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par LADISLAV MIŠÍK et JÁNOS T. TÓTH

RÉSUMÉ. Nous donnons des conditions suffisantes pour que l'ensemble $R(A)$ des fractions d'un ensemble d'entiers $A$ soit dense dans $\mathbb{R}^+$, en termes des densités logarithmiques de $A$. Ces conditions diffèrent sensiblement de celles précédemment obtenues en termes des densités asymptotiques.

ABSTRACT. In the paper sufficient conditions for the $(R)$-density of a set of positive integers in terms of logarithmic densities are given. They differ substantially from those derived previously in terms of asymptotic densities.

1. Preliminaries

Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and positive real numbers, respectively. For $A \subseteq \mathbb{N}$ and $x \in \mathbb{R}^+$ let $A(x) = \{a \in A; a \leq x\}$. Denote by $R(A) = \{\frac{a}{b}; a \in A, b \in A\}$ the ratio set of $A$ and say that a set $A$ is $(R)$-dense if $R(A)$ is (topologically) dense in the set $\mathbb{R}^+$ (see [3]). Let us notice that the $(R)$-density of a set $A$ is equivalent to the density of $R(A)$ in the set $(1, \infty)$.

Define

$$d(A) = \liminf_{x \to \infty} \frac{\#A(x)}{x}, \quad \overline{d}(A) = \limsup_{x \to \infty} \frac{\#A(x)}{x}, \quad d(A) = \lim_{x \to \infty} \frac{\#A(x)}{x}.$$ 

the lower asymptotic density, upper asymptotic density, and asymptotic density (if defined), respectively.

Similarly, define

$$\delta(A) = \liminf_{x \to \infty} \frac{\sum_{a \in A(x)} \frac{1}{a}}{\ln x}, \quad \overline{\delta}(A) = \limsup_{x \to \infty} \frac{\sum_{a \in A(x)} \frac{1}{a}}{\ln x}, \quad \delta(A) = \lim_{x \to \infty} \frac{\sum_{a \in A(x)} \frac{1}{a}}{\ln x},$$

the lower logarithmic density, upper logarithmic density, and logarithmic density (if defined), respectively.

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The following relations between asymptotic density and \((R)\)-density are known:

(S1) If \(d(A) > 2\) then \(A\) is \((R)\)-dense (see [3], [4]).

(S2) If \(d(A) \geq \frac{1}{2}\) then \(A\) is \((R)\)-dense and for all \(b \in (0, \frac{1}{2})\) there is a set \(B\) such that \(d(B) = b\) and \(B\) is not \((R)\)-dense (see [2], [5]).

(S3) If \(d(A) = 1\) then \(A\) is \((R)\)-dense and for all \(b \in (0, 1)\) there is a set \(B\) such that \(d(B) = b\) and \(B\) is not \((R)\)-dense (see [3], [4]).

Notice that the results (S1), (S2) and (S3) can be formulated in a common way as results about maximal sets (with respect to the corresponding density) which are not \((R)\)-dense as follows. Denote by \(\mathcal{D} = \{A \subset \mathbb{N} \mid A\) is not \((R)\) - dense\}. Then we have:

(S1) \(\{d(A) \mid A \in \mathcal{D}\} = \{0\}\).

(S2) \(\{\bar{d}(A) \mid A \in \mathcal{D}\} = \{0, \frac{1}{2}\}\).

(S3) \(\{\bar{d}(A) \mid A \in \mathcal{D}\} = (0, 1)\).

The aim of this paper is to prove corresponding relations for logarithmic density. It appears (see Theorem 2 and Corollary 1) that they differ substantially from the above ones for asymptotic density.

2. Logarithmic density and \((R)\)-density

First, let us introduce a useful technique for calculation densities. It can be easily seen that in practical calculation of densities of a set \(A\), the following method can be used.

Write the set \(A\) as

\[ A = \bigcup_{n=1}^{\infty} (p_n + 1, q_n) \cap \mathbb{N}, \]

where \(0 \leq p_1 < q_1 \leq p_2 < q_2 \leq \cdots\) are integers. Then

\[ d(A) = \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} (q_i - p_i)}{p_{n+1}}, \quad \bar{d}(A) = \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} (q_i - p_i)}{q_n}, \]

and

\[ \delta(A) = \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} \ln \frac{q_i}{p_i}}{\ln p_{n+1}}, \quad \bar{\delta}(A) = \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} \ln \frac{q_i}{p_i}}{\ln q_n}. \]

In practice the bounds \(p_n, q_n\) of intervals determining the set \(A\) are often real numbers instead of integers. Then it may be convenient to use the following lemma. In fact, we will use it in later calculations.

**Lemma 1.** Let \(0 \leq p_1 < q_1 \leq p_2 < q_2 \leq \cdots\) be given real numbers such that \(\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty\) and let \(|a_n| \leq d, \ |b_n| \leq d\) for each \(n \in \mathbb{N}\) and some
Proof. For all $n \in \mathbb{N}$ let $a_n, b_n$ be such that

1. $|a_n| \leq d$, $|b_n| \leq d$,
2. both $p_n + a_n$ and $q_n + b_n$ are integers,
3. $A = \bigcup_{n=1}^{\infty} (p_n + a_n + 1, q_n + b_n) \cap \mathbb{N}$.

First, let us notice that it is known that if for an increasing sequence of positive integers $\{p_n\}$ the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges then $\lim_{n \to \infty} \frac{n}{p_n} = 0$ ([1], 80. Theorem, p.124) and trivially also $\lim_{n \to \infty} \frac{n}{p_n + r} = 0$ for any fixed $r \in \mathbb{R}$.

Now a simple analysis shows that for each $n \in \mathbb{N}$

$$
\frac{\sum_{i=1}^{n} (q_i - p_i - 2d)}{p_{n+1} + d} \leq \frac{\sum_{i=1}^{n} ((q_i + b_i) - (p_i + a_i))}{p_{n+1} + a_{n+1}} \leq \frac{\sum_{i=1}^{n} (q_i - p_i + 2d)}{p_{n+1} - d}
$$

and, using Lemma 1,

$$
\liminf_{n \to \infty} \frac{\sum_{i=1}^{n} (q_i - p_i - 2d)}{p_{n+1} + d} \leq \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} ((q_i + b_i) - (p_i + a_i))}{p_{n+1} + a_{n+1}} = \delta(A)
$$

$$
\leq \liminf_{n \to \infty} \frac{\sum_{i=1}^{n} (q_i - p_i + 2d)}{p_{n+1} - d}
$$

or, rewritten,

$$
\liminf_{n \to \infty} \left( \frac{\sum_{i=1}^{n} (q_i - p_i)}{p_{n+1} + d} - \frac{2dn}{p_{n+1} + d} \right) \leq \delta(A)
$$

$$
\leq \liminf_{n \to \infty} \left( \frac{\sum_{i=1}^{n} (q_i - p_i)}{p_{n+1} - d} + \frac{2dn}{p_{n+1} - d} \right).
$$
As both \( \lim_{n \to \infty} \frac{n}{p_{n+1} + d} \) and \( \lim_{n \to \infty} \frac{n}{p_{n+1} - d} \) equal 0 an application of the Sandwich Theorem completes the proof for \( d(A) \). In a very similar way one can prove the corresponding statement for \( \overline{d}(A) \).

Let \( s \) be the first positive integer \( i \) such that \( p_i - d > 0 \). The following inequalities hold for every \( i = s, s + 1, \ldots \).

\[
(I) \quad \ln \frac{q_i - d}{p_i + d} = \ln \frac{q_i}{p_i} - \int_{p_i}^{p_i + d} \frac{1}{x} \, dx - \int_{q_i - d}^{q_i} \frac{1}{x} \, dx \geq \ln \frac{q_i}{p_i} - \frac{2d}{p_i - d}
\]

and

\[
(II) \quad \ln \frac{q_i + d}{p_i - d} = \ln \frac{q_i}{p_i} + \int_{p_i}^{p_i - d} \frac{1}{x} \, dx + \int_{q_i}^{q_i + d} \frac{1}{x} \, dx \leq \ln \frac{q_i}{p_i} - \frac{2d}{p_i - d}
\]

Again, a simple analysis shows that

\[
\liminf_{n \to \infty} \frac{\sum_{i=s}^{n} \ln \frac{q_i - d}{p_i + d}}{\ln (p_{n+1} + d)} \leq \liminf_{n \to \infty} \frac{\sum_{i=s}^{n} \ln \frac{q_i + b_i}{p_i + a_i}}{\ln (p_{n+1} + a_{n+1})} = \delta(A)
\]

and,

\[
\liminf_{n \to \infty} \left( \frac{\sum_{i=s}^{n} \ln \frac{q_i}{p_i}}{\ln (p_{n+1} + d)} - 2d \frac{\sum_{i=s}^{n} \frac{1}{p_i - d}}{\ln (p_{n+1} + d)} \right) \leq \delta(A)
\]

As the series \( \sum_{i=1}^{n} \frac{1}{p_i - d} \) is convergent and \( \lim_{n \to \infty} \frac{\ln (p_{n+1} + d)}{\ln (p_{n+1} - d)} = 1 \) an application of the Sandwich Theorem completes the proof for \( \delta(A) \). In a very similar way one can prove the corresponding statement for \( \overline{\delta}(A) \). \( \square \)

The following simple lemma will be used in later calculations.

**Lemma 2.** Let \((a_i)_{1 \leq i \leq n}\) and \((b_i)_{1 \leq i \leq n}\) such that \(1 \leq a_1, a_i < b_i, i = 1, 2, \ldots, n, b_i \leq a_{i+1}, i = 1, 2, \ldots, n - 1, \) and \(b_n \leq w\). Then \( \ln w \geq \sum_{i=1}^{n} \ln \frac{b_i}{a_i} \).
Proof. The statement of the Lemma is a straightforward consequence of the following relations

\[ w \geq b_n = \frac{b_n}{a_n} \frac{a_n}{b_{n-1}} \frac{b_{n-1}}{a_{n-1}} \cdots \frac{b_2}{a_2} \frac{a_2}{b_1} \frac{b_1}{a_1} = a_1 \left( \prod_{i=1}^{n} \frac{b_i}{a_i} \right) \left( \prod_{i=1}^{n-1} \frac{a_{i+1}}{b_i} \right) \geq \prod_{i=1}^{n} \frac{b_i}{a_i} . \]

The following class of sets plays an important role in our consideration.

\[ \mathbb{A} = \left\{ A(a, b) = \bigcup_{n=s}^{\infty} (a^n b^n + 1, a^{n+1} b^n) \cap \mathbb{N} \mid 1 < a < b \right\} \]

where \( s = \min\{n \in \mathbb{N} \mid a^n b^n + 1 \leq a^{n+1} b^n \} \).

**Theorem 1.** Let \( 1 < a < b \) and \( A = A(a, b) \in \mathbb{A} \). Then

(i) \( R(A) \cap (a, b) = \emptyset \),

(ii) \( \bar{d}(A) = \frac{a - 1}{ab - 1} \), \( \overline{d}(A) = \frac{b(a - 1)}{ab - 1} \),

(iii) \[ \delta(A) = \frac{\ln a}{\ln a + \ln b} . \]

**Proof.** (i) Let \( x \in A \), \( y \in A \) and \( x < y \). First, let there be a \( n \in \mathbb{N} \) such that both \( x \) and \( y \) belong to the block \( (a^n b^n + 1, a^{n+1} b^n) \). Then

\[ \frac{y}{x} \leq \frac{a^{n+1} b^n}{a^n b^n + 1} < \frac{a^{n+1} b^n}{a^n b^n} = a. \]

On the other hand, let \( x \in (a^n b^n + 1, a^{n+1} b^n) \) and \( y \in (a^n b^n + 1, a^{n+1} b^n) \) for \( m > n \). Then

\[ \frac{y}{x} \geq \frac{a^m b^m + 1}{a^{n+1} b^n} > \frac{a^m b^m}{a^{n+1} b^n} \geq b. \]

In both cases \( \frac{y}{x} \) does not belong to \( (a, b) \).

(ii) Calculate, using Lemma 1,

\[ d(A) = \liminf_{n \to \infty} \frac{\sum_{i=s}^{n} (a^{i+1} b^i - a^i b^i)}{a^{n+1} b^{n+1}} = (a - 1)a^s b^s \liminf_{n \to \infty} \frac{\sum_{i=0}^{n-s} a^i b^i}{a^{n+1} b^{n+1}} \]

\[ = \frac{a - 1}{ab - 1} \lim_{n \to \infty} \frac{a^{n+1} b^{n+1} - a^s b^s}{a^{n+1} b^{n+1}} = \frac{a - 1}{ab - 1}. \]

The corresponding value of \( \overline{d}(A) \) can be calculated in a very similar way.
(iii) Again, using Lemma 1, we have

\[ \delta(A) = \liminf_{n \to \infty} \frac{\sum_{i=s}^{n} \ln \frac{a^{i+1}b^i}{a^ib^i}}{\ln(a^{n+1}b^{n+1})} = \lim_{n \to \infty} \frac{(n-s+1) \ln a}{(n+1)(\ln a + \ln b)} = \frac{\ln a}{\ln a + \ln b}. \]

Remark 1. A simple analysis of equalities (ii) in Theorem 1 in comparison to the results (S1), (S2), (S3) shows

\begin{itemize}
  \item[(A1)] \{d(A); A \in \mathbb{D}\} = \{0\} = \{d(A); A \in \mathbb{A}\}.
  \item[(A2)] \{d(A); A \in \mathbb{D}\} = (0, \frac{1}{2}) = \{d(A); A \in \mathbb{A}\}.
  \item[(A3)] \{\overline{d}(A); A \in \mathbb{D}\} = (0, 1) = \{\overline{d}(A); A \in \mathbb{A}\}.
\end{itemize}

A similar analysis of equality (iii) in Theorem 1 leads to the following.

Conjecture. The following equalities hold

\[ \{\delta(A); A \in \mathbb{D}\} = \{\delta(A); A \in \mathbb{A}\} = \{\overline{\delta}(A); A \in \mathbb{D}\} = \{\overline{\delta}(A); A \in \mathbb{A}\} = (0, \frac{1}{2}). \]

The purpose of the rest of this paper is to prove this conjecture. All the corresponding results will be corollaries to the following.

Theorem 2. Let \(1 < a < b\) and \(A = A(a, b) \in \mathbb{A}\). Then the set \(A\) is maximal element in the set \(\{X \subset \mathbb{N}; R(X) \cap (a, b) = \emptyset\}\) with respect to the partial order induced by any of \(\delta, \delta, \delta\).

Proof. Let \(X \subset \mathbb{N}\) be an infinite set such that \(R(X) \cap (a, b) = \emptyset\). Then \(X\) can be written in the form

\[ X = \bigcup_{n=1}^{\infty} (p_n + 1, q_n) \cap \mathbb{N}; \quad 0 \leq p_1 < q_1 < p_2 < q_2 < \cdots \text{ are integers.} \]

For the proof it is sufficient to show (taking into account Theorem 1 (iii))

\[ \overline{\delta}(X) \leq \overline{\delta}(A(a, b)) = \delta(A(a, b)) = \frac{\ln a}{\ln a + \ln b}. \]

Thus we can also suppose

\[ (0) \quad \overline{\delta}(X) > 0. \]

The proof will be carried in several steps.

Step 1. In this step we will prove

\[ (1) \quad \text{if } p_n \geq \frac{1}{b-a} \text{ then } q_n < a(p_n + 1). \]
Proof of (1). Suppose $q_n \geq a(p_n + 1)$. Then $q_n \geq a(p_{n+1} + 1)$ and also $q_{n+1} \leq a$ and, as $R(X) \cap (a, b) = \emptyset$, there exists $m \in (p_n + 1, q_n) \cap \mathbb{N}$ such that $\frac{m}{p_n + 1} < a$ and $\frac{m+1}{p_n + 1} > b$. Consequently $\frac{m+1}{p_n + 1} - \frac{m}{p_n + 1} > b - a$ which implies $p_n < \frac{1}{b-a}$, a contradiction.

Step 2. In this step we will prove

\[(2) \quad \text{if } p_n \geq \frac{ab}{b-a} \text{ then } X \cap (a(p_n + 1), bq_n) = \emptyset.\]

Proof of (2). Let $p_n \geq \frac{ab}{b-a}$. Then also $p_n \geq \frac{1}{b-a}$ and, by the previous step, $q_n < a(p_n + 1)$. Suppose on the contrary that there exists $x \in X \cap (a(p_n + 1), bq_n)$. Then $\frac{x}{p_{n+1}} \geq \frac{a(p_{n+1})}{p_{n+1}} = a$, $\frac{x}{q_n} \leq \frac{bq_n}{q_n} = b$ and, as $R(X) \cap (a, b) = \emptyset$, there exists $m \in (p_n + 1, q_n) \cap \mathbb{N}$ such that $\frac{x}{m} > b$ and $\frac{x}{m+1} < a$. Consequently

\[
b - a < \frac{x}{m} - \frac{x}{m+1} = \frac{x}{m(m+1)} \quad \frac{x}{m^2} < \frac{bq_n}{(p_n + 1)^2} \leq \frac{ba(p_n + 1)}{(p_n + 1)^2} = \frac{ab}{p_{n+1}}\]

which implies $p_n < \frac{ab}{b-a}$, a contradiction.

Step 3. In this step we will introduce some useful notation. Denote by $k_0$ the smallest integer $k$ such that $p_k \geq \frac{ab}{b-a}$ and let $K_0 = \{k_0, k_0 + 1, k_0 + 2, \ldots\}$. From (2) we have for every $k \in K_0$

\[(3) \quad (a(p_k + 1), bq_k) \cap \mathbb{N} \subset \mathbb{N} - X\]

and so we can define a function $\varphi : K_0 \rightarrow K_0$ by

\[(4) \quad (a(p_k + 1), bq_k) \subset (a(p_{\varphi(k)}), b(p_{\varphi(k)} + 1)).\]

The range of this function $\varphi(K_0) = \{l_1 < l_2 < \ldots\}$ is infinite, denote $K_n = \varphi^{-1}(l_n)$ for each $n \in \mathbb{N}$. Evidently $K_0 = \bigcup_{n=1}^{\infty} K_n$ and for every $m < n, x \in K_m$ and $y \in K_n$ it is $x < y$. Let us call a big gap in $X$ any interval of the form $(a(p_k), b(p_{k+1}) + 1)$ where $k \in K_n$ for $n \in \mathbb{N}$. Finally, let us introduce two sequences \{$u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$ by

\[u_n = p_{\min} K_n, \quad v_n = q_{\max} K_n \quad \text{for} \quad n = 1, 2, 3, \ldots .\]

The above definitions imply that both $a(u_n + 1)$ and $bv_n$ belong to the same big gap in $X$ and, consequently,

\[(5) \quad v_n < a(u_n + 1) \quad \text{for} \quad n = 1, 2, 3, \ldots .\]

Step 4. In this step we will present and prove (if necessary) some simple relations and statement which will be used in the final calculation.
An easy analysis proves

\[
\sum_{a=p+1}^{q} \frac{1}{a} \leq \ln \frac{q}{p} \quad \text{for all positive integers } p < q.
\]

From Lemma 1 and (0) it can be seen that

\[
\text{(7) the series } \sum_{i=1}^{\infty} \ln \frac{q_i}{p_i} \text{ is divergent.}
\]

As \( a(u_{i+1} + 1) \) belongs to the big gap next to the big gap in which \( b(u_i + 1) \) lies we have \( u_{i+1} + 1 \geq \frac{b}{a}(u_i + 1) \) for every \( i \in \mathbb{N} \). Therefore the series \( \sum_{i=1}^{\infty} \frac{1}{u_i} \) is convergent and, consequently,

\[
\text{(8) } \sum_{i=1}^{\infty} \ln \frac{u_i + 1}{u_i} < \infty, \quad \lim_{i \to \infty} \ln (1 + \frac{1}{u_i}) = 0.
\]

**Step 5.** For the rest of proof suppose that \( m \) is a sufficiently large fixed positive integer and denote by \( n \) the greatest (fixed from this moment) positive integer \( k \) for which \( a(u_k + 1) \leq m \). Thus, using (5), we have

\[
\text{(9) } b v_n \leq m < b v_{n+1} < a(b(u_{n+1} + 1)).
\]

The considerations in Step 3 imply that the intervals \( (p_i + 1, q_i) \) for \( i = 1, 2, \ldots, \max K_n \) and \( (a(u_j + 1), \max b v_j) \) for \( j = 1, 2, \ldots, n \) are mutually disjoint and, by definition of numbers \( m \) and \( n \), they are all contained in the interval \( (1, m) \). Thus, by Lemma 2, we have

\[
\text{(10) } \ln m \geq \sum_{i=1}^{\max K_n} \ln \frac{q_i}{p_i} + \sum_{j=1}^{n} \ln \frac{b v_j}{a u_j + 1}.
\]

For similar reason we have

\[
\text{(11) } \sum_{j=1}^{\max K_n} \ln \frac{q_j}{p_j} \leq \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j} + \sum_{i=1}^{n} \ln \frac{v_i}{u_i}.
\]

The last inequality together with (5) imply

\[
\text{(12) } \sum_{j=1}^{\max K_n} \ln \frac{q_j}{p_j} \leq \sum_{j=1}^{k_0-1} \ln \frac{q_j}{p_j} + n \ln a + \sum_{i=1}^{n} \ln \left( 1 + \frac{1}{u_i} \right).
\]
Step 6. Denote \( c = \sum_{j=1}^{k_n-1} \frac{\ln q_j}{p_j} \). Now we are able to estimate

\[
s(m) = \frac{\sum_{a \in X, a \leq m} \frac{1}{a}}{\ln m} = \frac{\sum_{a \in X, a \leq v_n} \frac{1}{a} + \sum_{a \in X, u_{n+1} < a \leq m} \frac{1}{a}}{\ln m} \leq \frac{\max K_n \sum_{i=1}^{\ln m} \frac{q_i}{p_i} + \ln \frac{u_{n+1}+1}{u_{n+1}}}{\ln m} \]

by (6).

\[
\leq \frac{\sum_{i=1}^{\ln m} \ln \frac{q_i}{p_i} + n \ln \frac{b}{a} + \sum_{i=1}^{n} \ln \frac{v_i}{u_i} + \sum_{i=1}^{n} \ln \frac{u_i}{u_{i+1}}}{\frac{1 + \frac{\ln \frac{ab+1}{u_{n+1}}}{\max K_n \ln \frac{q_i}{p_i}}}{\ln m}}
\]

by (9), (10).

\[
= \frac{1 + \frac{\sum_{i=1}^{n} \ln \frac{b}{a}}{\max K_n \ln \frac{q_i}{p_i}} + \sum_{i=1}^{\frac{n}{u_i}} \ln \frac{v_i}{u_i} + \sum_{i=1}^{\frac{n}{u_i+1}} \ln \frac{u_i}{u_{i+1}}}{\frac{1 + \frac{\ln \frac{ab+1}{u_{n+1}}}{\max K_n \ln \frac{q_i}{p_i}}}{\ln m}}
\]

by (11), (12).

\[
= S(m).
\]

Step 7. In this step we will complete the proof by limit process. Let \( m \to \infty \) (and consequently \( n \to \infty \)). Then, using (7) and (8), we have

\[
\bar{\delta}(X) = \limsup_{m \to \infty} s(m) \leq \limsup_{m \to \infty} S(m) = \lim_{m \to \infty} \frac{1 + 0}{1 + \frac{\ln b - \ln a}{\ln a}} = \frac{\ln a}{\ln a + \ln b}.
\]

The following corollary is a direct consequence of the previous theorem and it shows that the relations between \((R)\)-density and logarithmic densities are completely different from those between \((R)\)-density and asymptotic densities.
Corollary. The following relations hold

\[ \{ \delta(A); \ A \in \mathbb{D} \} = \{ \bar{\delta}(A); \ A \in \mathbb{D} \} = \{ \delta(A); \ A \in \mathbb{D} \} = \left( 0, \frac{1}{2} \right). \]

References


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