible by $\ell$. Let

$$\mathcal{L} = \text{the Fourier transform of } C^\infty_c(O^*_F, \overline{\mathbb{Z}_\ell}).$$

We have $\mathcal{L} \subset C^\infty_c(\hat{O}_F^*, \overline{\mathbb{Z}_\ell})$ and $\mathcal{L} = C^\infty_c(\hat{O}_F^*, \overline{\mathbb{Z}_\ell})$ if and only if $q \not\equiv 1 \pmod{\ell}$ [V2]. In general, we separate the $\ell$-regular part $X$ of $O^*_F$ from the $\ell$-part $Y$ of $O^*_F$, which is a cyclic group of order $m = \ell^n$. The volume of $X$ for $dx$ should be a unit in $\mathbb{Z}_\ell^*$; we can suppose it is equal to 1. The group of $\overline{\mathbb{Q}}_\ell$-characters satisfies $\hat{O}_F^* \simeq \hat{X} \times \hat{Y}$. A general character in $\hat{O}_F^*$ is now written as $\chi\mu$ where $\chi \in \hat{X}$ and $\mu \in \hat{Y}$, and a function $v : \hat{O}_F^* \to \overline{\mathbb{Q}}_\ell$ is thought as a function $v : \hat{X} \to C(\hat{Y}, \overline{\mathbb{Q}}_\ell)$ with $v(\chi)(\mu) := v(\chi\mu)$.

The $\mathbb{Z}_\ell$-module $\mathcal{L}$ consists of all functions $v : \hat{X} \to L$ with compact support, where

$$L \subset C^\infty_c(\hat{Y}, \mathbb{Z}_\ell)$$

is the free $\mathbb{Z}_\ell$-module with basis the characters $y : \mu \to \mu(y^{-1})$ of $\hat{Y}$ for all $y \in Y$.

We need some elementary linear algebra. The $\mathbb{Z}_\ell$-module $L$ is the set of functions $v \in C^\infty_c(\hat{Y}, \mathbb{Q}_\ell)$ such that

$$y \mapsto <v, y> := |Y|^{-1} \sum_{\mu \in \hat{Y}} v(\mu) \mu(y)$$

belongs to $C(Y, \mathbb{Z}_\ell)$. The orthogonality formula of characters gives

$$v = \sum_{y \in Y} <v, y> y$$

for all $v \in C(\hat{Y}, \mathbb{Q}_\ell)$. For the usual product, $C^\infty_c(\hat{Y}, \mathbb{Q}_\ell)$ is an algebra.

**Lemma.** Let $v \in C^\infty_c(\hat{Y}, \mathbb{Q}_\ell)$.

(i) The inclusion $vL \subset L$ is equivalent to $v \in L$.

(ii) The equality $vL = L$ is equivalent to $v \in L$ and $v(\mu) \in \mathbb{Z}_\ell$ for all $\mu \in \hat{Y}$.

(iii) The inclusion $vL \subset \Lambda L$ is equivalent to $<v, y> \in \Lambda$ for all $y \in Y$ ($\Lambda$ is the maximal ideal of $\mathbb{Z}_\ell$).

**Proof.** (i) The inclusion $vL \subset L$ is equivalent to $<v_z, z'> = <v, z^{-1}z'> \in \mathbb{Z}_\ell$ for all $z, z' \in Y$, which is equivalent to $v \in L$.

(ii) $vL = L$ means that $v_z$ for $z \in Y$ is a basis of $L$. We have $v_z = \sum_{z' \in Y} <v, z^{-1}z'> z'$, hence $vL = L$ means that

$$<v, z^{-1}z'>_{z, z'} \in SL(m, \mathbb{Z}_\ell).$$

The Dedekind determinant $\text{det}(<v, z^{-1}z'>_{z, z'})$ is equal to $\prod_{\mu \in \hat{Y}} v(\mu)$ (see [L] exercise 28 page 495).

(iii) see the proof of (i).
Let \( \pi \in \text{Cusp}_{\mathbb{Q}_\ell} G \) integral. As \( \pi(w) \) is an isomorphism of the integral Kirillov model, the function

\[
c(\pi \otimes \chi) : \mu \in \hat{Y} \to c(\pi \otimes \chi \mu) \in \mathbb{Q}_\ell
\]
satisfies \( c(\pi \otimes \chi)L = L \) for all character \( \chi \in \hat{X} \). We apply the lemma to \( c(\pi \otimes \chi) \). We define \textbf{new epsilon factors}

\[
\varepsilon(\pi, y) := \langle c(\pi), y > X^{f(\pi)}; \quad < c(\pi), y > = |Y|^{-1} \sum_{\mu \in \hat{Y}} c(\pi \otimes \mu) \mu(y),
\]
for all \( y \in Y \). As have \( f(\pi) \geq 2 \) for \( \pi \in \text{Cusp}_{\mathbb{Q}_\ell} G \), we have \( f(\pi) = f(\pi \otimes \mu) \geq 2 \) for all \( \mu \in \hat{Y} \). When \( Y \) is trivial (i.e. \( q \equiv 1 \mod \ell \)), they are simply the usual ones.

\textbf{Theorem.} (1) Let \( \pi \in \text{Cusp}_{\mathbb{Q}_\ell} G \) integral. Then the constant of the epsilon factor is a unit \( c(\pi) \in \mathbb{Z}_\ell^* \) and the new constants \( < c(\pi), y > \in \mathbb{Z}_\ell \) are integral, for all \( y \in Y \).

(2) Let \( \pi, \pi' \in \text{Cusp}_{\mathbb{Q}_\ell} G \) integral with central characters \( \omega_\pi, \omega_{\pi'} \). Then \( r_\ell \pi = r_\ell \pi' \) if and only if \( r_\ell \omega_\pi = r_\ell \omega_{\pi'} \) and their new epsilon factors have the same reduction modulo \( \ell \): the conductors \( f(\pi \otimes \chi) = f(\pi' \otimes \chi) \) are equal, and the new constants have the same reduction modulo \( \ell \):

\[
r_\ell < c(\pi \otimes \chi), y > = r_\ell < c(\pi' \otimes \chi), y >
\]
for all \( y \in Y \), and all \( \mathbb{Q}_\ell \)-characters \( \chi \in \hat{X} \).

\textbf{Proof.} With the last corollary of the paragraph (1), \( r_\ell \pi = r_\ell \pi' \) if and only if \( r_\ell \omega_\pi = r_\ell \omega_{\pi'} \) and

\[
c(\pi \otimes \chi) f_m(\chi^{-1} \omega_\pi^{-1}) = c(\pi' \otimes \chi) f'_m(\chi^{-1} \omega_{\pi'}^{-1}) \quad \text{modulo } \Lambda \mathcal{L}
\]
for all \( f_n \in C^\infty_c(O_F^*, \mathbb{Z}_\ell) \) and all \( n \in \mathbb{Z} \). With the lemma, we deduce the theorem. \( \square \)

We apply now the theorem to representations over \( \overline{F}_\ell \). Any \( \pi \in \text{Cusp}_{\overline{F}_\ell} G \) lifts to \( \overline{Q}_\ell \) and we can define epsilon factors

\[
\varepsilon(\pi \otimes \chi, y) := \langle c(\pi \otimes \chi), y > X^{f(\pi \otimes \chi)}
\]
for all \( y \in Y \) and all \( \chi \in \text{Hom}(O_F^*, \overline{F}_\ell^*) = \text{Hom}(X, \overline{F}_\ell^*) \), by reduction modulo \( \ell \). They are not zero for any \( (y, \chi) \).

\textbf{Corollary.} \( \pi, \pi' \in \text{Cusp}_{\overline{F}_\ell} G \) are isomorphic if and only if they have the same central character and the same epsilon factors

\[
\varepsilon(\pi \otimes \chi, y) = \varepsilon(\pi' \otimes \chi, y)
\]
for all \( y \in Y \), and for all character \( \chi \in \text{Hom}(O_F^*, \overline{F}_\ell^*) \).
Final remarks. a) When \( n > 2 \), the groups \( GL(m, O_F)^* \) for \( m \leq n - 1 \) replace \( O_F^* \).

b) Using the explicit description for the irreducible representations of dimension \( n \) of \( WF \) \([V3]\), one could try to prove a similar theorem for the irreducible integral \( \mathbb{Q}_F \)-representations of \( WF \) of dimension \( n \). To my knowledge this is a known and harder problem, which is not solved in the complex case.

References


[H2] G. Henniart, Une preuve simple des conjectures de Langlands pour \( GL(n) \) sur un corps \( p \)-adique. Prépublication 99-14, Orsay.


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