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RÉSUMÉ. On donne une condition nécessaire pour qu'une représentation surjective $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{PGL}_2(F_3)$ provienne de la 3-torsion d'une $\mathbb{Q}$-courbe. Nous étudions plus particulièrement le cas des $\mathbb{Q}$-courbes quadratiques.

ABSTRACT. We give a necessary condition for a surjective representation $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{PGL}_2(F_3)$ to arise from the 3-torsion of a $\mathbb{Q}$-curve. We pay a special attention to the case of quadratic $\mathbb{Q}$-curves.

1. Introduction

Let $C$ be a $\mathbb{Q}$-curve defined over a number field $k$, that is an elliptic curve over $k$ without complex multiplication and isogenous to all its Galois conjugates. Throughout, we will denote by $G_k$ and $G_\mathbb{Q}$ the absolute Galois groups $\text{Gal}(\overline{\mathbb{Q}}/k)$ and $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, respectively. Let

$$\rho_C : G_k \to \text{GL}_2(F_3)$$

be the representation given by the Galois action on the 3-torsion points of $C$. Then, there exists an odd representation

$$\varrho_C : G_\mathbb{Q} \to \text{PGL}_2(F_3)$$

whose restriction to $G_k$ is lifted to $\text{GL}_2(F_3)$ by $\rho_C$. The representation $\varrho_C$ comes from the Galois action on the 3-torsion of the abelian varieties of $\text{GL}_2$-type having the curve $C$ as a quotient (see the proof of Theorem 2.1).

The fixed field of $\varrho_C$, which we will denote by $K_C$, has Galois group over $\mathbb{Q}$ inside the symmetric group $S_4$, since this last group is isomorphic to $\text{PGL}_2(F_3)$. The behaviour of the restriction of $\varrho_C$ to $G_k$ implies the following property for the field $K_C$: its compositum with $k$ is the extension
generated by the $x$-coordinates of the 3-torsion points of $C$, with respect to any Weierstrass equation for $C$ over $k$. Whenever $k/\mathbb{Q}$ is Galois and $\rho_C$ is surjective, $K_C$ is the only $S_4$-extension of $\mathbb{Q}$ satisfying that property; a proof of this claim is given in the appendix. We recall that $\rho_C$ being odd amounts to $K_C$ not being real.

We will say that a representation $\rho : G_\mathbb{Q} \to \text{PGL}_2(\mathbb{F}_3)$ arises from a $\mathbb{Q}$-curve $C$ if $\rho = \rho_C$, where this last equality is considered up to conjugation inside $\text{PGL}_2(\mathbb{F}_3)$. Any such representation $\rho$ must be odd.

In section 2 we give a necessary condition for a surjective representation $\rho$ as above to arise from a given $\mathbb{Q}$-curve, in terms of the trace quadratic form attached to any quartic subextension of the fixed field of $\rho$. In section 3 we focus on the case of elliptic curves defined over quadratic fields.

2. The sign component in $\text{Br}_2(\mathbb{Q})$ attached to a $\mathbb{Q}$-curve

Let $C/k$ be a $\mathbb{Q}$-curve. From any locally constant set of isogenies from $C$ to its $G_\mathbb{Q}$-conjugates, one can attach to $C$ an invariant $\xi_C \in H^2(G_\mathbb{Q}, \mathbb{Q}^*)$ of its isogeny class (see [5], Proposition 2.1, and also [7], section 6). The sign component of $\xi_C$, denoted by $\xi_C^\pm$, is an element in $\text{Br}_2(\mathbb{Q})$, the 2-torsion of the Brauer group of $\mathbb{Q}$. This element is related to the complete definition of the $\mathbb{Q}$-curve (cf. [5] and [6]): assuming $k$ to be the minimal field of definition for $C$ up to isogeny, which is a polyquadratic extension of $\mathbb{Q}$, the existence of a $k$-twist $C'$ of $C$ with all the isogenies defined over $k$ amounts to the existence of a double cover of the group $\text{Gal}(k/\mathbb{Q})$ whose corresponding embedding problem has obstruction given by $\xi_C^\pm$.

The sign component $\xi_C^\pm$ is explicitly given in [5], Theorem 3.1, as a product of quaternion algebras in terms of the minimal field of definition for $C$ up to isogeny and the degrees of the isogenies between $C$ and its Galois conjugates. Our first result gives another expression for $\xi_C^\pm$ depending only on the octahedral extension $K_C/\mathbb{Q}$ in the introduction.

**Theorem 2.1.** Assume that $\rho_C$ is surjective, and let $K_C$ be defined as in section 1. Then, the sign component of $\xi_C$ is given by the following product in $\text{Br}_2(\mathbb{Q})$:

$$\xi_C^\pm = w_C (-2, d_C) (-1, -3),$$

where $d_C$ and $w_C$ are, respectively, the discriminant and the Witt invariant of the trace quadratic form attached to any quartic subextension of $K_C/\mathbb{Q}$. 
**Proof.** For any character \( \eta : G_\mathbb{Q} \rightarrow F^* \), with \( F \) an algebraically closed field, let \( [\eta] \) be the element in \( \text{Br}_2(\mathbb{Q}) \) giving the obstruction to the existence of a character \( \psi : G_\mathbb{Q} \rightarrow F^* \) such that \( \psi^2 = \eta \). Let us consider these two particular cases:

- For the mod 3 cyclotomic character
  \[ \chi : G_\mathbb{Q} \rightarrow \text{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q}) \simeq \{ \pm 1 \}, \]
  \( [\chi] \) is given by the quaternion algebra \((-1, -3)\).
- For any lifting
  \[ \rho : G_\mathbb{Q} \rightarrow \text{GL}_2(\overline{\mathbb{F}_3}) \]
  of the projective representation \( \varrho_C \), the element \( [\det \rho] \) gives, by Theorem 6 in [8] and Proposition 1.2 in [4], the obstruction to the solvability of the embedding problem
  \[ 2S_4^- \rightarrow S_4 \simeq \text{Gal}(K_C/\mathbb{Q}), \]
  where \( 2S_4^- \) is the only double cover of \( S_4 \) which can be embedded into \( \text{SL}_2(\mathbb{F}_3) \). That obstruction can be expressed (see [9], Théorème 1, and [10], section 2) in terms of the trace quadratic form attached to any quartic subextension of \( K_C/\mathbb{Q} \), so that we have the equality
  \[ [\det \rho] = w_C(-2, d_C) \]
  in \( \text{Br}_2(\mathbb{Q}) \).

Every lifting of \( \varrho_C \) into \( \text{GL}_2(\overline{\mathbb{F}_3}) \) is obtained, up to isomorphism, as follows. Let \( A/\mathbb{Q} \) be an abelian variety having the \( \mathbb{Q} \)-curve \( C \) as a quotient and with \( \mathbb{Q} \)-endomorphism algebra \( \mathbb{Q} \otimes \text{End}_\mathbb{Q}(A) \) a number field of degree \( \dim(A) \). By [7], Theorem 6.1, such an abelian variety exists, and we can also assume, replacing \( A \) by a \( \mathbb{Q} \)-isogenous abelian variety if necessary, that the \( \mathbb{Q} \)-endomorphism ring of \( A \) is the maximal order in \( \mathbb{Q} \otimes \text{End}_\mathbb{Q}(A) \). For every prime ideal \( p \) over 3 in that order, the intersection of the kernels of all the endomorphisms in \( p \) becomes then a 2-dimensional vector space over \( \overline{\mathbb{F}_3} \), and the representation

\[ \rho_{A,p} : G_\mathbb{Q} \rightarrow \text{GL}_2(\overline{\mathbb{F}_3}) \]
given by the Galois action on \( A[p] \) is a lifting of \( \varrho_C \); for another description of the representations \( \rho_{A,p} \), we refer to [2], section 2.

From [1], Proposition 2.15 (see also [5], Theorem 4.2), we have the identity

\[ [\chi \det \rho_{A,p}] = \xi_C^\pm. \]

By combining it with the above equalities, we obtain

\[ \xi_C^\pm = [\det \rho_{A,p}] [\chi] = w_C(-2, d_C)(-1, -3) \]
in \( \text{Br}_2(\mathbb{Q}) \), as desired. \( \square \)
Remark 2.1. Given an octahedral extension $K/Q$, let $d \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ and $w \in \text{Br}_2(Q)$ be, respectively, the discriminant and the Witt invariant of the quadratic form $\text{Tr}_{K_1/Q}(x^2)$ attached to a quartic subextension $K_1/Q$ of $K/Q$. Then, $w$ can easily be computed from any reduced polynomial $f(X) = X^4 + aX^2 + bX + c$ defining the extension $K_1/Q$ (cf. [2], Remark 4.2, and [3], section 2):

- If $a = 0$ or $a = 2d$ up to squares, then $w = (-1, -d)$.
- Otherwise, $\delta = 2a^3 + 9b^2 - 8ac$ is non-zero, and then $w = (-1, -d)(2a, \delta)$.

Notice also that $d$ is the discriminant of $f$ up to squares.

The following corollary, which is just a restatement of Theorem 2.1, gives the necessary condition announced in the introduction.

Corollary 2.1. Let $\varrho: G_Q \to \text{PGL}_2(\mathbb{F}_3)$ be a surjective representation, and let $d$ and $w$ be the invariants defined by the fixed field of $\varrho$ (as in Remark 2.1). If $\varrho$ arises from a $Q$-curve $C$, with attached sign component $\xi_C^\pm$ in $\text{Br}_2(Q)$, then the equality

$$\xi_C^\pm = w(-2, d)(-1, -3)$$

must hold.

3. Ellipticity over quadratic fields of projective mod 3 Galois representations

Given a projective representation

$$\varrho: G_Q \to \text{PGL}_2(\mathbb{F}_3),$$

we will say that $\varrho$ is elliptic over a quadratic field $k$ if its restriction to $G_k$ is given by

$$\varrho|_{G_k} = \overline{\rho_E}$$

for some elliptic curve $E$ defined over $k$. Here $\rho_E$ denotes as above the representation of $G_k$ attached to the 3-torsion points of $E$, and $\overline{\rho_E}$ stands for its associated projective representation. It is clear from the definitions that ellipticity is a necessary condition for the representation $\varrho$ to arise from a $Q$-curve defined over $k$.

Whenever $\text{det} \varrho: G_Q \to \mathbb{F}_3$ is the cyclotomic character $\chi$, the representation $\varrho$ arises from an elliptic curve defined over $Q$ (and hence it is elliptic over any quadratic field) if and only if it can be lifted to $\text{GL}_2(\mathbb{F}_3)$ [3]. If this last condition is not fulfilled, there are still a priori infinitely many quadratic fields $k$ over which $\varrho$ could be elliptic.

On the other hand, the quadratic field $k$ is uniquely determined by $\varrho$ provided that $\text{det} \varrho \neq \chi$: it corresponds necessarily to the quadratic character $\chi \text{det} \varrho$, so that the restriction $\text{det} \varrho|_{G_k}$ becomes cyclotomic. In this
case, the following result gives a characterization of those surjective representations \( \rho \) which are elliptic over \( k \).

**Theorem 3.1.** Let \( \rho : G_Q \to \text{PGL}_2(\mathbb{F}_3) \) be a surjective representation with non-cyclotomic determinant, and let \( d \) and \( w \) be, respectively, the discriminant and the Witt invariant of the trace quadratic form attached to any quartic subextension of the fixed field of \( \rho \). The following conditions are equivalent:

1. The representation \( \rho \) is elliptic over a quadratic field.
2. For every prime \( p \) that splits in \( \mathbb{Q}(\sqrt{-3d}) \), the local component at \( p \) of the quaternion algebra \( w(-1,-d) \) is trivial.
3. Every prime \( p \) for which \( w_p \neq (-1,-d)_p \) in \( \text{Br}_2(\mathbb{Q}_p) \) satisfies the following (where we regard \( d \) and \( -3d \) as squarefree integers):
   - If \( p \) is odd, then the Legendre symbol \( \left( -3d \middle| p \right) \) is not 1.
   - If \( p = 2 \), then \( d \equiv 5 \pmod{8} \).
4. The fixed field of \( \rho \) is the splitting field of a polynomial of the form \( X^4 - 6X^2 + bX + c \), for some \( b,c \in \mathbb{Q} \).

**Proof.** The hypothesis on \( \det \rho \) amounts to saying that \( d \neq -3 \) in \( \mathbb{Q}^* / \mathbb{Q}^* -2 \). As we have noticed above, the only quadratic field \( k \) over which \( \rho \) can be elliptic is the fixed field of \( \chi \det \rho \), namely \( \mathbb{Q} \left( \sqrt{-3d} \right) \).

Let \( K_1/\mathbb{Q} \) be a quartic subextension of the fixed field of \( \rho \). Consider the quadratic form \( \text{Tr}_{K_1/\mathbb{Q}}(x^2) \) on \( K_1 \), with invariants \( d \in \mathbb{Q}^* / \mathbb{Q}^* -2 \) and \( w \in \text{Br}_2(\mathbb{Q}) \), and denote by \( T \) its restriction to the 3-dimensional subspace

\[
\left\{ x \in K_1 \mid \text{Tr}_{K_1/\mathbb{Q}}(x) = 0 \right\}.
\]

The quadratic form \( T \) has also discriminant \( d \) and Witt invariant \( w \). Regarded as a real quadratic form, its signature is \( (r_1 + r_2 - 1, r_2) \), where \( r_1 \) (resp. \( 2r_2 \)) is the number of real (resp. non-real) embeddings of \( K_1 \) into \( \overline{\mathbb{Q}} \); in particular, it represents any positive real number, which in terms of Hilbert symbols means that \( w_{\infty} = (-1,-d)_{\infty} \) in \( \text{Br}_2(\mathbb{R}) \) whenever \( d \) is negative.

Condition (4) amounts to the existence of an element \( \alpha \in K_1 \) with \( \text{Tr}_{K_1/\mathbb{Q}}(\alpha) = 0 \) and \( \text{Tr}_{K_1/\mathbb{Q}}(\alpha^2) = 3 \), i.e. to the representability of 3 by \( T \). In terms of Hilbert symbols, the obstruction to that is given by the equality

\[
w_p = (-1,-d)_p
\]

in \( \text{Br}_2(\mathbb{Q}_p) \) for every prime \( p \) such that \( -3d \) is a square in \( \mathbb{Q}_p^* \). This is in turn equivalent to condition (2), whose translation into Legendre-Kronecker symbols is given by (3).

Consider now the natural morphisms

\[
\mathbb{Q}^* / \mathbb{Q}^* -2 \to k^* / k^* -2 \quad \text{Br}_2(\mathbb{Q}) \to \text{Br}_2(k).
\]
The discriminant and the Witt invariant of the quadratic form (over \(k\)) \(Tr_{k_1, k/k}(x^2)\) are the image of \(d\) and \(w\), respectively, by these maps. By Theorem 3 in [3] and Theorem 4.2, Lemma 4.1 in [2], \(\rho\) is elliptic over \(k\) if and only if \(w = (-1, -d)\) in \(Br_2(k)\). Since \(k\) is imaginary whenever \(w_\infty \neq (-1, -d)_\infty\) in \(Br_2(\mathbb{R})\), this amounts again to condition (2).

**Remark 3.1.** The representation \(\rho\) need not be odd to satisfy the equivalent conditions in the proposition. Also, the surjectivity assumption can be relaxed by only asking the fixed field of \(\rho\) to be the normal closure of a quartic extension of \(\mathbb{Q}\), and the result remains the same.

Let us now apply the above result, along with the one in the previous section, to \(\mathbb{Q}\)-curves of degree \(N\), that is to \(\mathbb{Q}\)-curves defined over a quadratic field, with non-rational \(j\)-invariant, and having an isogeny of degree \(N\) to its conjugate curve.

**Proposition 3.1.** Let \(\rho\), \(d\) and \(w\) be as in Theorem 3.1. If \(\rho\) arises from a \(\mathbb{Q}\)-curve of degree \(N\), then the following two equivalent conditions are satisfied:

(i) The Witt invariant \(w \in Br_2(\mathbb{Q})\) is given by

\[
\begin{align*}
  w &= (-1, -d) (2N, -3d).
\end{align*}
\]

(ii) For every polynomial \(X^4 - 6X^2 + bX + c \in \mathbb{Q}[X]\) having the fixed field of \(\rho\) as splitting field, \(\delta = 3(16c + 3b^2 - 144)\) is non-zero and the quaternion algebra \((2\delta N, -3d)\) is trivial in \(Br_2(\mathbb{Q})\).

**Proof.** The existence of polynomials as in (ii) is ensured by Theorem 3.1, and the equivalence between the two conditions is a straightforward consequence of Remark 2.1. Let \(C\) be a \(\mathbb{Q}\)-curve of degree \(N\) attached to \(\rho\). Since the quadratic field of definition for \(C\) is \(\mathbb{Q}(\sqrt{-3d})\), the sign component \(\xi_C^\pm\) is given by the quaternion algebra \((N, -3d)\) [5]. Condition (i) follows then from Corollary 2.1.

**Appendix**

We look closer here at the uniqueness of the octahedral extension \(K_C/\mathbb{Q}\) attached in the introduction to a \(\mathbb{Q}\)-curve \(C\) defined over a Galois number field \(k\), in the case of surjective 3-torsion. The precise statement amounts to the following octahedral exercise. Its proof is obtained directly from Galois theory and the lemma below.

**Proposition.** Let \(K/\mathbb{Q}\) and \(k/\mathbb{Q}\) be normal extensions such that the Galois groups \(\text{Gal}(K/\mathbb{Q})\) and \(\text{Gal}(K/k)\) are isomorphic to the symmetric group \(S_4\). Then, there are no other \(S_4\)-extensions of \(\mathbb{Q}\) having the same compositum with \(k\) as \(K\).
Lemma. Let $G_1$ and $G_2$ be groups, and identify them with their respective canonical images inside the product group $G_1 \times G_2$. Assume the center of $G_2$ to be trivial. Let $H$ be a normal subgroup of $G_1 \times G_2$ with the same order as $G_1$ and having trivial intersection with $G_2$. Then, $H$ must be equal to $G_1$.

Proof. From the assumptions on $H$, this subgroup must be of the form
$$\{(g, \sigma_g) \mid g \in G_1\}.$$ All we must see then is that $\sigma_g = 1$ for all $g$ in $G_1$. Assume that $\sigma_g \neq 1$ for some $g$. Since $G_2$ has trivial center, there must be some $\tau$ in $G_2$ such that $\tau^{-1}\sigma_g \tau \neq \sigma_g$. Then, the element
$$(g, \tau^{-1}\sigma_g \tau) = (1, \tau)^{-1}(g, \sigma_g)(1, \tau)$$ lies in $H$ and is different from $(g, \sigma_g)$, which yields a contradiction. \hfill \Box

References