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S-integral points on elliptic curves - Notes on a paper of B. M. M. de Weger


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S-integral points on elliptic curves - Notes on a paper of B. M. M. de Weger

par Emanuel Herrmann* et Attila Pethő**

1. Introduction

In a recent paper [12] B.M.M. de Weger solved the Diophantine equation

\[ y^2 = x^3 - 228x + 848 \]

completely in rational numbers \( x, y \) such that their denominator in the lowest form is a power of 2. With other words, he solved (1) in \( S \)-integers where \( S = \{2, \infty\} \). De Weger uses in the proof algebraic number theoretical considerations and lower estimates for linear forms in complex and \( p \)-adic elliptic logarithms.

In the present paper we will give a much shorter proof of a generalization of Theorem 1 of [12]. Here we use the theory of elliptic curves and linear forms in elliptic logarithms. More precisely, we are using a theorem of Rémond and Urfels [6], which can be applied for curves of rank at most 2. An alternative method which avoids lower bounds for linear forms in \( q \)-adic elliptic logarithms is given in [5]. However the bounds coming from [5] are

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in the actual case much larger as working directly with the Theorem of Rémond and Urfels (cf. Section 3).

We now state our result.

**Theorem 1.** Let \( S = \{2, 3, 5, 7, \infty\} \). Then the equation
\[
y^2 = x^3 - 228x + 848
\]
has only 65 \( S \)-integer solutions \((x, \pm y)\) listed in Table 2 at the end of this paper.

2. Notations and Auxiliary Results

Let the elliptic curve be defined by the equation
\[
y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{Z}.
\]
Let \( S = \{q_1, \ldots, q_{s-1}, q_s = \infty\} \) be a set of primes including the infinite prime. To simplify the presentation we assume that the equation (2) is minimal for every finite prime \( q \in S \). For the general case we refer to the paper [5].

Let \( P_1, \ldots, P_r \) denote a basis of the Mordell-Weil group \( E(\mathbb{Q}) \) and let \( g \) be the order of the torsion subgroup \( E_{\text{tors}}(\mathbb{Q}) \) of \( E(\mathbb{Q}) \). Let \( \hat{h} \) denote the Néron-Tate height on \( E(\mathbb{Q}) \). Designate by \( \lambda \) the smallest eigenvalue of the positive definite regulator matrix \((\hat{h}(P_i, P_j))_{1 \leq i, j \leq r}\).

Let \( \wp(u) \) be the Weierstrass \( \wp \)-function corresponding to the curve \( E(\mathbb{C}) \). Let \( \Omega = \langle \omega_1, \omega_2 \rangle \) be its fundamental lattice and \( \omega_1 \) its real period. There exists, for any \( P = (x, y) \in E(\mathbb{C}) \), an element \( u \in \mathbb{C}/\Omega \) such that \( (x, y) = (\wp(u), \frac{1}{2} \wp'(u)) \). This is called the (complex) elliptic logarithm of \( P \). In the sequel \( u_{i, \infty} \) denotes the elliptic logarithm of \( P_i \) for \( i = 1, \ldots, r \). We put \( u_{i, \infty} = g^{u_{i, \infty}}_{\omega_1} \).

For a finite prime \( q \in S \) let \( E_0(\mathbb{Q}_q) \) denote the points of \( E(\mathbb{Q}_q) \) with non-singular reduction modulo \( q \). Then the index \([E(\mathbb{Q}_q) : E_0(\mathbb{Q}_q)]\) is finite, and equal to the Tamagawa number \( c_q \) because by our assumption equation (2) is minimal at \( q \). Let further \( \tilde{E} \) denote the reduced curve \( E \) modulo \( q \). Let \( N_q = \#\tilde{E}(\mathbb{F}_q) \) be the number of rational points of \( \tilde{E}/\mathbb{F}_q \). With the order \( g \) of the torsion group, we define the number
\[
m = m_q = \text{lcm}(g, c_q \cdot N_q).
\]
Finally for the finite places \( q \in S \), let \( u_{i, q} \) denote the \( q \)-adic elliptic logarithm of \( mP_i \) for \( i = 1, \ldots, r \). For the definition and basic properties of \( q \)-adic elliptic logarithms we refer to Silverman [7] and to [5]. Now we state the main result of [5] in the special case considered, i.e. for curves given in short Weierstrass form.

**Theorem A.** Let the elliptic curve \( E(\mathbb{Q}) \) be defined by equation (2), which is minimal for every finite prime \( q \in S \). Assume that the \( S \)-integral point
$P = (x, y) \in E(Z_S)$ has the representation

\begin{equation}
P = \sum_{i=1}^{r} n_i P_i + T
\end{equation}

with \( n_i \in \mathbb{Z}, i = 1, \ldots, r, \) and \( T \) a torsion point of \( E(Q) \). For \( N(P) = \max\{|n_i|, i = 1, \ldots, r\} \), we have

\begin{equation}
N(P) \leq N_0 = \sqrt[\lambda]{(\frac{k_1}{2} + k_2)}
\end{equation}

with \( k_2 = \log \max\{|2A|^{1/2}, |4B|^{1/3}\} \),

\[ k_1' = 7 \cdot 10^{38s+49} s^{20s+15} Q^{24} (\log^* Q)^{4s-2} k_3 (\log k_3)^2 ((20s - 19) k_3 + \log(ek_4)), \]

\[ k_1 = k_1' + 2 \log 6, \]

where \( \log^* Q = \max\{\log Q, 1\} \) for \( Q = \max\{q_1, \ldots, q_{s-1}\} \), \( s = \#S \),

\[ k_3 = \frac{32}{3} \sqrt{|\Delta_0|} (8 + \frac{1}{2} \log |\Delta_0|)^4, \]

\[ k_4 = 10^4 \max\{16A^2, 256 \sqrt{|\Delta_0|^3}\} \]

with \( \Delta_0 = 4A^3 + 27B^2 \). Moreover, there exists a place \( q \in S \) such that

\begin{equation}
\left| \sum_{i=1}^{r} n_i u_{i,q} + n_{r+1} \right|_q \leq k_5 \exp \left\{ \frac{1}{s} N(P)^2 + \frac{k_2}{s} \right\}
\end{equation}

with \( n_{r+1} \in \mathbb{Z} \) if \( q = \infty \) and \( n_{r+1} = 0 \) otherwise, and with \( k_5 = \frac{2q}{3\omega_1} \) if \( q = \infty \) and \( k_5 = 1 \) otherwise.

Theorem A together with numerical Diophantine approximation techniques is sufficient to prove our Theorem 1. However it was pointed out already in \[5\] that combining the method of Smart \[8\] with results of David \[2\] and of Remond and Urfels \[6\] one can obtain a much better estimate for \( N(P) \) as by the one implied by Theorem A. In the sequel we assume \( r \leq 2 \). To formulate the next theorem we have to introduce further notations. Let
For a finite place $q \in S'$ let $j = \frac{j_1}{j_2}$ with $j_1, j_2 \in \mathbb{Z}$ and $\gcd(j_1, j_2) = 1$ be the $j$–invariant of $E(\mathbb{Q})$. Put

$$h = \log \max\{4|A_j|, 4|B_j|, |j_1|\},$$

$$\log V_i = \max \left\{ \hat{h}(P_i), h, \frac{3\pi |u'_{i,\infty}^2}{\Im \tau} \right\}, \quad i = 1, 2,$$

$$\log V_0 = \max \left\{ h, \frac{3\pi}{\Im \tau} \right\},$$

$$k_{6,\infty} = \frac{k_2 + s \log k_5}{\lambda},$$

$$k_{7,\infty} = \frac{2 \cdot 10^{68} \cdot s \cdot h^5}{\lambda} \prod_{i=0}^{2} \log V_i.$$

For a finite place $q \in S$ let

$$\alpha_q = \begin{cases} 3, & \text{if } q = 2 \\ \frac{1}{q-1}, & \text{otherwise} \end{cases},$$

$$\sigma_q = \left( q^\alpha_q \max\{|u'_{1,q}|, |u'_{2,q}|\} \right)^{-1},$$

$$d_q = \max\{1, 1/\log \sigma_q\},$$

$$a_i = \max\{1, \hat{h}(P_i)\}, \quad i = 1, 2,$$

$$\beta = \max\{\log N(P), \log |A|_\infty, \log |B|_\infty, a_1, a_2, d_q\},$$

$$\gamma = \max\{\log |A|_\infty, \log |B|_\infty, \log \beta\},$$

$$k_{6,q} = \frac{k_2}{\lambda},$$

$$k_{7,q} \geq \frac{(3.6 \cdot 10^{25} \cdot a_1 a_2 q^6 \log \sigma_q)}{\lambda}.$$

**Theorem B.** Assuming that $r \leq 2$ and using the notations introduced in Theorem A and above we have

$$N(P) \leq N_1 := \max\{N_q : q \in S\},$$

where

$$N_q = \begin{cases} 2^5 \sqrt{k_{6,\infty} k_{7,\infty}} (\log 5^5 k_{7,\infty})^{5/2}, & \text{if } q = \infty, \\ 2^4 \sqrt{k_{6,q} k_{7,q}} (\log 4^4 k_{7,q})^2, & \text{if } q \in S \setminus \{\infty\}. \end{cases}$$

**Proof.** Combining inequality (5) with the lower bounds for linear forms in elliptic logarithms due to David [2] and for linear forms in at most two $q$–adic elliptic logarithms due to Rémond and Urfels [6] one obtains the upper bound for $N(P)$ analogously as described for example in Gebel, Pethő and Zimmer [3, 4]. Therefore we omit the details. \qed
3. Proof of Theorem 1

3.1. Basic data of the elliptic curve. In the sequel we denote by $E$ the elliptic curve over $\mathbb{Q}$ defined by equation (1). Let $S = \{2, 3, 5, 7, \infty\}$. It is easy to check, that (1) is minimal for every finite prime $q \in S$. Actually, it is a global minimal model of $E$. The discriminant of $E$ is $\Delta = -16\Delta_0$ with $\Delta_0 = -27993600$. We have
\[
E(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^2,
\]
where the only non-trivial torsion point is $(4,0)$ and a basis of the infinite part of the Mordell-Weil group is $P_1 = (-2, 36), P_2 = (-11, 45)$. (See Tzanakis [10], or one of of the programs apecs [13], Magma$^1$ [1], mwrank [14] or Simath [15].)

Now we can compute the fundamental parallelogram of the associated Weierstrass $\wp$-function and get
\[
\omega_1 = 0.767848, \quad \omega_2 = -0.631356 \cdot i \quad \text{and} \quad \tau = \frac{\omega_1}{\omega_2} = 1.216188 \cdot i.
\]
The regulator matrix of $E$ is
\[
R = \begin{pmatrix}
0.423441 & -0.158771 \\
-0.158771 & 0.906408
\end{pmatrix},
\]
hence its smallest eigenvalue is given by $\lambda = 0.375922$.

Using Tate’s algorithm [9] we compute the Tamagawa numbers
\[
c_2 = 4, \quad c_3 = 4, \quad c_5 = 2 \quad \text{and} \quad c_7 = 1.
\]
The curve $E$ has additive reduction at the primes 2 and 3, multiplicative reduction at 5 and good reduction at 7. Hence,
\[
N_2 = 2, \quad N_3 = 3, \quad N_5 = 6 \quad \text{and} \quad N_7 = 12.
\]
Using these data we can compute the numbers $m_q$ and obtain
\[
m_2 = 8, \quad m_3 = 12, \quad m_5 = 12 \quad \text{and} \quad m_7 = 12.
\]

3.2. Upper Bounds for $N(P)$.

(i) The first way to obtain an upper bound for $N(P)$ is to calculate $N_0$ of Theorem A. We have actually $Q = 7, s = 5$,
\[
k_2 = \log \max\{456^{1/2}, 3392^{1/3}\} = 3.061246,
\]
\[
k_3 = \frac{32}{3} \sqrt{|\Delta_0|} \left(8 + \frac{1}{2} \log |\Delta_0|\right)^4 = 4.258342 \cdot 10^9,
\]
\[
k_4 = 10^4 \max\{16 \cdot 228^2, 256 \cdot |\Delta_0|^{3/2}\} = 3.791649 \cdot 10^{17}
\]
and $k_1 = 3.730724 \cdot 10^{369}$, hence $N(P) \leq N_0 = 7.044216 \cdot 10^{184}$.

(ii) Another, a bit more complicated, way to find an upper bound for $N(P)$ is to compute $N_1 = \max\{N_q : q \in S\}$ as defined in Theorem B.

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$^1$Magma version 2.6 will have an implementation of the algorithm described in [5].
Consider first the case $q = \infty$. Then we have

$$h = \log \max \{4 \cdot 228 \cdot 75, 4 \cdot 848 \cdot 75, 2^5 \cdot 19^3 \} = 12.446663,$$

$$\log V_0 = \max \{h, \frac{3\pi}{\text{Im}(\tau)} \} = 12.446663,$$

$$\log V_1 = \max \{\hat{h}(P_1), h, \frac{3\pi g^2 |\omega_{1,\infty}|^2}{\omega_{2,\infty}^2 \text{Im}(\tau)} \} = 21.645104,$$

$$\log V_2 = \max \{\hat{h}(P_2), h, \frac{3\pi g^2 |\omega_{2,\infty}|^2}{\omega_{1,\infty}^2 \text{Im}(\tau)} \} = 28.279603,$$

$$k_{5,\infty} = \frac{4}{3\omega_{1}} = 1.736455,$$

$$k_{6,\infty} = 15.483196,$$

$$k_{7,\infty} = 6.054145 \cdot 10^{78}.$$

Thus we obtain $N_{\infty} \leq 1.530526 \cdot 10^{47}$ after a simple computation.

Next we have to consider the cases $q = 2, 3, 5$ and $7$. In Table 1 below you find the actual values of $\alpha_q, \sigma_q$ and $d_q$.

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<td>$k_{7,q}$</td>
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<td>$9.5742 \cdot 10^{27}$</td>
<td>$5.779766 \cdot 10^{26}$</td>
<td>$7.76455 \cdot 10^{26}$</td>
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</tbody>
</table>

The following values are independent of $q \in \{2, 3, 5, 7\}$

$$a_1 = \max \{1, \hat{h}((-2, 36))\} = \max \{1, 0.423441\} = 1,$$

$$a_2 = \max \{1, \hat{h}((-11, 45))\} = \max \{1, 0.906408\} = 1,$$

$$k_{6,q} = k_2/\lambda = 8.143301.$$

Choosing the worst cases from Table 1 we see that we can take

$$k_{7,q} = k_{7,3} = 9.5742 \cdot 10^{27}, \quad q = 2, 3, 5, 7,$$

thus

$$N_q = N_3 = 2.187487 \cdot 10^{19}, \quad q = 2, 3, 5, 7.$$

These inequalities imply

$$N(P) \leq N_I = \max \{N_q : q \in S\} = 1.530526 \cdot 10^{47}$$

by Theorem B. Since $N_I$ is much smaller than $N_0$ we use this value in the sequel.
3.3. Reduction of the large upper bound for $N(P)$. By Theorem 1, and by the last section we have to solve the Diophantine approximation problem

$$|n_1u'_1,q + n_2u'_2,q + n_3|_q \leq k_5\exp\{0.075184 \cdot N(P)^2 + 0.6122492\},$$

$$N(P) \leq N_1 = 1.530526 \cdot 10^{47}$$

for each $q \in S$.

To solve these systems we use the well known reduction procedure of de Weger [11]. (See also Smart [8].) For details about the high precision computation of $q$-adic elliptic logarithms we refer to Pethő et al. [5]. We shall also use the notations introduced there.

We first take $q = \infty$ and perform a de Weger reduction with $C = 10^{142}$. We obtain the new upper bound $N(P) \leq \mathcal{M}_\infty = 67$ in the case $q = \infty$. Comparing this bound with $N_q, q = 2, 3, 5, 7$ we obtain

$$N(P) \leq N_3 = 2.187487 \cdot 10^{19},$$

i.e. we may perform the $q$-adic reduction steps with this value.

To do this we compute for each $q \in S \setminus \{\infty\}$, the $q$-adic elliptic logarithms of $m_qP_i, i = 1, 2$, with precision at least

$$n_2 = 129, \quad n_3 = 82, \quad n_5 = 56, \quad n_7 = 46.$$ 

This precision is necessary to carry out the $q$-adic de Weger reduction. For this purpose we use the method of [5].

$$u'_{1,2} = 134584334573222732131510464853384888320 + O(2^{128})$$

$$u'_{2,2} = 224603122385055121905025779589746548856 + O(2^{128})$$

$$u'_{1,3} = 35130898366670225251067311603381664587 + O(3^{81})$$

$$u'_{2,3} = 32674326287561878726624624078558984866 + O(3^{81})$$

$$u'_{1,5} = 118414103305724592543524002578287458095 + O(5^{55})$$

$$u'_{2,5} = 193714651202697832194263283063279750580 + O(5^{55})$$

$$u'_{1,7} = 49086609441793589144883973076015987885 + O(7^{46})$$

$$u'_{2,7} = 723939447229120403790851561285560713079 + O(7^{46})$$

Now we perform the $q$-adic de Weger reduction with the values $C_2 = 2^{128}, \quad C_3 = 3^{81}, \quad C_5 = 5^{55}$ and $C_7 = 7^{46}$ and obtain the new bound

$$N(P) \leq \max\{\mathcal{M}_\infty = 67, \mathcal{M}_2 = 12, \mathcal{M}_3 = 13, \mathcal{M}_5 = 13, \mathcal{M}_7 = 13\}.$$ 

This new upper bound for $N(P)$ can be further reduced. On repeating this reduction process 3-times, we eventually get $N(P) \leq 13$, which cannot be reduced any further.
Table 2

$S$-integral points $P = (x, y) = \left( \frac{\xi}{\zeta}, \frac{\eta}{\zeta^2} \right) = \sum_{i=1}^{2} n_i P_i + T_j$, $j = 0, 1$

on $E$: $y^2 = x^3 - 228x + 848$ for $S = \{2, 3, 5, 7, \infty\}$

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