

SOROOSH YAZDANI

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Multiplicative functions and k -automatic sequences

par SOROOSH YAZDANI

RÉSUMÉ. Une suite est dite k -automatique si son n^e terme peut être engendré par une machine à états finis lisant en entrée le développement de n en base k . Nous prouvons que, pour de nombreuses fonctions multiplicatives f , la suite $(f(n) \bmod v)_{n \geq 1}$ n'est pas k -automatique. C'est en particulier le cas pour les fonctions multiplicatives $\tau_m(n)$, $\sigma_m(n)$, $\mu(n)$ et $\phi(n)$.

ABSTRACT. A sequence is called k -automatic if the n 'th term in the sequence can be generated by a finite state machine, reading n in base k as input. We show that for many multiplicative functions, the sequence $(f(n) \bmod v)_{n \geq 1}$ is not k -automatic. Among these multiplicative functions are $\tau_m(n)$, $\sigma_m(n)$, $\mu(n)$, and $\phi(n)$.

We call a function $f : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{C}$ multiplicative, if for all $m, n \in \mathbb{N} \setminus \{0\}$, m and n coprime, we have $f(mn) = f(m)f(n)$. As usual let $\tau(n)$, $\sigma(n)$, $\phi(n)$, $\mu(n)$ represent the number of divisors of n , sum of the divisors of n , number of numbers less than or equal to n and prime to n , and the Möbius function respectively. We know that $\tau(n)$, $\sigma(n)$, $\phi(n)$, and $\mu(n)$ are multiplicative. Also let $\tau_m(n)$ be number of elements in $\{(a_1, a_2, \dots, a_m) \mid a_1 a_2 \cdots a_m = n \text{ and } a_1, a_2, \dots, a_m \in \mathbb{N} \setminus \{0\}\}$. Then we have

$$(1) \quad \tau_m(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}) = \prod_{i=1}^t \binom{m + \alpha_i - 1}{m - 1},$$

where p_i 's are distinct primes (see for example [9, p. 72]). Furthermore let

$$\sigma_m(n) = \sum_{k|n} k^m.$$

Recall that $\sigma_m(n)$ is multiplicative for all integers m . Note that $\sigma_1(n) = \sigma(n)$, and $\tau_2(n) = \tau(n)$.

Given $k \geq 2$, we say a sequence $\mathbf{T} = (t(n))_{n \geq 1}$ is k -automatic if and only if

$$\mathbf{T}^{(k)} = \left\{ \mathbf{T}_{l,r}^{(k)} \mid l \geq 0 \text{ and } 0 \leq r < k^l \right\}$$

is finite, where $\mathbf{T}_{l,r}^{(k)} = (t(k^l n + r))_{n \geq 1}$. The set $\mathbf{T}^{(k)}$ is called the k -kernel of \mathbf{T} . We say a set $S \subset \mathbb{N} \setminus \{0\}$ is k -automatic if the sequence $(\chi_S(n))_{n \geq 1}$ is k -automatic, where

$$\chi_S(n) = \begin{cases} 1 & \text{if } n \in S, \\ 0 & \text{otherwise.} \end{cases}$$

If $t : \mathbb{N} \setminus \{0\} \rightarrow X$ for some set X , and if there is a mapping $\Phi : X \rightarrow Y$, then we can extend Φ to sequences in X with $\Phi(\mathbf{T}) = (\Phi(t(n)))_{n \geq 1}$. Note that

$$\Phi : \mathbf{T}^{(k)} \rightarrow (\Phi(\mathbf{T}))^{(k)}$$

is an onto mapping. Specifically note that the cardinality of $\mathbf{T}^{(k)}$ is greater than or equal to the cardinality of $(\Phi(\mathbf{T}))^{(k)}$, and hence if \mathbf{T} is k -automatic, then so is $\Phi(\mathbf{T})$. Therefore we have the following,

Lemma 1. *Let $(f(n))_{n \geq 1}$ be a sequence of integers. If there exist integers $v, k \geq 2$ such that $(f(n) \bmod v)_{n \geq 1}$ is k -automatic, then for all $q \mid v$ we have that the sequence $(f(n) \bmod q)_{n \geq 1}$ is also k -automatic.*

The term k -automatic is used because one can compute $t(n)$ by feeding the base k representation of n as an input to a finite state machine [5]. In [3], see also [4], it is shown that given prime p and a sequence $(t(n))_{n \geq 1}$ with values in \mathbb{F}_p , then

$$F(X) = \sum_{n \geq 0} t(n)X^n \in \mathbb{F}_p[[X]]$$

is algebraic over $\mathbb{F}_p(X)$ if and only if $(t(n))_{n \geq 1}$ is p -automatic.

Now we proceed to prove the first theorem in this paper, whose proof is a variation of a proof suggested by J. Shallit.

Theorem 2. *Let $v > 1$ be an integer and f a multiplicative function. Assume that for some integer $h \geq 1$ there exist infinitely many primes q_1 such that $f(q_1^h) \equiv 0 \pmod{v}$. Furthermore assume that there exist relatively prime integers b and c such that for all primes $q_2 \equiv c \pmod{b}$ we have $f(q_2) \not\equiv 0 \pmod{v}$. Then the sequence $F = (f(n) \bmod v)_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Proof. Choose an arbitrary integer $k \geq 2$. Since $\gcd(b, c) = 1$, by Dirichlet's theorem there exists some integer m such that $bm + c > k$, and $bm + c$ is prime. Letting $a = bm + c$, we get $\gcd(a, bk) = 1$.

Now we will show that given $l, r_1, r_2 \in \mathbb{N} \setminus \{0\}$ such that $k^l > 2bk$, $0 \leq r_1 \neq r_2 < k^l$, and $r_2 \equiv a \pmod{bk}$, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $f(k^l n + r_1) \not\equiv f(k^l n + r_2) \pmod{v}$, hence $F_{l,r_1}^{(k)} \neq F_{l,r_2}^{(k)}$. This in turn means that the k -kernel of F is infinite, which means that F is not k -automatic.

Choose a prime $q_1 > k^l$ such that $f(q_1^h) \equiv 0 \pmod{v}$. Observe that $\gcd(bk^l, q_1^h) = 1$ since q_1 is prime and $q_1 > k^l > b$. Hence there exists an integer n_0 such that

$$(2) \quad n_0 bk^l + r_1 \equiv q_1^h \pmod{q_1^{h+1}}.$$

Furthermore observe that

$$\begin{aligned} n_0 bk^l + r_2 &\equiv r_2 \pmod{b} \\ &\equiv a \pmod{b}, \end{aligned}$$

for all n_0 . Therefore for all $j \in \mathbb{N} \setminus \{0\}$, we have

$$q_1^h \parallel bk^l(n_0 + jq_1^{h+1}b) + r_1$$

and

$$(3) \quad bk^l(n_0 + jq_1^{h+1}b) + r_2 \equiv a \pmod{b}.$$

We need to show that for some j , $n_0 + jq_1^{h+1}b > 0$ and the left-hand side of Equation (3) is prime. To do so we will show that $\gcd(n_0 bk^l + r_2, k^l q_1^{h+1} b^2) = 1$, and apply Dirichlet's theorem.

Note that $r_2 \equiv a \pmod{k}$, and $\gcd(a, k) = 1$. Therefore

$$\gcd(k^l, n_0 bk^l + r_2) = \gcd(k^l, r_2) = 1.$$

Also $n_0 bk^l + r_2 \equiv a \pmod{b}$. Since $\gcd(a, b) = 1$, we get

$$\gcd(b, n_0 bk^l + r_2) = \gcd(b, r_2) = 1.$$

Finally from Equation (2) we have that $n_0 bk^l + r_2 \equiv q_1^h + r_2 - r_1 \pmod{q_1^{h+1}}$. We know that $r_1 \neq r_2$, and $0 \leq r_1, r_2 < k^l < q_1$. Since q_1 is prime, we get that

$$\gcd(q_1, n_0 bk^l + r_2) = \gcd(q_1, r_2 - r_1) = 1.$$

Therefore $\gcd(n_0 bk^l + r_2, k^l q_1^{h+1} b^2) = 1$. Hence by Dirichlet's theorem, we can find an integer $j > |n_0|$ such that

$$\begin{aligned} k^l q_1^{h+1} b^2 j + n_0 bk^l + r_2 &\equiv a \pmod{b} \\ &\equiv c \pmod{b} \end{aligned}$$

is prime. By hypothesis, $f(k^l(q_1^{h+1} b^2 j + bn_0) + r_2) \pmod{v} \neq 0$.

On the other hand we have that by Equation (2)

$$q_1^h \parallel k^l(q_1^{h+1} b^2 j + bn_0) + r_1.$$

Since f is multiplicative, we have $f(k^l(q_1^{h+1} b^2 j + bn_0) + r_1) \pmod{v} = 0$. Letting $n = q_1^{h+1} b^2 j + bn_0$, we get $f(k^l n + r_1) \not\equiv f(k^l n + r_2) \pmod{v}$.

Therefore F is not k -automatic for any $k \geq 2$. □

From this theorem we immediately get the following corollaries.

Corollary 3. *Given $m \geq 1$ and $v \geq 3$, the sequence $(\sigma_m(n) \bmod v)_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Proof. Given an integer $v \geq 3$, there are infinitely many primes $q_1 \equiv 1 \pmod{v}$. Taking $h = v - 1$ we get

$$\begin{aligned} \sigma(q_1^{v-1}) &\equiv \sum_{k|q_1^{v-1}} k^m \pmod{v} \\ &\equiv \sum_{k=0}^{v-1} q_1^{km} \pmod{v} \\ &\equiv \sum_{k=0}^{v-1} 1 \pmod{v} \\ &\equiv 0 \pmod{v}. \end{aligned}$$

Also for primes $q_2 \equiv 1 \pmod{v}$, we have $\sigma_m(q_2) \bmod v = 2$, since $v \geq 3$. So the hypotheses of Theorem 2 are satisfied, and hence $(\sigma_m(n) \bmod v)_{n \geq 1}$ is not k -automatic. □

Corollary 3 answers the question raised by Allouche and Thakur of whether

$$(4) \quad \sum_{n \geq 1} \sigma_m(n) X^n \in \mathbb{F}_p[[X]]$$

is always transcendental over $\mathbb{F}_p(X)$ for odd primes p [2]. They proved the transcendence of Equation (4) for many cases of p and m in order to give a proof of the function field analogue of Mahler-Manin conjecture. Since $(\sigma_m(n) \bmod p)_{n \geq 1}$ is not p -automatic for primes $p \geq 3$, using Christol's theorem [3] and [4] we get that the formal power series $\sum_{n \geq 1} \sigma_m(n) X^n$ in Equation (4) is always transcendental over $\mathbb{F}_p(X)$.

Corollary 4. *Given $v \geq 3$, the sequence $(\phi(n) \bmod v)_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Proof. Note that

$$\phi(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}) = \prod_{i=1}^t (p_i^{\alpha_i} - p_i^{\alpha_i - 1}).$$

Hence given a prime $q_1 \equiv 1 \pmod{v}$ we have $\phi(q_1) \equiv 0 \pmod{v}$. Also, given a prime $q_2 \equiv -1 \pmod{v}$ we have $\phi(q_2) \equiv -2 \pmod{v}$. Since $v \geq 3$ the hypotheses of Theorem 2 are satisfied. Hence $(\phi(n))_{n \geq 1}$ is not k -automatic. □

Note that $(\phi(n) \bmod 2)_{n \geq 1}$ is k -automatic for all k , since $\phi(n)$ is even for all $n > 2$, and hence $(\phi(n) \bmod 2)_{n \geq 1}$ is constant for $n > 2$.

We also get the following well-known result, which is a direct consequence of the fact that square-free numbers are not k -automatic [5, p. 183].

Corollary 5. *Given an integer $v \geq 2$, the sequence $(\mu(n) \bmod v)_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Proof. This is a direct consequence of Theorem 2. □

The proof of Theorem 2 relied heavily on the existence of primes q such that $f(q) \not\equiv 0 \pmod{v}$. Now we look at another set of multiplicative functions f , where $v|f(q)$ for all primes q , and some integer v . The technique used in this section is different from that used in the proof of the previous theorem, and we need to give the following definition.

Definition 1. Let $T = (t(n))_{n \geq 1}$, and $\#S$ be the number of elements in the set S . Then the *density* of the symbol a in the sequence T is defined to be

$$d(T, a) = \lim_{n \rightarrow \infty} \frac{\#\{i \leq n \mid t(i) = a\}}{n},$$

if the limit exists, and is undefined otherwise.

Using this definition we will cite the following lemma due to Minsky and Papert [8], see also [5, p. 184].

Lemma 6. *For any k -automatic sequence F , if $d(F, a) = 0$ then*

$$\limsup_{j \rightarrow \infty} \frac{\alpha_{j+1}}{\alpha_j} > 1,$$

where α_j is the position of the j 'th occurrence of a .

We now proceed to prove the following theorem.

Theorem 7. *Let $v > 1$ be an integer, and let f be a multiplicative function such that $f(\prod p_i^{\beta_i}) = \prod g(\beta_i)$ for some function g , where the p_i are distinct primes. Also suppose that $g(1) \equiv 0 \pmod{v}$ and that there exists some integer $h \geq 1$ such that $g(h) \not\equiv 0 \pmod{v}$. Then $F = (f(n) \bmod v)_{n \geq 1}$ is not k -automatic for any integer $k \geq 2$.*

Proof. First, we need the following lemma.

Lemma 8. *Let f be a multiplicative function such that $f(q) \equiv 0 \pmod{v}$ for all primes q . Then*

$$d(F, a) = \begin{cases} 1, & \text{if } a = 0; \\ 0, & \text{otherwise.} \end{cases}$$

where $F = (f(n) \bmod v)_{n \geq 1}$.

Proof. Note that if $f(n) \not\equiv 0 \pmod{v}$, then n is a powerful number (a number where each of its prime factor occurs to a power greater than 1). Choose $a \not\equiv 0 \pmod{v}$. From [6], see also [7, p. 178], we have that for any $\epsilon > 0$

$$\#\{i \mid i < n; i \text{ is a powerful number}\} < n^{\frac{1}{2} + \epsilon},$$

for large enough n . Choosing $\epsilon < \frac{1}{2}$ we get that $d(F, a) = 0$ for $a \neq 0$. Hence the desired result follows. \square

Now we are ready to prove our next Theorem. Let $a = g(h) \pmod{v}$ and α_j be the j 'th occurrence of a in F . By definition of h , we get $a \neq 0$. From Lemma 8 we have $d(F, a) = 0$. So if we show that $\limsup_{j \rightarrow \infty} \frac{\alpha_{j+1}}{\alpha_j} = 1$, we are done.

On the other hand, note that $(p_i^h)_{i \geq 1}$ is a subsequence of $(\alpha_i)_{i \geq 1}$, where p_i is the i 'th prime. Therefore for all j there exists i such that

$$p_i^h \leq \alpha_j < \alpha_{j+1} \leq p_{i+1}^h.$$

Hence

$$1 \leq \frac{\alpha_{j+1}}{\alpha_j} \leq \frac{p_{i+1}^h}{p_i^h}.$$

Therefore

$$\begin{aligned} 1 \leq \limsup_{j \rightarrow \infty} \frac{\alpha_{j+1}}{\alpha_j} &\leq \limsup_{i \rightarrow \infty} \frac{p_{i+1}^h}{p_i^h} \\ &= \left(\limsup_{i \rightarrow \infty} \frac{p_{i+1}}{p_i} \right)^h. \end{aligned}$$

But $\limsup_{i \rightarrow \infty} \frac{p_{i+1}}{p_i} = \lim_{i \rightarrow \infty} \frac{p_{i+1}}{p_i} = 1$, this is an immediate consequence of the prime number theorem $\lim_{i \rightarrow \infty} p_i/i \log i = 1$.

Therefore $\limsup_{j \rightarrow \infty} \alpha_{j+1}/\alpha_j = 1$. It follows that F is not k -automatic for any $k \geq 2$. \square

Corollary 9. *Given an integer $m \geq 1$, the sequence $(\sigma_m(n) \pmod{2})_{n \geq 1}$ is not k -automatic for any $k \geq 2$.*

Proof. Let $n = 2^\alpha d$, where d is odd. Then we have

$$\begin{aligned} \sigma_m(n) &= \sigma_m(2^\alpha)\sigma_m(d) \\ &= (1 + 2^m + \dots + 2^{m\alpha})\sigma_m(d) \\ &\equiv \sigma_m(d) \pmod{2} \\ &\equiv \tau(d) \pmod{2}. \end{aligned}$$

Furthermore, we know that $\tau(d)$ is odd only when d is a perfect square. So $\sigma_m(n) \pmod{2} = 1$ if and only if n is a perfect square times a power of 2. Let $S = (\sigma_m(n) \pmod{2})_{n \geq 1}$. Then we get $d(S, 1) = 0$ since

$$\#\{i \leq n \mid \sigma_m(i) \equiv 1 \pmod{2}\} = O(\sqrt{n}).$$

On the other hand if α_n represents the position of the n 'th occurrence of 1, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} &\leq \limsup_{n \rightarrow \infty} \frac{2^\alpha(n+2)^2}{2^\alpha n^2} \\ &= \limsup_{n \rightarrow \infty} \frac{(n+2)^2}{n^2} \\ &= 1. \end{aligned}$$

So we have $(\sigma_m(n) \bmod 2)_{n \geq 1}$ is not k -automatic by Theorem 7. □

Also combining Theorems 1 and 2 we get the following new result.

Corollary 10. *For all integers $v, m, k \geq 2$ the sequence $(\tau_m(n) \bmod v)_{n \geq 1}$ is not k -automatic.*

Proof. Assume that for some $v, m, k \geq 2$ the sequence $(\tau_m(n) \bmod v)_{n \geq 1}$ is k -automatic. Therefore by Lemma 1 we get given an integer $p|v$ the sequence $(\tau_m(n) \bmod p)_{n \geq 1}$ is also k -automatic. Therefore assume without loss of generality that v is a prime. Consider the following cases.

Case 1: $m \not\equiv 0 \pmod v$. Then if we choose α and h such that $v^\alpha \parallel m-1$ and $h \equiv 1 - m \pmod{v^{\alpha+1}}$ we get

$$v \mid \frac{m+h-1}{m-1}, \Rightarrow v \mid \frac{m+h-1}{m-1} \binom{m+h-2}{m-2} = \binom{m+h-1}{m-1}.$$

Therefore for any prime q we have $\tau_m(q^h) \bmod v = 0$ by (1). On the other hand for all primes q we have $\tau_m(q) \bmod v = m \bmod v \neq 0$. Hence by Theorem 2, we get that $(\tau_m(n) \bmod v)_{n \geq 1}$ is not k -automatic.

Case 2: $m \equiv 0 \pmod v$. Let $g(h) = \binom{m+h-1}{m-1}$. We have that $f(\prod p_i^{\alpha_i}) = \prod g(\alpha_i)$ by (1). Also we know $g(1) = m \equiv 0 \pmod v$. Assume that $v^\alpha \parallel m$. Then we get $g(v^\alpha) \not\equiv 0 \pmod v$. Therefore by Theorem 7, $(\tau_m(n) \bmod v)_{n \geq 1}$ is not k -automatic. □

Corollary 10 can be used to prove the transcendence of π_q (an analogue of π in the field $GF(q)((X))$) over the field $GF(q)(X)$ [1].

It is worth mentioning that both of our theorems relied on $v|f(n)$, for some n . If f is multiplicative and $v \nmid f(n)$ for any $n \geq 1$, then its the analysis becomes much more difficult. For example the Liouville function defined by

$$\lambda(p_1^{\alpha_1} \dots p_t^{\alpha_t}) = (-1)^{\alpha_1 + \dots + \alpha_t},$$

is never divisible by any prime. It seems that the question of whether or not $(\lambda(n))_{n \geq 1}$ is k -automatic is an open problem worth pursuing.

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Soroosh YAZDANI
Department of Mathematics
University of Illinois
273 Altgelt Hall, MC-382
1409 W. Green Street
Urbana, IL 61801
USA
E-mail : syazdani@math.uiuc.edu