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2–modular lattices from ternary codes

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ABSTRACT. The alphabet $\mathbb{F}_3 + v\mathbb{F}_3$ where $v^2 = 1$ is viewed here as a quotient of the ring of integers of $\mathbb{Q}(\sqrt{-2})$ by the ideal $(3)$. Self-dual $\mathbb{F}_3 + v\mathbb{F}_3$ codes for the Hermitian scalar product give 2–modular lattices by construction $A_K$. There is a Gray map which maps self-dual codes for the Euclidean scalar product into Type III codes with a fixed point free involution in their automorphism group. Gleason type theorems for the symmetrized weight enumerators of Euclidean self-dual codes and the length weight enumerator of Hermitian self-dual codes are derived. As an application we construct an optimal 2-modular lattice of dimension 18 and minimum norm 3 and new odd 2-modular lattices of norm 3 for dimensions 16, 18, 20, 22, 24, 26, 28 and 30.

1. Introduction

Recent years witnessed a burst of activity in codes over $\mathbb{Z}_4$ [6, 11, 12, 14, 2, 3, 20] with applications to (nonlinear) binary codes [14] and unimodular lattices [2, 20]. Another important alphabet of size 4 besides $\mathbb{Z}_4$ is $\mathbb{F}_2 + u\mathbb{F}_2$ introduced in [1] to construct lattices, and explored further in [10] to study self-dual binary codes with a fixed point free (fpf) involution in...
their automorphism groups. This last class of codes was introduced in [4]. The current work is a ternary analogue of [10]. Here self-dual ternary codes with a fixed point free involution are characterized as Gray images of self-dual codes over $R_3 := \mathbb{F}_3 + v\mathbb{F}_3$ for the Euclidean scalar product. For instance Pless symmetry codes admit a natural description as Gray images of extended cyclic codes over $R_3$. The natural weight is the Lee weight defined as the Hamming weight of the Gray image with values 0, 1, 2.

While $\mathbb{F}_2 + u\mathbb{F}_2$ is a local ring like $\mathbb{Z}_4$ the alphabet $\mathbb{F}_3 + v\mathbb{F}_3$ is a semi-local ring like $\mathbb{Z}_6$. It is, as noticed in [1] abstractly isomorphic to $\mathbb{F}_3 \times \mathbb{F}_3$. The main technical tool in that context is therefore the Chinese Remainder Theorem (CRT). Another way to look at it would be the $(u + v, u - v)$ construction [15, 16].

Following [1] we view $R_3$ (or $\mathbb{F}_3 \times \mathbb{F}_3$) as a quotient of the ring of integers of $\mathbb{Q}(\sqrt{-2})$ by the ideal (3). This induces a conjugation on $R_3$, making it necessary to introduce a Hermitian scalar product. The natural weight attached to that number field is the length function which takes values 0, 1, 2, 3. By construction $A$ of [17] (Chap. 7) we obtain odd 2-modular lattices as per the definition in [22].

2. Notation and Definitions

2.1. Codes. Let $R_3$ denote the ring with 9 elements $\mathbb{F}_3 + v\mathbb{F}_3$ where $v^2 = 1$. This ring contains two maximal ideals $(v - 1)$ and $(v + 1)$. Observe that both of $R_3/(v + 1)$ and $R_3/(v - 1)$ are $\mathbb{F}_3$. The CRT tells us that

$$R_3 = (v - 1) \oplus (v + 1).$$

More concretely, linear algebra over $\mathbb{F}_3$ shows that

$$a + vb = (a - b)(v - 1) - (a + b)(v + 1),$$

for all $a, b \in \mathbb{F}_3^n$.

A code over $R_3$ is an $R_3-$submodule of $R_3^n$. The Euclidean scalar product is

$$\sum_i x_i y_i.$$

The Gray map $\phi$ from $R_3^n$ to $\mathbb{F}_3^{2n}$ is defined as $\phi(x + y) = (x, y)$ for all $x, y \in \mathbb{F}_3^n$. The Lee weight of $x + y$ is the Hamming weight of its Gray image. Define the Lee composition of $x$ say $m_i(x), i = 0, 1, 2,$ as the number of entries in $x$ of weight $i$. The symmetrized length weight enumerator (slwe), whose name will be justified in the next subsection, is then

$$\text{slwe}_C(a, b, c) = \sum_{x \in C} a^{m_0(x)} b^{m_1(x)} c^{m_2(x)}.$$

The swap map on $\mathbb{F}_3^{2n}$ is defined as

$$s((x, y)) = (y, x)$$
for all $x, y \in \mathbb{F}_3^n$.

2.2. Lattices. Let $K$ be the quadratic number field $\mathbb{Q}(\sqrt{-2})$ with ring of integers $\mathcal{O} = \mathbb{Z}[\sqrt{-2}]$. Then define $R_3 := \mathcal{O} / (3)$. Denote by a bar the conjugation which fixes $\mathbb{F}_3$ and maps $b$ to $-b$. Consequently, the natural scalar product induced by the Hermitian scalar product of $\mathbb{C}^n$ is

$$\sum_i x_i \overline{y}_i.$$

The length function as defined in [1, p.96] is

$$l_K(a) := \inf \{x \overline{x} : x \equiv a \pmod{3}\}.$$

Noticing that $\sqrt{-2} \equiv v \pmod{3}$, and using the fact that $K$ is Euclidean we see that

$$l_K(\pm 1) = 1 < 9$$
$$l_K(\pm v) = 2 < 9.$$  

$$l_K(\pm 1 + \pm v) = 3 < 9.$$  

We then extend $l_K$ componentwise to $R_3^n$. Define the length composition $n_i(x)$ $i = 0, \ldots, 3$ of $x \in R_3^n$ as the number of coordinates of length $i$. The length weight enumerator (lwe) can then be defined as

$$\text{lwe}_C(a, b, c, d) = \sum_{x \in C} a^{n_0(x)} b^{n_1(x)} c^{n_2(x)} d^{n_3(x)}$$

Define the minimum length $l(C)$ of a code $C$ as the minimum of the length of a nonzero element.

Define construction $A_K(C)$ as the preimage in $\mathcal{O}^n$ of $C \subseteq R_3^n$. Recall that an integral lattice is $l$-modular [1, 18] for some prime $l$ if its dual is equivalent to itself by a similarity of rate $\sqrt{l}$.

**Theorem 2.1.** If $C \subseteq R_3^n$ is a self-dual code then the lattice $A_K(C)/\sqrt{3}$ is $2$-modular. Its norm is equal to the minimum of 3 and $l(C)/3$.

**Proof.** The first assertion follows by [1, Remark 3.8] and can alternatively be derived directly by checking that $\mathcal{O}$ is 2-modular for the bilinear form

$$(x, y) \mapsto Tr_K(x\overline{y})/2.$$  

The second assertion follows by observing that the lattice above contains vectors of the shape $3/\sqrt{3}(10^{n-1})$ whose norm is 3.  

$\square$
3. Structure and duality of codes over $R_3$

By the properties of CRT any code over $R_3$ is permutation-equivalent to a code generated by the following matrix:

$$
\begin{pmatrix}
I_{k_1} & (1-v)B_1 & (1+v)A_1 & (1+v)A_2 & (1-v)B_2 & (1+v)A_3 & (1-v)B_3 \\
0 & (1+v)I_{k_2} & 0 & (1+v)A_4 & 0 & (1-v)I_{k_3} & 0 \\
0 & 0 & (1-v)I_{k_3} & 0 & (1-v)B_4 & 0 & 0
\end{pmatrix}
$$

where $A_i$ and $B_j$ are ternary matrices. Such a code is said to have rank $\{3k_1, 3k_2, 3k_3\}$.

If $H$ is a code over $R_3$. Let $H^+$ (resp. $H^-$) be the ternary code such that $(1+v)H^+$ (resp. $(1-v)H^-$) is read $H \text{ mod } (1-v)$ (resp. $H \text{ mod } (1+v)$). We have: $H = (1+v)H^+ \oplus (1-v)H^-$ with:

$$H^+ = \{s \mid \exists t \in F_3^n \mid (1+v)s + (1-v)t \in H\}$$

and

$$H^- = \{t \mid \exists s \in F_3^n \mid (1+v)s + (1-v)t \in H\}$$

The code $H^+$ is permutation-equivalent to a code with generator matrix of the form:

$$\begin{pmatrix}
I_{k_1} & 0 & 2A_1 & 2A_2 & 2A_3 \\
0 & I_{k_2} & 0 & A_4 & 0
\end{pmatrix},
$$

where $A_i$ are ternary matrices. And the ternary code $H^-$ is permutation-equivalent to a code with generator matrix of the form:

$$\begin{pmatrix}
I_{k_1} & 2B_1 & 0 & 2B_2 & 2B_3 \\
0 & 0 & I_{k_3} & 0 & B_4
\end{pmatrix},
$$

where $B_i$ are ternary matrices.

The preceding statements show that any code $H$ over $F_3$ is completely characterized by its associated codes $H^+$ and $H^-$ and conversely. We give now a characterization of the dual of a code depending on the scalar product.

**Theorem 3.1.** Let $H$ be a code of length $n$ over $R_3$, with associated ternary codes $H^+$ and $H^-$ then for the Hermitian scalar product:

$$H^\perp = (1+v)(H^-)^\perp \oplus (1-v)(H^+)^\perp,$$

and the self-dual codes over $R_3$ are the codes $H$ with associated ternary codes $H^+$ and $H^-$ verifying $H^+ = (H^-)^\perp$.

**Proof.** Observe that if $c, c', d, d'$ are ternary vectors of length $n$ then

$$(c(1-v) + d(1+v))(c'(1-v) + d'(1+v)) = a(1-v) + b(1+v)$$

with $-a = cd'$ and $-b = dc'$. This shows that $a = b = 0$ if and only if $dc' = cd' = 0$.

Here is the analogue of the preceding theorem for Euclidean codes.
Theorem 3.2. Let H be a code of length n over $R_3$, with associated ternary codes $H^+$ and $H^-$, then for the Euclidean scalar product:

$$H^\perp = (1 + v)(H^+) \oplus (1 - v)(H^-)^\perp,$$

and the self-dual codes over $R_3$ are the codes H with associated ternary codes $H^+$ and $H^-$ such that $H^+$ and $H^-$ are self-dual ternary codes.

Proof. Observe that if $c, c', d, d'$ are ternary vectors of length n then

$$(c(1 - v) + d(1 + v))(c'(1 - v) + d'(1 + v)) = a(1 - v) + b(1 + v)$$

with $-a = cc'$ and $-b = dd'$. This shows that $a = b = 0$ if and only if $cc' = dd' = 0$.

This shows, in particular, that Euclidean self-dual codes exist in length n if and only if n is a multiple of 4, since self-dual codes over $F_3$ exist only for length a multiple of 4.

The number of distinct self-dual sub-spaces (and therefore the mass formula) for each duality can be deduced from the preceding theorems:

Theorem 3.3. Denote by $N_e(n)$ the number of distinct self-dual codes of length n over $R_3$ for the Euclidean scalar product then n is a multiple of 4 and:

$$N_e(n) = \left[ 2 \prod_{i=1}^{\frac{n-2}{2}} (3^i + 1) \right]^2.$$

Proof. Self-dual codes over $F_3$ are known to exist only for length n a multiple of 4 and the number $\sigma(n)$ of such sub-spaces has been calculated in [19]. In our case, applying the preceding theorem on the Euclidean duality, we deduce $N_e(n)$ only by squaring $\sigma(n)$.

Theorem 3.4. Denote by $N_h(n)$ the number of distinct self-dual codes of length n over $R_3$ for the Hermitian scalar product then:

$$N_h(n) = 1 + \sum_{k=1}^{n} \left[ \prod_{i=0}^{k-1} \frac{3^{n-i} - 1}{3^{i+1} - 1} \right].$$

Proof. Let $H(H^+, H^-)$ be a self-dual Hermitian code of length n. If $H^+$ is given, then $H^-$ has to be its dual. So that the number of distinct self-dual codes is equal to the number of possible ternary code of length n. The number of ternary codes of length n and dimension k, calculated by induction, is $\prod_{i=0}^{k-1} \frac{3^{n-i} - 1}{3^{i+1} - 1}$, the total number follows.

Corollary 3.5. If $(C, D)$ denotes a pair of self-dual ternary codes of length n then $\phi(C(1 - v) + D(1 + v))$ is a self-dual code with a fixed point free involution involutory automorphism.
Proof. By preceding Theorem we know that $C(1-v)+D(1+v)$ is self-dual. But $(a+vb)(a'+vb')=0$ yields $aa'+bb'=0$ i.e. $\phi(a+vb)\phi(a'+vb')=0$. The first assertion follows. The second assertion follows by noticing that the Gray image of multiplication by $v$ is the swap of the Gray image:

$$\phi(v(x+vy)) = (y,x) = s(\phi(x+vy)).$$


Now, we characterize a class of ternary self-dual codes with a special symmetry property.

**Theorem 3.6.** Up to permutation of coordinates, every self-dual ternary code $T$ of length $2n$ with a fixed point free involutory automorphism can be realized as $\phi(C)$ for some self-dual $C$ of length $n$ over $R_3$ for the Euclidean scalar product.

**Proof.** Let $\sigma$ be such an automorphism. Write an arbitrary element of $T$ as $(a, \sigma(a))$ with $a \in F_3^n$. Take $C$ to be the code of $R_3^n$ consisting of all $a + v\sigma(a)$. To check that $C$ is self-dual observe that if $t := (a, b)$ and $t' := (a', b')$ are in $T$ so is $s(t') = (b', a')$. Now the inner product $\phi^{-1}(t)\phi^{-1}(t') = (a + bv)(a' + b'v)$ is $tt' + v(ts(t'))$. \qed

**Examples of Euclidean self-dual $R_3$-codes**

1. Let $p$ be a prime $\equiv -11 \pmod{12}$. Consider the Pless symmetry code $S_{2p+2}$, of length $2p + 2$. It is held invariant by the natural swap map by [17], p. 511 (in particular, if $p = 11$ we get the ternary Golay code). We denote by $IS_{p+1}$ the inverse Gray image of length $p + 1$. In the next section this is constructed as a quadratic residue code over $R_3$.

2. Let $W$ be a $n$ by $n$ weighing matrix of weight $k$ (i.e. $WW^T = kI$) with $k \equiv \epsilon \pmod{3}$ with $\epsilon = \pm 1$. Assume that $W^T = \epsilon W$. Then, the $R_3$-span of $W - \epsilon vI$ is self-dual of length $n$.

Are there $R_3$-codes which are both Euclidean self-dual and Hermitian self-dual? The answer is simple.

**Proposition 3.7.** An $R_3$-code $C$ is self-dual for both the Hermitian and Euclidean scalar product if and only if it is self-conjugate. In particular it is the $R_3$-span of a ternary matrix the $F_3$-span of which is self-dual.

**Proof.** The first assertion is immediate from the definitions. The second assertion follows by combining Theorems 3.1 and 3.2 to get $C^+ = C$ a ternary self-dual code. \qed

This is the case in particular of Example 2 as the next section shows.

**4. Pless Symmetry Codes**

Pless defined symmetry codes over $F_3$. These codes have length $2(p+1)$ where $p$ is a prime congruent to 5 modulo 6. These can be expressed as
Gray images of extended quadratic residue codes defined over $R_3$ when $p$ is congruent to 11 modulo 12.

Let $p$ be a prime congruent to 5 modulo 6. Let $\epsilon = (-1/p)$.

If $p \equiv 11 \pmod{12}$ then $\epsilon = -1$ and let $\delta = v \in R_3$. Note that $\delta^2 = \epsilon p$ in $R_3$. Denote the action of natural involution of $R_3$ by a bar, so that $x + y\bar{v} = x - yv$ for $x, y \in F_3$. We shall construct quadratic residue codes of length $p + 1$ over $R_3$.

Let $S_p$ be the matrix

$$S_p = \begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
\epsilon & \epsilon & \cdots & \epsilon \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 \\
\end{pmatrix}$$

where $S_p'$ is the circulant matrix whose $(i,j)$-entry is $((j - i)/p)$. Then $S_p^t = \epsilon S_p$ and $S_p^2 = \epsilon p I$. Let $Q$ be the submodule of $R_3^{p+1}$ spanned by the rows of $\delta I + S_p$. We show that $Q$ is self-dual in an appropriate sense. If $\epsilon = -1$ then $R = R_3$ and $\delta I + S_p = v I + S_p$. Hence

$$(\delta I + S_p)(\delta I + S_p)^t = (v I + S_p)(v I - S_p) = I - S_p^2 = 0.$$

As it will become apparent that $|Q| = 3^{p+1}$, then $Q$ is self-orthogonal.

Recall the Gray code map $\phi : R^{p+1} \rightarrow F_3^{2(p+1)}$ as above for $R_3$. This map preserves orthogonality and so $\phi(Q)$ is self orthogonal. In each case $\phi(Q)$ contains the code with generator matrix $(S_p, I)$, and so $|Q| \geq 3^{p+1}$. Consequently $\phi(Q)$ has this generator matrix and is the Pless symmetry code.

5. The MacWilliams Relations

The complete weight enumerator for a code over $R_3$ is given by:

$$W_C(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) = \sum A(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \prod a_i^{a_i}$$

where there are $A(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ vectors in $C$ with $a_i$ appearing $a_i$ times in the vector.

5.1. The Euclidean Inner Product. Notice there is no generating character for the ring, hence the MacWilliams relations in [24] do not apply. Instead we use a slightly modified approach using a symmetric character table for the additive group of the ring as is done in [7]. Index the matrix by the elements of $R_3$ in the following order:

$$0, 1, 2, v, 1 + v, 2 + v, 2v, 1 + 2v, 2 + 2v$$
Then the MacWilliams relations for the complete weight enumerator are given by the following matrix where $\omega = e^{\frac{2\pi i}{3}}$.

$$M_C = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\
1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\
1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\
1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega & 1 \\
1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega & 1 \\
1 & \omega^2 & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \\
1 & \omega^2 & \omega^2 & \omega^2 & \omega & 1 & \omega & 1
\end{pmatrix}$$

Specializing the variables to the symmetric (i.e. grouping the variables with their symmetric as one variable) and indexing the matrix by

$$0, \pm 1, \pm v, \pm (1 + v), \pm (1 + 2v),$$

we obtain the MacWilliams relations for the symmetrized weight enumerator:

$$M_S = \frac{1}{3} \begin{pmatrix}
1 & 2 & 2 & 2 & 2 \\
1 & 2 & -1 & -1 & -1 \\
1 & -1 & 2 & -1 & -1 \\
1 & -1 & -1 & -1 & 2 \\
1 & -1 & -1 & 2 & -1
\end{pmatrix}$$

Further specialization gets the MacWilliams relations for the Hamming weight enumerator:

$$M_H = \frac{1}{3} \begin{pmatrix}
1 & 8 \\
1 & -1
\end{pmatrix}$$

The following matrix gives the weight enumerator for the length weight enumerator and is indexed by $0, \pm 1, \pm v, \pm 1 \pm v$.

$$M_L = \frac{1}{3} \begin{pmatrix}
1 & 2 & 2 & 4 \\
1 & 2 & -1 & -2 \\
1 & -1 & 2 & -2 \\
1 & -1 & -1 & 1
\end{pmatrix}$$

The symmetrized length weight enumerator is given by the following matrix, where $\pm 1$ and $\pm v$ are grouped together:

$$M_{SL} = \frac{1}{3} \begin{pmatrix}
1 & 4 & 4 \\
1 & 1 & -2 \\
1 & -2 & 1
\end{pmatrix}$$
5.2. **The Hermitian Inner Product.** The complete weight enumerator for the Hermitian inner product can be determined from the MacWilliams relations for the standard inner product and are given by the matrix:

\[
M'_C = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\
1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega \\
1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 \\
1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega \\
1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\
1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\
1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega \\
\end{pmatrix}
\]

We specialize variables to get the MacWilliams relations for the symmetrized weight enumerator.

\[
M'_S = \frac{1}{3} \begin{pmatrix}
1 & 2 & 2 & 2 & 2 \\
1 & 2 & -1 & -1 & -1 \\
1 & -1 & 2 & -1 & -1 \\
1 & -1 & -1 & 2 & -1 \\
1 & -1 & -1 & -1 & 2 \\
\end{pmatrix}
\]

Then we have \(M'_H = M_H, M'_L = M_L,\) and \(M'_{SL} = M_{SL} .\)

**Example:** Let \(C\) be the code \(\{0, 1 + 2v, 2 + v\}.\) Its weight enumerator is \(W = a_0 + a_5 + a_7.\) Applying \(M_C\) to \(W\) gives \(a_0 + a_4 + a_8\) corresponding to its orthogonal in the ordinary inner product, i.e. the code \(\{0, 1 + v, 1 + 2v\}.\) Applying \(M'_C\) to \(W\) gives \(a_0 + a_5 + a_7\) corresponding to its orthogonal in the Hermitian inner product, i.e. the code \(C.\)

5.3. **Gleason Relations.** Define the matrices \(P_3\) and \(P_4\) as diagonal matrices with entries respectively \(1, \omega, \omega^2\) and \(1, \omega, \omega^2, 1.\) Define the matrix groups \(G_3 := \langle M_{SL}, P_3, iI_3 \rangle,\) and \(G_4 := \langle M_L, P_4 \rangle.\) The following lemma is easily dealt with by Magma.

**Lemma 5.1.** The groups \(G_3\) and \(G_4\) are of respective orders 48, 24 and have Molien series (corresponding to Hironaka decomposition of their ring of invariants) respectively

\[
\frac{1 + 2t^8 + t^{12}}{(1 - t^4)(1 - t^{12})} = 1 + 2t^4 + 5t^8 + 10t^{12} + 15t^{16} + 22t^{20} + O(t^{21}),
\]

and

\[
\frac{1}{(1 - t)(1 - t^2)(1 - t^3)(1 - t^4)} = 1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + 11t^7 + 15t^8 + O(t^9).
\]
The group $G_4$ is abstractly isomorphic to the group number 2 with name $G(1,1,4)$ in the list of [23].

We are now in a position to state the following analogues of Gleason theorem. The Gleason polynomials are easy to obtain in Magma and too unwieldy to be recorded.

**Theorem 5.2.** The symmetrized length weight enumerator of a Euclidean code is invariant under $G_3$. It belongs to the ring $S \oplus h_8S \oplus h_8'S \oplus h_{12}S$ with

$$S = C[g_4,g_4',g_{12}]$$

with $g_4,g_4',g_{12}$ are primary invariants of degree 4, 4, 12 respectively and $h_8,h_8',h_{12}$ are secondary invariants of degree 8, 8, and 12 respectively.

**Proof.** The slwe is invariant under $P_3$ by self-duality of the Gray image. Invariance under $iI_3$ follows by the fact that the length must be a multiple of 4 by Theorem 3.2.

**Theorem 5.3.** The length weight enumerator of a Hermitian code is invariant under $G_4$. It belongs to the ring

$$C[f_1,f_2,f_3,f_4]$$

where $f_i$ is an homogeneous polynomial of degree $i$ in $a,b,c,d$.

**Proof.** The lwe is invariant under $P_4$ by the integrality of the corresponding lattice.

6. Some odd 2-modular lattices

In this section we give some codes over $R_3$ for the lengths $n = 4, 6, 8, 9, 10, 11, 12, 13, 14$ and 15, which are Hermitian self-dual and have minimum length weight 9. All these codes give by construction $A_K$ examples of odd 2-modular lattices of dimension $2n$ and minimum norm 3 by Theorem 2.1.

The following upper bound was given in [22]:

**Theorem 6.1.** If $L$ is a strongly 2-modular lattice with norm $\mu$ in dimension $n$ then:

$$\mu \leq 2[\frac{n}{16}] + 2.$$ 

Thus by theorem 2.1 a direct construction $A_K$ can only give extremal odd lattices of norm 3 for lengths strictly less than 8.

The code of length 8 leads to the unique 2-modular lattice of dimension 16 and norm 3, the so called 'odd Barnes-Wall' lattice of [22], the code of length 9 leads to a new optimal 2-modular lattice of dimension 18 since for this length there is no extremal lattice (i.e. norm 4) [22]. The other codes
lead to norm 3 odd 2-modular lattices of dimension $2n$. All the lattices constructed for $n \geq 9$ are new.

The codes of lengths lower than 7 are easy to find since we only need a minimum length weight of 6 to obtain extremal codes. We now describe how we found the codes of length 8 or more: by Theorem 3.1 we know that the self-dual Hermitian codes of length $n$ are the codes $H$ which are written:

$$H = (1 + v)C \oplus (1 - v)C^\perp,$$

with $C$ a ternary code of length $n$. In order to find such codes $H$ with length weight 9 or more, we first notice that if $C \cap C^\perp$ is non null then $H$ contains words of length weight equal to 3, and also that if $C$ or $C^\perp$ contain non null words of Hamming weight 2 or less then $H$ contains words of length weight 3 or 6.

We therefore searched for ternary codes $C$ with the following necessary conditions:

$$W_H(C) \geq 3, \quad W_H(C^\perp) \geq 3, \quad C \cap C^\perp = 0.$$ 

The codes were found, starting from binary codes with good minimum weight read-off $\pmod{3}$ and when the code $H$ did not have good minimum length weight, we exchanged some 1 by $-1$ in the ternary code $C$. The minimum length weight was checked by exhaustive search, using the Magma system.

- $n = 4 \quad C_4 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

- $n = 6 \quad C_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 \end{pmatrix}$

- $n = 8 \quad C_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 \end{pmatrix}$

- $n = 9 \quad C_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$
References


\[
\begin{align*}
\bullet \ n &= 10 \quad C_{10} &= \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\bullet \ n &= 11 \quad C_{11} &= \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\bullet \ n &= 12 \quad C_{12} &= \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\bullet \ n &= 13 \quad C_{13} &= \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\bullet \ n &= 14 \quad C_{14} &= \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\bullet \ n &= 15 \quad C_{15} &= \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\end{align*}
\]


S. T. Dougherty, Some thought about codes over groups. preprint.


