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par YINGCHUN CAI

1. Introduction

Let \( \Gamma = PSL(2, Z) \). By definition an element \( P \in \Gamma \) is hyperbolic if as a linear fractional transformation

\[
Pz = \frac{az + b}{cz + d}, \quad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

it has two distinct real fixed points. By a conjugation any hyperbolic element \( P \) can be given in a form \( P = \sigma^{-1}P'\sigma \) with \( \sigma \in SL(2, R) \) and \( P' = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t > 1 \). Here \( P' \) acts as multiplication by \( t^2 \), \( P'z = t^2z \).

The factor \( t^2 \) is called the norm of \( P \), let us denote it by \( NP \), it depends only on the class \( \{P\} \) of elements conjugate to \( P \), \( NP = N\{P\} = t^2 \). \( P \) and \( \{P\} \) are called primitive if they are not essential powers of other hyperbolic elements and classes respectively. For primitive \( P \) the norms can be viewed as "pseudoprimes", they have the same asymptotic distribution as the rational primes,

\[
\pi_r(x) = \#\{\{P\} - \text{primitive}; NP \leq x\} \sim \text{liz},
\]

where

\[
\text{liz} = \int_2^x \frac{dt}{\log t}.
\]

The problem of finding a formula with good error term was intensively studied by many mathematicians, first of all by H. Huber [4], D. Hejhal [2, 2001], P. Sarnak [2, 1993], W. Luo et al. [2, 1993].
3], A. B. Venkov [13] and N. V. Kuznetzov [7] before the eighties, not always for the same group. The result were of the type
\[
\pi_\Gamma(x) = \text{li}x + O(x^{3/4} \log^\alpha x), \quad \alpha > 0,
\]
this result was also known to A. Selberg and P. Sarnak [12] gave a “direct” proof of it.

In order to investigate the asymptotic distribution of \( \pi_\Gamma(x) \) A. Selberg introduced the Selberg zeta-function which mimics the classical zeta-function of Riemann in various aspects. The Selberg zeta-function is defined by
\[
Z(s) = \prod_{\{P\}} \prod_{k=0}^{\infty} (1 - (NP)^{-s-k})
\]
for \( \text{Re}(s) > 1 \) where \( \{P\} \) runs over the set of all primitive hyperbolic classes of conjugate elements in \( \Gamma \). The most fascinating property of the Selberg zeta-function \( Z(s) \) is that the analogue of the Riemann hypothesis is true. In view of this property one should expect an error term \( O(x^{1/2+\varepsilon}) \). Let us explain why this result is not obvious. It is convenient to speak of the allied sum
\[
\Psi_\Gamma(x) = \sum_{N\{P\} \leq x} \Lambda P
\]
where \( \Lambda P = \log NP \) if \( \{P\} \) is a power of a primitive hyperbolic class and \( \Lambda P = 0 \) otherwise. Then like in the theory of rational primes, we have the following explicit formula:

**Lemma 1 ([5]).**

(1.1) \[
\Psi_\Gamma(x) = x + \sum_{|t_j| \leq T} \frac{x^{s_j}}{s_j} + O \left( \frac{x \log^2 x}{T} \right), \quad 1 \leq T \leq x^{1/2} \log^{-2} x
\]
where \( s_j = 1/2 + t_j \) runs over the zeroes of \( Z(x) \) on \( \text{Re}(s) = 1/2 \) counted with their multiplicities.

Here, in the sum each term \( \frac{x^{s_j}}{s_j} \) has the order \( \frac{x^{1/2}}{|t_j|} \) but the number of terms is (refer to [2])
\[
\#\{j; |t_j| \leq T\} \sim \frac{T^2}{12},
\]
therefore, treat (1.1) trivially yields
\[
\Psi_\Gamma(x) = x + O \left( x^{1/2}T + \frac{x \log^2 x}{T} \right).
\]
On taking the optimal value $T = x^{1/4} \log x$ we obtain the error term $O(x^{3/4} \log x)$. At this point the situation differs very much from the one concerning rational primes because the Selberg zeta-function $Z(x)$ has much more zeroes than does the Riemann $\zeta(s)$.

From the above arguments one sees that in order to reduce the exponent $3/4$ one cannot simply handle the sum over the zeroes in (1.1) by summing up the terms with absolute values but a significant cancellation of terms must be taken into account. Only after N. V. Kuznetzov [8] published his summation formula does this suggestion become realizable. In 1983 H. Iwaniec [5] realized such a treatment and proved that

$$\pi_T(x) = \text{li}x + O(x^{3/4 + \varepsilon}), \quad \varepsilon > 0$$

by means of Kuznetzov trace formula incorporated with estimates for sums of real character of a special type. More precisely, the basic ingredients in Iwaniec’s arguments is the following mean value estimate for the Rankin zeta-function:

$$(1.2) \quad \sum_{|t_j| \leq T} \frac{|R_j(s)|}{\cosh \pi t_j} \ll T^{3/5} |s|^A \log^2 T$$

where $R_j(s)$ is the Rankin zeta-function and $Re(s) = \frac{1}{2}$; and the estimate for the following sum

$$(1.3) \quad \sum_{B < a \leq B + A} \sum_{R < r \leq 2R} \left( \frac{a^2 - 4}{r} \right)$$

where $\left( \cdot \right)$ denotes the Jacobi’s symbol.

In the meantime P. G. Gallagher obtained

$$\pi_T(x) = \text{li}x + O(x^{3/4} \log^{-1} x).$$

In 1994 W. Z. Luo and P. Sarnak [9] proved Iwaniec’s mean value conjecture: (1.2) holds with the exponent $5/2$ replaced by $2 + \varepsilon$. By this mean value estimate and A. Weil’s upper bound for Kloosterman sum,

$$\left| \sum_{\substack{a \pmod{c} \atop \gcd(a,c) = 1}} e \left( \frac{ma + n\bar{a}}{c} \right) \right| \leq 2(m, n, c)^{1/2} c^{1/2} d(c)$$

they proved

$$\pi_T(x) = \text{li}x + O(x^{7/6 + \varepsilon}), \quad \varepsilon > 0$$

for the modular group $\Gamma = PSL(2, \mathbb{Z})$.

In 1994 W. Z. Luo, Z. Rudnick and P. Sarnak [10] made considerable advance in the research of Selberg’s eigenvalue conjecture. They showed
that for any congruence subgroup $\Gamma \subset SL(2, \mathbb{Z})$ the least nonzero eigenvalue $\lambda_1(\Gamma) \geq \frac{23}{100}$. As a by-product they obtained

$$\pi_\Gamma(x) = lx + O(x^{\frac{7}{10}})$$

for any congruence subgroup $\Gamma \subset SL(2, \mathbb{Z})$.

In this paper we insert Burgess's bound for the character sum estimate and the mean value estimate (1.2) for the Rankin zeta-function into the arguments of Iwaniec and obtain the following result.

**Theorem.** For the modular group $\Gamma = PSL(2, \mathbb{Z})$,

$$\pi_\Gamma(x) = lx + O(x^{\frac{71}{102} + \varepsilon}), \quad \varepsilon > 0.$$  

2. Some preliminary lemmas

**Lemma 2 ([9]).** Let $\rho_j(n)$ denote the $n$-th Fourier coefficients for the Fourier expansion at $\infty$ of the $j$-th Maass cusp form, and

$$R_j(s) = \sum_{n=1}^{\infty} \frac{|\rho_j(n)|^2}{n^s}$$

denote the Rankin-Selberg $L$-function for the $j$-th Maass cusp form. Then

$$\sum_{|t_j| \leq T} \frac{|R_j(s)|}{\cosh \pi t_j} \ll |s|^{4T^2 + \varepsilon}.$$  

**Lemma 3 ([1]).** If $q$ is not a square then

$$\sum_{r \leq H} \left(\frac{q}{r}\right) \ll H^{\frac{2}{3}} q^{\frac{1}{3} + \varepsilon},$$

where $\left(\frac{q}{r}\right)$ denotes the Jacobi's symbol and the constant implied in $\ll$ depends on $\varepsilon$ at most.

We infer from Lemma 3 a simple corollary.

**Lemma 4.** Let $1 \leq R_1 < R_2, a \geq 1, q \geq 1$. Then

$$\sum_{R_1 < r \leq R_2, (r,a)=1} \mu^2(r) \left(\frac{a^2 - 4}{r}\right) \ll a^{\frac{2}{3}} R_2^{\frac{3}{2}} (aqR_2)^\varepsilon.$$  

**Lemma 5 ([8]).** Let $\varphi(x)$ be a smooth function on $[0, \infty]$ such that

$$\varphi(0) = \varphi'(0), \varphi^{(l)}(x) \ll x^{-3}, x \to \infty, (l = 0, 1, 2, 3).$$
Define
\[ \varphi_0 = \frac{1}{2\pi} \int_0^\infty J_0(y) \varphi(y) dy, \]
\[ \varphi_B(x) = \int_0^1 \int_0^\infty \xi x J_0(\xi x) J_0(\xi y) \varphi(y) dy d\xi, \]
\[ \varphi_H(x) = \int_1^\infty \int_0^\infty \xi x J_0(\xi x) J_0(\xi y) \varphi(y) dy d\xi, \]
\[ \varphi(t) = \frac{\pi}{2i \sinh \pi t} \int_0^\infty (J_{2it}(x) - J_{-2it}(x)) \frac{\varphi(x)}{x} dx, \]
where \( J_\nu(x) \) is the Bessel function. Then
\[ \sum_{t_j} \frac{\rho(m) \overline{\rho_j(n)}}{\cosh \pi t_j} \varphi(t_j) + \frac{2}{\pi} \int_0^\infty \frac{\varphi(t)}{|\zeta(1 + 2it)|^2} d_{it}(m) d_{-it}(n) dt \]
\[ = \delta_{mn} \varphi_0 + \sum_{c=1}^\infty \frac{S(m, n, c)}{c} \varphi_H \left( \frac{4\pi \sqrt{mn}}{c} \right), \]
where
\[ d_{it}(n) = \sum_{d \mid n} d^it, S(m, n, c) = \sum_{a \equiv 1 (mod c)} e \left( \frac{ma + n\overline{a}}{c} \right), \]
\( \delta_{mn} \) is the Kronecker's delta symbol.

Proof. This is Kuznetzov Trace Formula. See Theorem 9.5 in [6]. \( \square \)

3. A mean value theorem for Fourier coefficients

Let
\[ \Lambda_j(N) = \sum_n \frac{h(n)|\rho_j(n)|^2}{\cosh \pi t_j} \]
where \( h(\xi) \) is a smooth function supported in \([N, 2N]\) such that
\[ |h^{(l)}(\xi)| \ll N^{-l}, l = 0, 1, 2, \ldots \]
\[ \int h(\xi) d\xi = N. \]
By Rankin [11] we know that \( R_j^2(s) = \zeta(2s)R_j(s) \) has meromorphic continuation onto the whole complex with only a simple pole at \( s = 1 \) with residue \( 2 \cosh \pi t_j \). And by [11] we know that \( R_j(s) \) is of polynomial growth in \(|s|\).
Lemma 6. We have

$$\Lambda_j(N) = \frac{12}{\pi^2} N + r(t_j, N)$$

with

$$\sum_{|t_j| \leq T} |r(t_j, N)| \ll T^2 N^{\frac{1}{2}} \log^2 NT.$$ 

Proof. Consider the Mellin transformation

$$\Omega(s) = \int h(\xi) \xi^{s-1} d\xi \ll (1 + |\tau|)^{-1999} N^\sigma$$

for \( s = \sigma + i\tau \) by partial integration 1999 times. By the inverse Mellin transformation and Cauchy's theorem we have

$$\Lambda_j(N) \cosh \pi t_j = \frac{1}{2\pi i} \int_{(2)} \Omega(s) R_j(s) ds$$

$$= 2\zeta^{-1}(2) \Omega(1) \cosh \pi t_j + O\left( N^{\frac{1}{2}} \int_0^\infty \frac{|R(\frac{1}{2} + i\tau)|}{(1 + \tau)^{1999}} d\tau \right),$$

and Lemma 6 follows from Lemma 2.

\[\square\]

4. A mean value theorem for \( \rho(c,a) \)

Let \( \rho(c,a) \) stand for the number of solutions \( d \pmod{c} \) of

$$d^2 - ad + 1 \equiv 0 \pmod{c},$$

and

$$F(A, B, C) = \sum_{B < a \leq A + B} \sum_{c \leq C} \rho(c, a)$$

for \( 1 \leq A \leq B \) and \( C \geq 1 \). We have

$$\sum_{a \pmod{c}} \rho(c, a) = \varphi(c)$$

where \( \varphi(c) \) is the Euler's function. Now

$$F(A, B, C) = \sum_{c \leq C} \left( \frac{A}{c} + O(1) \right) \varphi(c) = \frac{6}{\pi^2} AC + O(A + C^2).$$

Lemma 7. For \( 1 \leq A \leq B, C \geq 1 \) and any \( \varepsilon > 0 \),

$$F(A, B, C) = \frac{6}{\pi^2} AC + O((A^{\frac{3}{2}} B^{\frac{1}{5}} C + AC^{\frac{1}{2}})(BC)^\varepsilon).$$
Proof. Write $c = kl$ where $k$ is a squarefree odd number and $4l$ is a square-full number coprime with $k$. By the multiplicativity of $\rho(c, a)$ in $c$ we get

$$\rho(c, a) = \rho(k, a)\rho(l, a).$$

$d^2 - ad + 1 \equiv 0 \pmod{c}$ in $d \pmod{k}$ is equivalent to $x^2 \equiv a^2 - 4 \pmod{k}$ in $x \pmod{k}$ and the number of incongruent solutions of the latter is

$$\rho(k, a) = \prod_{p \mid k} \left(1 + \left(\frac{a^2 - 4}{p}\right)\right) = \sum_{r \mid k} \left(\frac{a^2 - 4}{r}\right).$$

Let $Q$ stand for the set of squarefull numbers. Then

$$F(A, B, C) = \sum_{l \in Q, (rs, 4l) = 1} \mu^2(rs) \sum_{B < a \leq A + B} \rho(l, a) \left(\frac{a^2 - 4}{r}\right)$$

$$= \sum_{lr \leq R} + \sum_{lr > R} = F_0(A, B, C) + F_\infty(A, B, C).$$

By (44) in [5] we have

$$F_0(A, B, C) = \lambda AC + O((CR^{\frac{1}{3}} + ACR^{-\frac{1}{3}} + AC^{\frac{1}{3}})(BC)^\varepsilon).$$

with some absolute constant $\lambda$.

By Lemma 4, after splitting up the summation over $r$ in $F_\infty(A, B, C)$ into intervals of the form $(R_1, R_2)$ with $l^{-1}R_1 \leq R_2 \leq 2R_1$ we deduce that

$$F_\infty(A, B, C) \ll \sum_{l \in Q} \sum_{B < a \leq A + B} \rho(l, a) \sum_{s \leq \frac{C}{lR_1}} R_1^{\frac{2}{3}} B^{\frac{2}{3}} (BC)^\varepsilon$$

$$\ll \sum_{l \in Q} \sum_{B < a \leq A + B} \rho(l, a) \frac{CR_1^{\frac{2}{3}} B^{\frac{2}{3}}}{lR_1} (BC)^\varepsilon$$

$$\ll R^{-\frac{1}{3}} B^{\frac{2}{3}} C (BC)^\varepsilon \sum_{B < a \leq A + B} \sum_{l \in Q} \rho(l, a) l^{-\frac{2}{3}}.$$
Since for $p > 2, \alpha > 2$ we have $\rho(p^\alpha, a) \leq 2p^{\left(\frac{\alpha}{2}\right)}$ where $p^\beta = (a^2 - 4, p^\alpha)$ it follows that

$$\sum_{\substack{l \leq C \\mathbb{I} \in \mathbb{Q}}} \frac{\rho(l, a)}{l^{\frac{3}{2}}} \ll \prod_{p \leq C} \left(1 + \sum_{\alpha \geq 2} \frac{\rho(p^\alpha, a)}{p^{\frac{3\alpha}{2}}}ight)$$

\[ \ll \prod_{1000 < p \leq C} \left(1 + \frac{2}{p^{\frac{3}{2}}}ight) \prod_{1000 < p | (a^2 - 4)} \left(1 + \frac{10}{p^{\frac{3}{2}}}ight) \]

\[ \ll 2^{d(a^2 - 4)} \ll (BC)^{\varepsilon}, \]

and finally

$$F_{\infty}(A, B, C) \ll AR^{-\frac{1}{2}}B^{\frac{2}{3}}C(BC)^{\varepsilon}. \tag{4.3}$$

By (4.3) and (4.4) with $R = A^{\frac{6}{5}}B^{\frac{4}{5}}$ we get that

$$F(A, B, C) = \lambda AC + O((A^{\frac{2}{3}}B^{\frac{2}{15}}C + AC^{\frac{1}{2}})(BC)^{\varepsilon}). \tag{4.4}$$

Comparing (4.1) and (4.5) for $B = A^2, C = A^{\frac{1}{2}}, A \to \infty$ one finds $\lambda = \frac{6}{\pi^2}$, which completes the proof of Lemma 7.

\[ \Box \]

5. An application of Kuznetzov trace formula

Let

$$\varphi(x) = -\frac{\sinh \beta}{\pi} x \exp(ix \cosh \beta),$$

$$2\beta = \log X + \frac{i}{T}. \tag{4.5}$$

Then (cf. [9])

$$\varphi(t) = \frac{\sinh(\pi + 2i\beta)t}{\sinh \pi t},$$

$$\varphi_0 = -\frac{\cosh \beta}{2\pi^2 \sinh^2 \beta},$$

$$\varphi_B(x) = -\frac{\sinh 2\beta}{2\pi} \int_0^1 \xi x J_0(\xi x)(\cosh^2 \beta - \xi^2)^{-\frac{3}{2}} d\xi,$$

$$\varphi_H(x) = -\frac{\sinh 2\beta}{2\pi} \int_1^\infty \xi x J_0(\xi x)(\cosh^2 \beta - \xi^2)^{-\frac{3}{2}} d\xi.$$
Then we have \( \varphi = \varphi_B + \varphi_H \). It is easy to show that
\[
\hat{\varphi}(t) = X^{it} \exp \left( -\frac{t}{T} \right) + O(e^{-\pi t}), \quad \varphi_0 = O(X^{-\frac{1}{2}}),
\]
\[
\int_0^\infty \frac{\hat{\varphi}(t)|d_\Theta(n)|^2}{|\zeta(1+2it)|^2} dt = O(T \log^2 T d^2(n)).
\]
Let
\[
S_n(\varphi) = \sum_{t_j>0} \frac{|\rho_j(n)|^2}{\cosh \pi t_j} \hat{\varphi}(t_j),
\]
\[
T_n(\varphi) = 2 \frac{2}{\pi} \int_0^\infty \frac{\hat{\varphi}(t)|d_\Theta(n)|^2}{|\zeta(1+2it)|^2} dt,
\]
\[
W_n(\varphi_H) = \sum_{c=1}^\infty \frac{S(n,c)}{c} \varphi_H \left( \frac{4\pi n}{c} \right),
\]
where \( S(n,c) = S(n,n;c) \). Then by Lemma 6 we have
\[
\sum_n h(n)S_n(\varphi) = \sum_{t_j>0} \hat{\varphi}(t_j) \Lambda_j(N) = \frac{12N}{\pi^2} \sum_{t_j>0} \hat{\varphi}(t_j) + \sum_{t_j>0} \hat{\varphi}(t_j) r(t_j,N)
\]
\[
= \frac{12N}{\pi^2} \sum_{t_j>0} X^{it_j} \exp \left( -\frac{t_j}{T} \right) + O(N^{\frac{1}{2}} T^2 \log^2 T),
\]
\[
\sum_n h(n)T_n(\varphi) \ll NT \log^{10} NT.
\]
By \( J_0(y) \ll \min(1, y^{-\frac{1}{2}}) \) and \(|S(n,c)| \leq 2(n,c)^{\frac{1}{2}} c\frac{1}{2} d(c)\) we get
\[
W_n(\varphi_B) \ll N^{\frac{1}{2}} X^{-\frac{1}{2}} \log^2 N.
\]
By the above arguments and Kuznetzov Trace formula we get
\[
(5.1) \quad \frac{12}{\pi^2} \sum_{t_j>0} X^{it_j} \exp \left( -\frac{t_j}{T} \right)
= \frac{1}{N} \sum_n h(n)W_n(\varphi) + O(T \log^{10} T + N^{-\frac{1}{2}} T^2 \log^2 T)
= \frac{1}{N} \sum_n h(n)W_n(\varphi) + O(T^{\frac{3}{2}} \log^2 T),
\]
where and below, we take
\[
X^{\frac{1}{10}} \leq T \leq X^{\frac{1}{2}}, N = T \log^{10} X.
\]
6. An estimation for $\sum_n h(n)W_n(\varphi)$

Since

$$S(n, c) = \sum_{a (\mod c)}^* \rho(c, a) e \left( \frac{na}{c} \right)$$

we have

$$\sum_n h(n)W_n(\varphi) = -4\sinh \beta \sum_{1}^{\infty} \frac{1}{c^2} \sum_{-c < 2a \leq c} \rho(c, a) \times \sum_n h(n) n e \left( \frac{n(2 \cosh \beta - a)}{c} \right).$$

Notice that

$$2 \cosh \beta = (X^{1/2} + X^{-1/2}) \cos \frac{1}{2T} + i(X^{1/2} - X^{-1/2}) \sin \frac{1}{2T}$$

$$= B + iE, \text{ say}$$

thus

$$B = X^{1/2} + O(X^{1/2}T^{-2}), \quad E = \frac{X^{1/2}}{2T} + O(X^{1/2}T^{-3}).$$

Moreover we have

$$\left| e \left( \frac{n(2 \cosh \beta - a)}{c} \right) \right| = \exp \left( -\frac{2\pi n}{c} E \right) \leq \exp \left( -\frac{NX^{1/2}}{cT} \right).$$

Let $C_1 = 18X^{1/2}, C_2 = X$. Then

$$\sum_{c=1}^{C_1} \frac{1}{c^2} \sum_{-c < 2a \leq c} \rho(c, a) \sum_n h(n) n e \left( \frac{n(2 \cosh \beta - a)}{c} \right) \ll N^2 X^{-10}.$$

And Weil's bound $|S(n, c)| \leq 2(n, c)^{1/2} c^{1/2} d(c)$ implies

$$\sum_{c=C_2}^{\infty} \frac{1}{c^2} \sum_{-c < 2a \leq c} \rho(c, a) \sum_n h(n) n e \left( \frac{n(2 \cosh \beta - a)}{c} \right) \ll \sum_n h(n) n \sum_{c=C_2}^{\infty} \frac{|S(n, c)|}{c^2} \ll \sum_n h(n) n \sum_{c=C_2}^{\infty} \frac{(n, c)^{1/2} d(c)}{c^{3/2}} \ll C_2^{-1/2} N^2 \log^{10} C_2 \ll X^{-1/2} N^2 \log^{10} X.$$
Now
\[ \sum_{c_1}^{C_2} \sum_{-c < 2a \leq c \leq C_2} \frac{1}{c^2} \rho(c, a) \sum_n h(n) ne \left( \frac{n(2 \cosh \beta - a)}{c} \right) \]
falls into one of at most \( O(\log X) \) partial sums of the form
\[ \sum_{C < c \leq 2C} \frac{1}{c^2} \sum_{-c < 2a \leq c} \rho(c, a) \sum_n h(n) ne \left( \frac{n(2 \cosh \beta - a)}{c} \right) \]
with \( C_1 \leq C < 2C \leq C_2 \). We have
\[ \left| \frac{2 \cosh \beta - a}{c} \right| \leq \frac{5}{8}. \]
By Poisson summation formula we have
\[ \sum_n h(n) ne \left( \frac{n(2 \cosh \beta - a)}{c} \right) = \sum_{x = -\infty}^{\infty} \int h(x) xe \left( \frac{x(2 \cosh \beta - a)}{c} - kx \right) dx \]
\[ = \int h(x) xe \left( \frac{x(2 \cosh \beta - a)}{c} - kx \right) dx + \sum_{k \neq 0} \int h(x) xe \left( \frac{x(2 \cosh \beta - a)}{c} - kx \right) dx. \]
For \( k \neq 0 \) we have
\[ \int h(x) xe \left( \frac{x(2 \cosh \beta - a)}{c} - kx \right) dx = \frac{1}{4\pi^2} \left( \frac{2 \cosh \beta - a}{c} - k \right)^2 \]
\[ \times \int (h(x) x)^{\prime \prime} e \left( \frac{x(2 \cosh \beta - a)}{c} - kx \right) dx \ll \frac{1}{k^2}. \]
Hence
\[ (6.4) \sum_{C < c \leq 2C} \frac{1}{c^2} \sum_{-c < 2a \leq c} \rho(c, a) \sum_n h(n) ne \left( \frac{n(2 \cosh \beta - a)}{c} \right) \]
\[ = \sum_{C < c \leq 2C} \frac{1}{c^2} \sum_{-c < 2a \leq c} \rho(c, a) \int h(x) xe \left( \frac{x(2 \cosh \beta - a)}{c} \right) dx + O(1). \]
For \( |B - a| > CN^{-1+\varepsilon} = A \) by a multiple partial integration we have
\[ \int h(x) xe \left( \frac{x(2 \cosh \beta - a)}{c} \right) dx \]
\[ = \frac{(-1)^l}{(2\pi i(2 \cosh \beta - a))^{\prime}} \int (h(x) x)^{(l)} e \left( \frac{x(2 \cosh \beta - a)}{c} \right) dx \]
\[ \ll \left( \frac{C}{|B - a|} \right)^l N^{2-l} \ll 1, \]
where \( l = \left[ \frac{2}{\varepsilon} \right] + 2 \). Thus

\[
(6.5) \quad \sum_{C < c \leq 2C} \frac{1}{c^2} \sum_{-c < a \leq c} \rho(c, a) \sum_{n} h(n) n e \left( \frac{n(2 \cosh \beta - a)}{c} \right)
= \sum_{C < c \leq 2C} \frac{1}{c^2} \sum_{|B - a| \leq A} \rho(c, a) \int h(x) e \left( \frac{x(2 \cosh \beta - a)}{c} \right) dx + O(1)
= \int \sum_{C < c \leq 2C} \frac{1}{c^2} \sum_{|B - a| \leq A} \rho(c, a) e \left( \frac{(B - a)x}{c} \right) \exp \left( -\frac{2\pi F_{\varepsilon}}{c} \right) h(x) dx + O(1)
= \int \sum_{C < c \leq 2C} \sum_{|B - a| \leq A} \rho(c, a) e \left( (B - a)x \right) \exp \left( -2\pi F_{\varepsilon} \right) h(cx) dx + O(1).
\]

Let

\[
F_{\varepsilon}(A, B, C) = \sum_{C < c \leq 2C} \sum_{|B - a| \leq A} \rho(c, a) e \left( (B - a)x \right)
= \int_{-A}^{A} e(\alpha x) d\alpha F_{\varepsilon}(A + \alpha, B - A, C)
= \frac{6C \sin(2\pi Ax)}{\pi^3 x} + O((1 + xA)(A^{3/4}B^{3/4}C + AC^{1/2})(BC)^{\varepsilon}),
\]

then

\[
\sum_{C < c \leq 2C} \sum_{|B - a| \leq A} \rho(c, a) e \left( (B - a)x \right) h(cx)
= \int_{C}^{2C} h(y) dy F_{\varepsilon}(A, B, y)
= \frac{6 \sin(2\pi Ax)}{\pi^3 x} \int_{C}^{2C} h(y) dy + O((A^{3/4}B^{3/4}C + AC^{1/2})(BC)^{\varepsilon})
= M + R, \text{ say}
\]

the integration of \( M \) over \( x \) is

\[
(6.6) \quad \frac{6}{\pi^3} \int_{C}^{2C} \left( \int \frac{\sin(2\pi Ax)}{\exp(2\pi Ey)} h(y) dy \right) dx
= \frac{6}{\pi^3} \int_{C}^{2C} \left( \int \frac{\sin(2\pi Axy^{-1})}{\exp(2\pi Exy^{-1})} h(x) dx \right) \frac{dy}{y}
\ll \frac{1}{N^2},
\]

where we have used the multiple integration by parts

\[
\int \frac{\sin(2\pi Axy^{-1})}{\exp(2\pi Exy^{-1})} h(x) dx = \text{Im} \int \frac{e(Axy^{-1})}{\exp(2\pi Exy^{-1})} h(x) dx
= \text{Im} \left( \frac{(-1)^l}{(2\pi i A y^{-1})^l} \int \left( \frac{h(x)}{\exp(2\pi Exy^{-1})} \right)^{(l)} e(Axy^{-1}) dx \right)
\ll \frac{N}{(ANC^{-1})^l} \ll \frac{1}{N^2},
\]
with $l = \left[ \frac{4}{3} \right] + 2$, since $\frac{ANC}{C}^{-1} \geq X^\varepsilon$.

Integration over $x$ from $\left( \frac{N}{2C}, \frac{2N}{C} \right)$ of $R$ yields

$$(6.7) \quad \ll N^2 C^{-2} (A \frac{3}{3} B^{3/2} C + AC^{3/2}) (BC)^\varepsilon) \ll X^{1/5 + \varepsilon} N^{7/5} C^{-3/5}.$$

Combining all the above arguments (6.1)—(6.7) we get

$$(6.8) \quad \sum_n h(n) W_n (\varphi) \ll X^{11/30 + \varepsilon} N^{7/5} C^{-3/5} + X^{1/3} + N^2 \log^{10} X$$

$$\ll X^{11/30 + \varepsilon} N^{7/5} C^{-3/5}.$$

### 7. Proof of the theorem

By (5.1) and (6.8) we get

$$(7.1) \quad \sum_{t_j > 0} X^{it_j} \exp \left( -\frac{t_j}{T} \right) \ll T^{\frac{2}{3}} X^{11/30 + \varepsilon} + T^{\frac{3}{2}} \log^2 T \ll T^{\frac{2}{3}} X^{11/30 + \varepsilon}. $$

By (7.1) and the Fourier Technique used in [9] we get

$$(7.2) \quad \sum_{|t_j| \leq T} X^{it_j} \ll T^{\frac{2}{3}} X^{11/30 + \varepsilon}. $$

By (58) in [9]

$$(7.3) \quad \sum_{|t_j| \leq T} X^{it_j} \ll T^{\frac{5}{4}} X^{1/3} \log^2 T.$$ 

By (7.2) and (7.3) and the inequality $\min(A, B) \leq A^\alpha B^\beta$ for $A > 0, B > 0, \alpha > 0, \beta > 0, \alpha + \beta = 1$ we get

$$\sum_{|t_j| \leq T} X^{it_j} \ll \min(T^{\frac{2}{3}} X^{11/30 + \varepsilon}, T^{\frac{5}{4}} X^{\frac{1}{8}} \log^2 T)$$

$$\ll (T^{\frac{2}{3}} X^{11/30 + \varepsilon})^{\frac{5}{17}} (T^{\frac{5}{4}} X^{\frac{1}{8}} \log^2 T)^{\frac{12}{17}} \ll TX^{10/51 + \varepsilon},$$

and the theorem follows from (7.4), Lemma 1 and summation by parts.

### References


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