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The diophantine equation $ax^2 + bxy + cy^2 = N$, 
$D = b^2 - 4ac > 0$


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The Diophantine equation
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par KEITH MATTHEWS

RéSUMÉ. Nous revisitons un algorithme dû à Lagrange, basé sur le développement en fraction continue, pour résoudre l'équation \( ax^2 + bxy + cy^2 = N \) en les entiers \( x, y \) premiers entre eux, où \( N \neq 0, \gcd(a, b, c) = \gcd(a, N) = 1 \) et \( D = b^2 - 4ac > 0 \) n'est pas un carré.

ABSTRACT. We make more accessible a neglected simple continued fraction based algorithm due to Lagrange, for deciding the solubility of \( ax^2 + bxy + cy^2 = N \) in relatively prime integers \( x, y \), where \( N \neq 0, \gcd(a, b, c) = \gcd(a, N) = 1 \) and \( D = b^2 - 4ac > 0 \) is not a perfect square. In the case of solubility, solutions with least positive \( y \), from each equivalence class, are also constructed.

Our paper is a generalisation of an earlier paper by the author on the equation \( x^2 - Dy^2 = N \). As in that paper, we use a lemma on unimodular matrices that gives a much simpler proof than Lagrange’s for the necessity of the existence of a solution.

Lagrange did not discuss an exceptional case which can arise when \( D = 5 \). This was done by M. Pavone in 1986, when \( N = \pm \mu \), where \( \mu = \min(x, y) \neq (0, 0) |ax^2 + bxy + cy^2| \). We only need the special case \( \mu = 1 \) of his result and give a self-contained proof, using our unimodular matrix approach.

1. Introduction

The standard approach to solving the equation
\[(1.1) \quad ax^2 + bxy + cy^2 = N\]
in relatively prime integers \( x, y \), is via reduction of quadratic forms, as in Mathews ([6, p 97]). There is a parallel approach in Faisant’s book ([2, pp 106–113]) which uses continued fractions.

However, in a memoir of 1770, Lagrange ([11, Oeuvres II, pp 655–726]), gave a more direct method for solving (1.1) when \( \gcd(a, b, c) = \gcd(a, N) = 1 \) and \( D = b^2 - 4ac > 0 \) is not a perfect square. This paper seems to have
been largely overlooked. (Admittedly, the necessity part of his proof is long and not easy to follow.)

M. Pavone ([10, p 271]) solved (1.1) when \( N = \pm \mu \), where

\[
\mu = \min_{(x,y) \neq (0,0)} |ax^2 + bxy + cy^2|.
\]

He had essentially solved (1.1) in general, as Lagrange showed how to reduce the problem to the case \( N = \pm 1 \). (See (4.2) and (4.6)).

Strangely Pavone made no mention of Lagrange’s paper, referring instead to Serret ([12, p 80]), who had earlier drawn attention to the possibility of an exceptional case.

A. Nitaj has also discussed the equation in his thesis, ([9, pp 57–88]), using a standard convergent sufficiency condition of Lagrange, which resulted in a restriction \( D \geq 16 \), thus making rigorous the necessity part of Lagrange’s discussion. Nitaj discussed only the case \( b = 0 \) in detail, along the lines of Cornacchia ([1, pp 66–70]).

Our contribution in this paper is to use the convergent criterion of Lemma 2, which results in no restriction on \( D \), while allowing us to deal with the non-convergent case, without having to appeal to the case \( \mu = 1 \) of Pavone, whose proof is somewhat complicated.

The continued fractions approach also has the attraction that it produces the solution \((x, y)\) with least positive \( y \) from each class, if \( \gcd(a, N) = 1 \).

Our treatment generalises an earlier paper by the author on the equation \( x^2 - Dy^2 = N \) (See Matthews [7]).

The assumption that \( \gcd(a, N) = 1 \) involves no loss of generality. For as pointed out by Gauss in his Disquisitiones (see [3, p 221] (also see Lemma 2 of Hua [5, pp 311–312]), there exist relatively prime integers \( \alpha, \gamma \) such that \( a\alpha^2 + b\alpha\gamma + c\gamma^2 = A \), where \( \gcd(A, N) = 1 \). Then if \( \alpha \delta - \beta \gamma = 1 \), the unimodular transformation \( x = \alpha X + \beta Y, y = \gamma X + \delta Y \) converts \( ax^2 + bxy + cy^2 \) to \( AX^2 + BXY + CY^2 \). Also the two forms represent the same integers.

2. The structure of the solutions

We outline the structure of the integer solutions of (1.1) as given in Skolem ([13, pp 42–45]).

The primitive solutions \( x + y\sqrt{D} \) of \( ax^2 + bxy + cy^2 = N \) (i.e. with \( \gcd(x, y) = 1 \)) fall into equivalence classes, with \( x + y\sqrt{D} \) and \( x' + y'\sqrt{D} \) being equivalent if and only if

\[
2ax + by + y\sqrt{D} = \frac{(u + v\sqrt{D})}{2}(2ax' + by' + y'\sqrt{D}),
\]

where \( u \) and \( v \) are integers satisfying \( u^2 - Dv^2 = 4 \).
This is equivalent to the equations

\begin{align}
\tag{2.2}
x &= \left(\frac{u-bv}{2}\right)x' - cvy', \quad y = avx' + \left(\frac{u+bv}{2}\right)y'.
\end{align}

It is easy to verify that (2.1) holds if and only if the following congruences hold:

\begin{align}
\tag{2.3}
2axx' + b(xy' + x'y) + 2cyy' & \equiv 0 \pmod{|N|} \\
\tag{2.4}
xy' - x'y & \equiv 0 \pmod{|N|}.
\end{align}

Each primitive solution gives rise to a root \( n \) of the congruence

\[ n^2 \equiv D \pmod{4|N|}. \]

In fact if \((\alpha, \gamma)\) is a solution of (1.1) and \(\alpha \delta - \beta \gamma = 1\), then

\[ n = (2a\alpha + b\gamma)\beta + (b\alpha + 2c\gamma)\delta. \]

Equivalent solutions give rise to congruent \( n \pmod{2|N|} \).

Conversely, primitive solutions which give rise to congruent \( n \pmod{2|N|} \) are equivalent. This follows from the equations

\[ -\gamma n + 2N\delta = 2a\alpha + b\gamma \]
\[ a\alpha n - 2N\beta = b\alpha + 2c\gamma \]

and congruences (2.3) and (2.4).

It is also straightforward to verify that if \(ax^2 + bxy + cy^2\) is replaced by an equivalent form \(AX^2 + BXY + CY^2\) under a unimodular transformation, then equivalent primitive representations \((x, y)\) and \((x', y')\) of \(N\) map into equivalent primitive representations \((X, Y)\) and \((X', Y')\). In fact the \( n \) of equation (2.5) is replaced by \(\Delta n\), where \(\Delta\) is the determinant of the transformation. (See Gauss [3, pp 130–131].)

3. Some lemmas

**Lemma 1.** Assume \(D > 0\) is not a perfect square and \(Q_0 \mid (P_0^2 - D)\).

If \((P_n + \sqrt{D})/Q_n\) is the \(n\)-th complete convergent in the simple continued fraction for \(x = (P_0 + \sqrt{D})/Q_0\) and \(G_{n-1} = Q_0A_{n-1} - P_0B_{n-1}\), where \(A_{n-1}/B_{n-1}\) denotes a convergent to \(x\), then

\[ G_{n-1}^2 - DB_{n-1}^2 = (-1)^n Q_0 Q_n, \]

or equivalently

\[ Q_0 A_{n-1}^2 - 2P_0 A_{n-1} B_{n-1} + \frac{P_0^2 - D}{Q_0} B_{n-1}^2 = (-1)^n Q_n. \]

**Proof.** See Mollin [8, pp 246–248].
Lemma 2. If \( \omega = \frac{P\zeta + R}{Q\zeta + S} \), where \( \zeta > 1 \) and \( P, Q, R, S \) are integers such that \( Q > 0, S > 0 \) and \( PS - QR = \pm 1 \), or \( S = 0 \) and \( Q = 1 = R \), then \( P/Q \) is a convergent \( A_n/B_n \) to \( \omega \). Moreover if \( Q \neq S > 0 \), then \( R/S = (A_{n-1} + kA_n)/(B_{n-1} + kB_n) \), \( k \geq 0 \). Also \( \zeta + k \) is the \((n + 1)\)th complete convergent to \( \omega \). Here \( k = 0 \) if \( Q > S \), while \( k \geq 1 \) if \( Q < S \).

Proof. This is an extension of Theorem 172, Hardy and Wright ([4, pp 140—141]), who dealt with the case \( Q > S > 0 \). See Matthews [7, pp 325–326]. \( \square \)

The following result is a special case (\( \mu(f) = 1 \)) of a result due to M. Pavone, [10, p 271]. Pavone’s proof is rather complicated and we give a self-contained proof using our Lemma 2 as cases (ii) and (iii) of the proof of Lemma 3 below and in the Appendix.

Lemma 3. Suppose \( X, y > 0, Q, n, R \) are integers and

\[
QX^2 + nXy + Ry^2 = 1,
\]

where \( D = n^2 - 4QR > 0 \) is not a perfect square. Also let \( \omega = \frac{-n+\sqrt{D}}{2Q} \) and \( \omega^* = \frac{-n-\sqrt{D}}{2Q} \) be the roots of \( Q\theta^2 + n\theta + R = 0 \). Then either

(i) \( X/y \) is a convergent \( A_{i-1}/B_{i-1} \) to \( \omega \) (resp. \( \omega^* \)) and if \( (P_i + \sqrt{D})/Q_i \) denotes the \( i \)th complete convergent to \( \omega \) (resp. \( \omega^* \)), then \( Q_i = (-1)^i2 \) (resp. \( Q_i = (-1)^{i+1}2 \)), or

(ii) \( D = 5, Q < 0 \) and

\[
X/y = (A_r - A_{r-1})/(B_r - B_{r-1}) = (A'_s - A'_{s-1})/(B'_s - B'_{s-1}),
\]

where \( A_r/B_r \) and \( A'_s/B'_s \) denote convergents to \( \omega \) and \( \omega^* \), respectively and

\[
\omega = [a_0, \ldots, a_r, \tilde{1}], \quad \omega^* = [b_0, \ldots, b_s, \tilde{1}],
\]

where \( a_r > 1 \) if \( r > 0 \) and \( b_s > 1 \) if \( s > 0 \).

Moreover \( X/y \) is not a convergent to \( \omega \) or \( \omega^* \).

Conversely, if (i) or (ii) hold, then \( X/y \) is a solution of (3.3).

Remarks. (a) In the Appendix, we prove that if \( D = 5 \) and \( Q < 0 \), then \( r - 1 \equiv s \pmod{2} \) and

\[
(A_r - A_{r-1})/(B_r - B_{r-1}) = (A'_s - A'_{s-1})/(B'_s - B'_{s-1}),
\]

the latter being obtained directly by an appeal to symmetry by Pavone ([10, p 277]).

(b) As Pavone points out, we have

\[
\frac{A_{r-2}}{B_{r-2}} < \frac{A_r - A_{r-1}}{B_r - B_{r-1}} < \frac{A_r}{B_r} < \omega \text{ if } r \text{ is even},
\]
The corresponding equations hold if \( w \) is replaced by \( w^* \), each \( A_r \) is replaced by \( A'_r \) and each \( B_r \) is replaced by \( B'_r \) etc.

Consequently, \( \frac{A_r - A_{r-1}}{B_r - B_{r-1}} \) is not a convergent to \( w \) or \( w^* \).

(c) If \( n \) is even, say \( n = 2P \), then \( w = \frac{-P + \sqrt{\Delta}}{Q} \) and \( w^* = \frac{-P - \sqrt{\Delta}}{Q} \), where \( \Delta = P^2 - QR \). If we then denote the \( n \)-th complete convergent to \( w \) (resp. \( w^* \)) by \( (P_n + \sqrt{\Delta})/Q_n \), condition (i) becomes \( Q_i = (-1)^i \) (resp \( Q'_i = (-1)^{i+1} \)).

**Proof.** Suppose (3.3) holds. Consider the matrix

\[
H = \begin{pmatrix} X & t \\ y & QX + Py \end{pmatrix},
\]

where \( t = -PX - Ry \) if \( n = 2P \), while \( t = -(P + 1)X - Ry \) if \( n = 2P + 1 \).

Then in both cases,

\[
\det H = QX^2 + nXY + Ry^2 = 1.
\]

Also it is straightforward to verify that

\[
\omega = \frac{X\alpha + t}{y\alpha + QX + Py},
\]

where

\[
\alpha = \begin{cases} \sqrt{\Delta} & \text{if } n = 2P, \\ \frac{\sqrt{D+1}}{2} & \text{if } n = 2P + 1. \end{cases}
\]

Case (i). Suppose \( QX + Py > 0 \). Then as \( \alpha > 1 \), Lemma 2 applies and \( X/y \) is a convergent to \( \omega \).

Case (ii). Suppose \( QX + Py = 0 \). On substituting for \( QX \) in (3.4), we get

\[
(-Py)X + nXY + Ry^2 = 1.
\]

Hence \( y = 1 \) and \( -PX + nX + R = 1 \). Also \( \omega = X - \frac{1}{\alpha} \).

Hence

\[
\omega^* = \begin{cases} X + \frac{1}{\sqrt{\Delta}} & \text{if } n = 2P, \\ X + \frac{1}{\sqrt{D+1}} & \text{if } n = 2P + 1. \end{cases}
\]

Hence \( X/y = \lfloor \omega^* \rfloor \) is a convergent to \( \omega^* \) if \( D \neq 5 \).

If \( D = 5 \), we see \( \omega^* = [X + 1, 1] (s = 0) \) and \( \omega = [X - 1, 2, 1] (r = 1) \).
Then
\[
\frac{A_1 - A_0}{B_1 - B_0} = \frac{(2X - 1) - (X - 1)}{2 - 1} = X,
\]
\[
\frac{A'_0 - A'_{-1}}{B'_0 - B'_{-1}} = \frac{(X + 1) - 1}{1 - 0} = X.
\]

Also \(QX^2 + (2P + 1)X + R = 1\) and \(P = -QX\) together give
\[-QX^2 + X + R = 1.
\]

Hence
\[
1 = \frac{D - 1}{4} = \frac{P^2 + P - QR}{QX^2 - X - R} = \frac{Q(QX^2 - X - R)}{-Q}
\]
and hence \(Q < 0\).

Case (iii) (a) Suppose \(QX + Py < 0\). Then \(-(QX + Py) > 0\) and
\[
\omega^* = \frac{X(-\alpha^*) - t}{y(-\alpha^*) - (QX + Py)},
\]
where \(-\alpha^* > 1\) if \(D \neq 5\).

Hence \(X/y\) is a convergent to \(\omega^*\), unless \(D = 5\).

(b) If \(D = 5\) and \(-(QX + Py) \geq y\), then
\[
\omega^* = \frac{X - t\alpha}{y - (QX + Py)\alpha}
\]
and again \(X/y\) is a convergent to \(\omega^*\) by Lemma 2.

In all cases where \(X/y\) is the convergent \(A_{n-1}/B_{n-1}\) to \(\omega\) (resp. \(\omega^*\)), it follows from (3.3) and equation (3.2) of Lemma 1, with \(P_0 = -n, Q_0 = 2Q\) (resp. \(P_0 = n, Q_0 = -2Q\)), that
\[
(-1)^nQ_n = Q_0A_{n-1}^2 - 2P_0A_{n-1}B_{n-1} + \frac{P_0^2 - D}{Q_0}B_{n-1}^2
\]
\[
= \begin{cases} 
2QX^2 + 2nXY + 2RY^2 = 2 & \text{for } \omega, \\
-2QX^2 - 2nXY - 2RY^2 = -2 & \text{for } \omega^*
\end{cases}
\]
and consequently \((-1)^nQ_n = 2\) (resp. \(-2\)) in all cases.

(c) Now suppose \(D = 5\) and \(y > -(QX + Py) > 0\).

Now, from (3.5), Lemma 2 tells us that
\[
\frac{X}{y} = \frac{A_{i-1} + kA_i}{B_{i-1} + kB_i}
\]
\[
(P + 1)X + Ry = -t = A_i
\]
\[-(QX + Py) = B_i,
\]
where \(k \geq 1\). Moreover \(\omega_{i+1}^* = \alpha + k = [k + 1, 1]\).
Hence \( \omega^* = [a_0, \ldots, a_s, 1] \), where \( s = i + 1 \) and \( a_s = k + 1 \).

Hence (3.6) gives

\[
\frac{X}{y} = \frac{A_{s-2} + (a_s - 1)p_{s-1}}{B_{s-2} + (a_s - 1)B_{s-1}} = \frac{A_s - A_{s-1}}{B_s - B_{s-1}}.
\]

Next we prove that \( Q < 0 \). We have from equation (1.1)),

\[
\begin{align*}
1 &= QX^2 + (2P + 1)XY + Ry^2 \\
Q &= Q^2X^2 + (2P + 1)QXY + QRy^2 \\
&= (QX + Py)^2 + (QX + Py - y)y \\
&= (-B_{s-1})^2 + (B_{s-1} - (B_s - B_{s-1}))(B_s - B_{s-1}) \\
&= B_{s-1}^2 + B_{s-1}B_s - B_s^2 \\
(3.7) &= -\frac{1}{4}((2B_s - B_{s-1})^2 - 5B_{s-1}^2).
\end{align*}
\]

However

\[
B_s = a_sB_{s-1} + B_{s-2} \geq 2B_{s-1} > \left(\frac{1 + \sqrt{5}}{2}\right)B_{s-1}
\]

so

\[
2B_s - B_{s-1} > \sqrt{5}B_{s-1}
\]

and hence

\[
(2B_s - B_{s-1})^2 - 5B_{s-1}^2 > 0.
\]

Then equation (3.7) gives \( Q < 0 \).

4. The main result

**Theorem 1.** Suppose

\[
ax^2 + bxy + cy^2 = N,
\]

where \( N \neq 0 \), \( \gcd(x, y) = 1 = \gcd(a, N) \) and \( y > 0 \) and \( D = b^2 - 4ac > 0 \) is not a perfect square.

Let \( \theta \) satisfy \( x \equiv y\theta \pmod{|N|} \), \( 0 \leq \theta < |N| \). Then

\[
a\theta^2 + b\theta + c \equiv 0 \pmod{|N|}.
\]

Let \( x = y\theta + |N|X \), \( n = 2a\theta + b \), \( Q = a|N| \), \( \omega = -n + \sqrt{D} \frac{2Q}{2Q} \) and \( \omega^* = \frac{-n - \sqrt{D}}{2Q} \).

Also let \( n = 2P \) or \( 2P + 1 \), according as \( b \) is even or odd. Then

(i) if \( QX + Py > 0 \), \( X/y \) is a convergent to \( \omega \);

(ii) Suppose \( QX + Py \leq 0 \). Then
(a) If \( D \neq 5 \), or \( D = 5 \) and \(- (QX + Py) \geq y \), then \( X/y \) is a convergent to \( \omega^* \).

(b) If \( D = 5 \) and \( y > -(QX + Py) \geq 0 \), then

\[
\frac{X}{y} = \frac{A_r - A_{r-1}}{B_r - B_{r-1}} = \frac{A'_s - A'_{s-1}}{B'_s - B'_{s-1}}
\]

which is not a convergent to \( \omega \) or \( \omega^* \).

Also \( aN < 0 \).

Conversely,

(a) if \( X/y \) is a convergent \( A_{i-1}/B_{i-1} \) to \( \omega \) (resp. \( \omega^* \)) and \( Q_i = (-1)^{i} 2N/|N| \) (resp. \( (-1)^{i+1} 2N/|N| \)), or

(b) if \( D = 5 \), \( aN < 0 \) and \( \frac{X}{y} = \frac{A_r - A_{r-1}}{B_r - B_{r-1}} \), where \( r \) is defined earlier,

then \((x, y)\), with \( x = y\theta + |N|X \), will be a solution to (4.1), possibly in-primitive.

Proof. Suppose

\[ ax^2 + bxy + cy^2 = N, \]

where gcd\((x, y) = 1 = gcd(a, N)\) and \( y > 0 \). Then clearly \( gcd(y, |N|) = 1 \).

Let \( x \equiv y\theta \mod{|N|} \) and

\[ x = y\theta + |N|X. \]  

Then

\[
a\theta^2 y^2 + b(y\theta)y + cy^2 \equiv 0 \mod{|N|}
\]

(4.3)

\[
a\theta^2 + b\theta + c \equiv 0 \mod{|N|}
\]

(4.4)

\[
4a^2 \theta^2 + 4ab\theta + 4ac \equiv 0 \mod{4|N|}
\]

\[
(2a\theta + b)^2 \equiv b^2 - 4ac \mod{4|N|},
\]

(4.5)

\[ or \ n^2 \equiv D \mod{4|N|}. \]

Also

\[
a(y\theta + |N|X)^2 + b(y\theta + |N|X)y + cy^2 = N
\]

\[
|N|^2 aX^2 + (2a\theta + b)|N|Xy + (a\theta^2 + b\theta + c)y^2 = N
\]

(4.6)

\[
QX^2 + nXY + R'y^2 = \frac{N}{|N|},
\]

where \( Q = a|N|, R = (a\theta^2 + b\theta + c)/|N| \) and \( n^2 - 4QR = D \).

The conclusions of the theorem then follow from Lemma 3, applied to the equation \( Q'X^2 + n'XY + R'y^2 = 1 \), where \( Q' = \epsilon Q, n' = \epsilon n, R' = \epsilon R \) and \( \epsilon = |N|/N \).
5. The algorithm

Let $\Delta = D/4$ if $b$ is even and let the $i$-th complete convergent to $\omega$ or $\omega^*$ be denoted by $(P_i + \sqrt{\Delta})/Q_i$ or $(P_i + \sqrt{D})/Q_i$, according as $b$ is even or odd.

If equation (4.1) is soluble with $x \equiv y \theta \pmod{|N|}$, $y > 0$, there will be infinitely many solutions because of equations (2.2). It follows that if $\omega$ and $\omega^*$ are not purely periodic, we need only examine the first period $m \leq i \leq m + l$ of the continued fractions for $\omega$ and $\omega^*$ to determine solubility of (4.1). For, with $\omega$ (resp. $\omega^*$) being $(-P \pm \sqrt{\Delta})/Q$ (resp. $(-2P + 1) \pm \sqrt{D})/2Q)$, the equation $Q_i = \pm 1$ (resp. $\pm 2$) will hold for infinitely many $i$ by periodicity and so there will be at least one such $i$ in the range $m \leq i \leq m + l$. Any such $i$ must have $Q_i = 1$ (resp. 2), as $(P_i + \sqrt{\Delta})/Q_i$ (resp. $(P_i + \sqrt{D})/Q_i$) is reduced for $i$ in this range and so $Q_i > 0$. Moreover if $l$ is even, the sign of $(-1)^i|N|/|N|$ is preserved from one period to the next. If $l$ is odd, then the first or second period will produce a solution. If $\omega$ or $\omega^*$ is purely periodic, we must examine $Q_2$, which corresponds to the third period.

Moreover there can be at most one $i$ in a period for which $Q_i = 1$ (resp. 2). For if $P_i + \sqrt{\Delta}$ (resp. $P_i + \sqrt{D})/2$ is reduced, then $P_i = \lfloor \sqrt{\Delta} \rfloor$ (resp. $P_i = 2\lfloor (\sqrt{D} - 1)/2 \rfloor + 1$) and hence two such occurrences of $Q_i = 1$ (resp. 2) within a period would give a smaller period.

Hence we have the following algorithm essentially due to Lagrange, apart from stage 1:

1. If $\gcd(a, N) > 1$, find a unimodular transformation of the given quadratic form into one in which $\gcd(a, N) = 1$. (See the last paragraph of the Introduction.)

2. Find all solutions $\theta$ of the congruence (4.3) in the range $0 \leq \theta < |N|$. (This can be done as follows:

First solve $t^2 \equiv b^2 - 4ac \pmod{4|N|}$, $-|N| < t \leq |N|$. (If there are no solutions $t$, then there is no primitive solution of (4.1) corresponding to $t$.)

Then solve $a\theta \equiv t-b \pmod{|N|}$, $0 \leq \theta < |N|.$

For each $\theta$, let $n = 2a\theta + b$, $P = \lfloor n/2 \rfloor$ and $Q = a|N|$.)

3. For each of the numbers $\omega = -P + \sqrt{\Delta}$ (resp. $-(2P + 1) + \sqrt{D}$), test the first period to see if $Q_i = 1$ (resp. 2) occurs. If $l$ is even, test additionally for $1 = (-1)^i|N|/|N|$ (resp. $2 = (-1)^i|N|/|N|$) to hold.

Similarly for each of the numbers $\omega^* = -P - \sqrt{\Delta}$ (resp. $-(2P + 1) - \sqrt{D}$), with $i$ replaced by $i + 1$.

If $D = 5$, test additionally to see if $aN < 0$ holds.

4. For each $\theta$ and corresponding $\omega$ for which test 3 succeeds, find the least $i$ for which the condition $Q_i = (-1)^i|N|/|N|$ (resp. $Q_i = (-1)^i2|N|/|N|$)
holds. If $l$ is even, this will occur in or before the first period, while if $l$ is odd, this will occur in or before the second period. Similarly for $\omega^*.$

For the corresponding convergent $A_{i-1}/B_{i-1}$ to $\omega$ or $\omega^*$, write $X = A_{i-1}, y = B_{i-1}$. If $D = 5$ and $aN < 0$, in relation to $\omega$, write $X = A_r - A_{r-1}, y = B_r - B_{r-1}$. Then $x = y\theta + |N|X$ produces a solution of (4.1) with $x \equiv y\theta \pmod{|N|}$.

Choose the solution $(x, y)$ with lesser of the $y$ values.

The algorithm will produce a solution $(x, y)$ from each class, with the additional feature that the least positive $y$ is chosen, if the quadratic form satisfies $\gcd(a, N) = 1$.

6. Examples

Example 1 (Gauss, Article 205). [3, p 189]

\begin{equation}
42x^2 + 62xy + 21y^2 = 585.
\end{equation}

As $\gcd(42, 585) = 3 = \gcd(21, 585)$, we make a suitable transformation

\[ x = -x' + y', y = 2x' - y', \]

which gives

\[ 42x^2 + 62xy + 21y^2 = 2x'^2 + 18x'y' + y'^2. \]

The latter form has $\Delta = 79$ and $\gcd(2, 585) = 1$.

We list the roots of $2\theta^2 + 18\theta + 1 \equiv 0 \pmod{585}$ and corresponding values $P = 2\theta + 9$:

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$\theta$ & 34 & 47 & 74 & 164 & 412 & 502 & 529 & 542 \\
\hline
$P$ & 77 & 103 & 157 & 337 & 833 & 1013 & 1067 & 1093 \\
\hline
\end{tabular}
\end{center}

We find that only $P = 157$ and $1013$ give solutions of equation (6.1):

(i) $\omega = (-157 + \sqrt{79})/1170$ gives $Q_3 = 1, A_2/B_2 = -1/7$

Then $y' = 7$ and $x' = 7\cdot 74 - 585 \cdot 1 = -67$. Hence $(x, y) = (74, -141)$ is a solution of (6.1).

$\omega^* = (-157 - \sqrt{79})/1170$ also gives the solution $(74, -141)$.

(ii) $\omega = (-1013 + \sqrt{79})/1170$ gives $Q_2 = 1, A_1/B_1 = -6/7$. Then $y' = 7$

and $x' = 7\cdot 502 - 585 \cdot 6 = 4$. Hence $(x, y) = (3, 1)$ is a solution of (6.1).

$\omega^* = (-1013 - \sqrt{79})/1170$ also gives the solution $(3, 1)$.

In fact Gauss gave solutions $(83, -87)$ and $(3, 1)$. In the notation of (2.2),
the solutions $(x, y) = (83, -87)$ and $(x', y') = (74, -141)$ are related by the
solution $(u, v) = (-80, 9)$ of the Pell equation $x^2 - 79y^2 = 1$.

Summarising:
Example 2. \(3x^2 - 3xy - 2y^2 = 202\).

Here \(D = 33\).

The solutions of \(3\theta^2 - 3\theta - 2 \equiv 0 \pmod{202}\) are 39, 63, 140, 164, with corresponding \(n\) values 231, 375, 837, 981.

(i) \(\theta = 39, \ \omega = (-231 + \sqrt{33})/1212\), \(Q_3 = -2, A_2/B_2 = -1/5\).

Then \(x = y\theta + |N|X = 5 \cdot 39 + 202 \cdot (-1) = -7\) and \((x, y) = (-7, 5)\).

\(\omega^* = (-231 - \sqrt{33})/1212\) produces the same solution.

(ii) \(\theta = 63, \ \omega = (-375 + \sqrt{33})/1212\), \(Q_6 = 2, A_5/B_5 = -7/23\).

Then \(x = 23 \cdot 63 + 202 \cdot (-7) = 35\) and \((x, y) = (35, 23)\).

\(\omega^* = (-375 - \sqrt{33})/1212\) gives \(Q_5 = 2, A_4/B_4 = -11/35\).

Then \(x = 35 \cdot 63 + 202 \cdot (-11) = -17\) and we get the equivalent solution \((-17, 35)\).

(iii) \(\theta = 140, \ \omega = (-837 + \sqrt{33})/1212\), \(Q_4 = 2, A_3/B_3 = -24/35\).

Then \(x = 35 \cdot 140 + 202 \cdot (-24) = 52\) and \((x, y) = (52, 35)\).

\(\omega^* = (-837 - \sqrt{33})/1212\) gives \(Q_5 = 2, A_4/B_4 = -16/23\).

Then \(x = 23 \cdot 140 + 202 \cdot (-16) = -12\) and we get the equivalent solution \((-12, 23)\).

(iv) \(\theta = 164, \ \omega = (-981 + \sqrt{33})/1212\), \(Q_2 = 2, A_1/B_1 = -4/5\).

Then \(x = 5 \cdot 164 + 202 \cdot (-4) = 12\) and \((x, y) = (12, 5)\).

\(\omega^* = (-981 - \sqrt{33})/1212\) produces the same solution.

Summarising:

\[
\begin{array}{|c|c|}
\hline
3x^2 - 3xy - 2y^2 = 202 \\
\hline
\text{Solution} & n \pmod{404} \\
\hline
(35, 23) & 29 \\
(-12, 23) & -29 \\
(12, 5) & 231 \\
(-7, 5) & -231 \\
\hline
\end{array}
\]

There are 4 equivalence classes of solutions.

Example 3. \(f(x, y) = 19x^2 - 85xy + 95y^2 = -671\).

Here \(D = 5\).

The solutions of \(19\theta^2 - 85\theta + 95 \equiv 0 \pmod{671}\) are 443, 454, 504, 515, with corresponding \(n\) values 16749, 17167, 19067, 19485.

The exceptional solutions give the solutions with smallest \(y\):

(i) \(\theta = 443, \ \omega = (-16749 + \sqrt{5})/25498 = [-1, 2, 1, 10, 1, 1, 2, 1], \ Q_7 = 2, A_5/B_5 = -44/67\). Also \(A_6/B_6 = -111/169\).
Exceptional solution:

\[(X, y) = (A_6 - A_5, B_6 - B_5) = (-67, 102), (x, y) = (229, 102).\]

\[\omega^* = [-1, 2, 1, 10, 3, 1] \text{ gives } Q_6 = 2 \text{ and correspondingly } (x, y) = (301, 137).\]

(ii) \(\theta = 454: \omega = (-17167 + \sqrt{5}) / 25498 = [-1, 3, 16, 1, 2, 1],\)

\[Q_5 = 2, A_3/B_3 = -35/52. \text{ Also } A_4/B_4 = -103/153.\]

Exceptional solution:

\[(X, y) = (A_4 - A_3, B_4 - B_3) = (-68, 101), (x, y) = (226, 101).\]

\[\omega^* = [-1, 3, 16, 3, 1] \text{ gives } Q_4 = 2 \text{ and correspondingly } (x, y) = (329, 150).\]

(iii) \(\theta = 504: \omega = (-19067 + \sqrt{5}) / 25498 = [-1, 3, 1, 26, 2, 1],\)

\[Q_5 = 2, A_3/B_3 = -80/107 \text{ and } A_4/B_4 = -163/218.\]

Exceptional solution:

\[(X, y) = (A_4 - A_3, B_4 - B_3) = (-83, 111), (x, y) = (251, 111).\]

\[\omega^* = [-1, 3, 1, 28, 1] \text{ gives } Q_4 = 2 \text{ and correspondingly } (x, y) = (254, 115).\]

(iv) \(\theta = 515: \omega = (-19485 + \sqrt{5}) / 25498 = [-1, 4, 4, 5, 2, 1],\)

\[Q_5 = 2, A_3/B_3 = -68/89. \text{ Also } A_4/B_4 = -149/195.\]

Exceptional solution:

\[(X, y) = (A_4 - A_3, B_4 - B_3) = (-81, 106), (x, y) = (239, 106).\]

\[\omega^* = [-1, 4, 4, 7, 1] \text{ gives } Q_4 = 2 \text{ and correspondingly } (x, y) = (271, 123).\]

Summarising:

<table>
<thead>
<tr>
<th>19x^2 - 85xy + 95y^2 = -671</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution</td>
</tr>
<tr>
<td>(226, 101)</td>
</tr>
<tr>
<td>(251, 111)</td>
</tr>
<tr>
<td>(239, 106)</td>
</tr>
<tr>
<td>(229, 102)</td>
</tr>
</tbody>
</table>

There are 4 equivalence classes of solutions.

7. Appendix

Lemma 4. Let

\[\omega = \frac{-2P + 1 + \sqrt{5}}{2Q} = [a_0, \ldots, a_r, 1],\]

\[\omega^* = \frac{-2P + 1 - \sqrt{5}}{2Q} = [b_0, \ldots, b_s, 1],\]

where \(a_r > 1\) if \(r > 0\) and \(b_s > 1\) if \(s > 0\). Then \(r - 1 \equiv s \pmod{2}\) and

\[A_r - A_{r-1} = A'_s - A'_{s-1} \text{ and } B_r - B_{r-1} = B'_s - B'_{s-1}.\]
Proof. We have
\[
\omega = \frac{-P + \alpha^{-1}}{Q} = \frac{A_r \alpha + A_{r-1}}{B_r \alpha + B_{r-1}}
\]
and hence
(7.1) \[ QA_r = -PB_r + B_{r-1} \]
(7.2) \[ -QA_{r-1} = -B_r + (P+1)B_{r-1}. \]
Then (7.1) gives
(7.3) \[ B_{r-1} = PB_r + QA_r, \]
and (7.2) and (7.3) give
\[
-QA_{r-1} = -B_r + (P+1)(PB_r + QA_r) = -B_r + P(P+1)B_r + Q(P+1)A_r = QRB_r + Q(P+1)A_r.
\]
Hence
(7.4) \[ -A_{r-1} = RB_r + (P+1)A_r. \]
Also (7.2) and (7.3) imply
(7.5) \[ -A_r = PA_{r-1} + RB_{r-1}. \]
Now let \( X = A_r - A_{r-1}, y = B_r - B_{r-1}. \)
Hence
\[
QX^2 + (2P+1)XY + Ry^2 = X(QX + Py) + y((P+1)X + Ry) = (A_r - A_{r-1})(2B_{r-1} - B_r) + (B_r - B_{r-1})(A_r - 2A_{r-1}) = A_rB_{r-1} - A_{r-1}B_r = (-1)^r - 1.
\]
Now \( y = B_r - B_{r-1} \geq B_r - 2B_{r-1} = -(QX + Py) \geq 0. \)
Similarly with
\[
\omega^* = \frac{-P - \alpha}{Q} = \frac{A_s' \alpha + A_{s-1}'}{B_s' \alpha + B_{s-1}'},
\]
and with \( X = A'_s - A_{s-1}' \), \( y = B'_s - B_{s-1}' \), we find
\[
QX + Py = -B_{s-1}' \quad \text{and} \quad (P+1)X + Ry = A_{s-1}',
\]
and
\[
QX^2 + (2P+1)XY + Ry^2 = (-1)^s.
\]
Also
\[
y = B'_s \alpha - B'_{s-1} \geq B'_{s-1} = -(QX + Py) \geq 0.
\]
It follows from cases (ii) and (iii)(c) in the proof of Lemma 3, that
\[ (-1)^{r-1}Q < 0, \quad (-1)^{s}Q < 0 \]
and
\[ A_r - A_{r-1} = A'_s - A'_{s-1} \quad \text{and} \quad B_r - B_{r-1} = B'_s - B'_{s-1}. \]

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The algorithm has been implemented in the author’s number theory calculator program CALC, available at http://www.maths.uq.edu.au/~krm/

References

[1] G. Cornacchia, Su di un metodo per la risoluzione in numeri interi dell’ equazione \[ \sum_{h=0}^{n} C_a x^{n-h} = P. \] Giornale di Matematiche di Battaglini 46 (1908), 33–90.

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