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A representation theorem for a class of rigid analytic functions

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1. Introduction

Let $p$ be a prime number, $\mathbb{Q}_p$ the field of $p$-adic numbers, $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ with respect to the $p$-adic absolute value. Let $t \in \mathbb{C}_p$ and set $E(t) = \mathbb{P}^1(\mathbb{C}_p) \setminus C(t, \varepsilon)$ which are equivariant with respect to the Galois group $G = \text{Gal}_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$, where $t$ is a Lipschitzian element of $\mathbb{C}_p$ and $C(t, \varepsilon)$ denotes the $\varepsilon$-neighborhood of the $G$-orbit of $t$. These functions are easily described in case $t$ is algebraic over $\mathbb{Q}_p$. For instance, if $t \in \mathbb{Q}_p$ then one can use the equivariant transformation $z \mapsto \frac{1}{z-t}$ to send $t$ to the point at infinity. Then the equivariant rigid analytic functions on $E(t)$ will correspond to the entire functions which are equivariant.
and these are simply power series \( f(z) = \sum_{n \geq 0} a_n z^n \) with \( a_n \in \mathbb{Q}_p \) for any \( n \) and such that \( \lim_{n \to \infty} |a_n|_p^{1/n} = 0 \).

If \( t \) is transcendental over \( \mathbb{Q}_p \) it is not obvious that there are any nonconstant equivariant rigid analytic functions on \( E(t) \). For certain elements \( t \) (called Lipschitzian) such a function \( z \mapsto F(t, z) \) is constructed in \([\text{APZ2}]\). In this paper we define for any Lipschitzian element \( t \) of \( \mathbb{C}_p \) and any natural numbers \( m, n \) an equivariant rigid analytic function \( F_{m,n}(t, z) \) on \( E(t) \), which is related to our basic trace series \( F(t, z) \). Then in Theorem 4.2 below we express any equivariant rigid analytic function on an affinoid \( E(t, \varepsilon) \) in terms of the above functions \( F_{m,n}(t, z) \).

2. Background material

2.1. Let \( p \) be a prime number and \( \mathbb{Q}_p \) the field of \( p \)-adic numbers endowed with the \( p \)-adic absolute value \( | \cdot | \), normalized such that \( |p| = 1/p \). Let \( \mathbb{Q}_p \) be a fixed algebraic closure of \( \mathbb{Q}_p \) and denote by the same symbol \( | \cdot | \) the unique extension of \( | \cdot | \) to \( \overline{\mathbb{Q}_p} \). Further, denote by \( (\mathbb{C}_p, | \cdot |) \) the completion of \( (\mathbb{Q}_p, | \cdot |) \) (see \([\text{Am}], [\text{Ar}]\)). Let \( G = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \) endowed with the Krull topology. The group \( G \) is canonically isomorphic with the group \( \text{Gal}_{\text{cont}}(\mathbb{C}_p/\mathbb{Q}_p) \) of all continuous automorphisms of \( \mathbb{C}_p \) over \( \mathbb{Q}_p \). For any \( x \in \mathbb{C}_p \) denote \( C(x) = \{ \sigma(x) | \sigma \in G \} \) the orbit of \( x \) and let \( \mathbb{Q}_p[x] \) be the closure of the ring \( \mathbb{Q}_p[x] \) in \( \mathbb{C}_p \). For any \( x \in \overline{\mathbb{Q}_p} \) denote \( \deg(x) = [\mathbb{Q}_p(x) : \mathbb{Q}_p] \).

2.2. Let \( x \in \mathbb{C}_p \). Given a real number \( \varepsilon > 0 \) let \( B(x, \varepsilon) = \{ y \in \mathbb{C}_p, |x - y| < \varepsilon \} \) the open ball of radius \( \varepsilon \) centered at \( x \). If \( M \) is a compact subset of \( \mathbb{C}_p \) and \( \varepsilon > 0 \) is a real number, denote by \( N(M, \varepsilon) \) the number of all disjoint balls of radius \( \varepsilon \) which have a non-empty intersection with \( M \). We say that \( M \) is Lipschitzian if \( \lim_{\varepsilon \to 0} \frac{N(M, \varepsilon)}{N(M, 2\varepsilon)} = 0 \). We call an element \( x \in \mathbb{C}_p \) Lipschitzian if \( C(x) \) is Lipschitzian. According to \([\text{APZ2}]\) if \( x \) is Lipschitzian then one can integrate Lipschitzian functions (see definition in 2.3 below) with respect to the \( p \)-adic Haar measure \( \pi_t \) induced by \( G \) on the set \( C(x) \).

Let \( G_x = \{ \sigma \in G : \sigma(x) = x \} \) and \( P \) a closed subgroup of \( G \) which contains \( G_x \). Then \( C_P(x) = \{ \sigma(x) : \sigma \in P \} \), the orbit of \( x \) with respect to \( P \), is a compact subset of \( C(x) = C_G(x) \). If \( x \) is Lipschitzian then \( C_P(x) \) is a Lipschitzian compact set for any \( P \) with \( G_x \subset P \). This follows from the fact that for any \( \varepsilon > 0 \), \( N(C_P(x), \varepsilon) \) divides \( N(C(x), \varepsilon) \).

Let \( x \in \mathbb{C}_p \) and \( P \) a closed subgroup of \( G \) which contains \( G_x \). For any \( \varepsilon > 0 \) let \( H_P(x, \varepsilon) = \{ \sigma \in P : |x - \sigma(x)| < \varepsilon \} \) and \( N_P(x, \varepsilon) = N(C_P(x), \varepsilon) \). Then \( H_P(x, \varepsilon) \) is an open subgroup of \( P \) and \( N_P(x, \varepsilon) = [P : H_P(x, \varepsilon)] \). In particular \( N(x, \varepsilon) = [G : H(x, \varepsilon)] \), where \( H(x, \varepsilon) = H_G(x, \varepsilon) \).
2.3. The notion of rigid analytic function is defined in [FP] (see also [Am]). According to [APZ2], a rigid analytic function defined on a subset \(D\) of \(C_p\) is said to be equivariant if for any \(z \in D\) one has \(C(z) \subset D\) and \(f(\sigma(z)) = \sigma(f(z))\) for all \(\sigma \in G\). A function \(f : C(t) \to C_p, t \in C_p\) is Lipschitzian if there exists a real number \(c > 0\) such that \(|f(x) - f(y)| \leq c|x - y|\) for all \(x, y \in C(t)\).

Let \(t\) be a Lipschitzian element of \(C_p\) and \(f : C(t) \to C_p\) a Lipschitzian function. Then the integral

\[
\int_{C(t)} f(x) d\pi_t(x)
\]

is well defined (see [APZ2]). In particular for any polynomial \(P(X) \in C_p[X]\), any \(z \in C_p \cup \{\infty\} \setminus C(t)\) and any natural number \(n\) the function \(z \mapsto f(x, z) = \frac{P(x)}{(z-x)^n}\) is Lipschitzian on \(C(t)\) and we consider the integral \(\int_{C(t)} f(x, z) d\pi_t(x)\). Let us denote

\[
F_{m,n}(t, z) = \int_{C(t)} \frac{x^m}{(z-x)^n} d\pi_t(x), \quad m \geq 0, n \geq 0.
\]

According to [APZ2] for any \(m \geq 0\) one has

\[
\int_{C(t)} x^m d\pi_t(x) = Tr(t^m).
\]

This shows that \(F_{m,0}(t, z) = Tr(t^m) \in Q_p, F_{0,0}(t, z) = 1\) and \(1 + F_{1,1}(t, \frac{1}{x}) = F(t, z)\), the trace function associated to \(t\). Also by the equality

\[
\frac{1}{(1-u)^m} = (1 + u + u^2 + \ldots + u^n + \ldots)^m = \sum_{s=0}^{\infty} \binom{m+s-1}{s} u^s
\]

valid for any positive integer \(m\) and any \(u\) with \(|u| < 1\), it follows that for \(|z| > |x|\) one has

\[
\frac{x^m}{(z-x)^n} = \sum_{s \geq 0} \binom{n+s-1}{s} \frac{x^{m+s}}{z^{n+s}}.
\]

Then one may write:

\[
F_{m,n}(t, z) = \sum_{s \geq 0} \binom{m+s-1}{s} \frac{Tr(t^{m+s})}{z^{n+s}}.
\]

This formula represents the expansion of \(F_{m,n}(t, z)\) in a suitable neighborhood of infinity. As in Theorem 6.1 of [APZ2] one shows that for all \(m \geq 0, n \geq 0\), \(F_{m,n}(t, z)\) is an equivariant rigid analytic function defined on \(C_p \cup \{\infty\} \setminus C(t)\).
Remark 2.1. $F_m^{n}(t, z) = -nF_{m,n+1}(t, z)$ for any $m > 0, n > 1$. As a consequence one has $F_{m,n+1}(t, z) = \frac{(-1)^n}{n!} F_{m,1}^{(n)}(t, z)$, where the derivative is taken with respect to $z$.

2.4. The above considerations can be generalized as follows: Let $\epsilon > 0$ be a real number and $S$ a system of right representatives of $G$ with respect to the subgroup $H(t, \epsilon)$. Assume that the identity element $e$ of $G$ belongs to $S$ and that $t$ is Lipschitzian. For any $\sigma \in S$ the subset $C_\sigma(t, \epsilon) = \{ \tau(t) : \tau \in \sigma H(t, \epsilon) \}$ is a compact subset of $C(t)$. For $m \geq 0, n \geq 0$ denote

$$F_{m,n}^\sigma(t, z) = \int_{C_\sigma(t, \epsilon)} \frac{x^m}{(z - x)^n} d\pi_t(x).$$

It is clear that

$$F_{m,n}(t, z) = \sum_{\sigma \in S} F_{m,n}^\sigma(t, z).$$

In fact this formula represents the Mittag-Leffler decomposition of $F_{m,n}(t, z)$ viewed as a rigid analytic function in the connected affinoid $\mathbb{C}_p \cup \{ \infty \} \setminus \cup_{\sigma \in S} B(\sigma(t), \epsilon) = E(t, \epsilon)$. In what follows we try to obtain a similar decomposition for any element of the set $A(E(t, \epsilon))$ of equivariant rigid analytic functions on $E(t, \epsilon)$.

3. A combinatorial Lemma

Let $\alpha, x, y, \{ a_m \}_{m \geq 1}$ be variables. For any $m \geq 1$, let us denote

$$h_m(\alpha) = a_1 \alpha^{m-1} + a_2 \binom{m-1}{1} \alpha^{m-2} + \ldots + \binom{m-1}{m-1} a_m$$

where as usually $\binom{m}{k} = \frac{m(m-1) \ldots (m-k+1)}{k!}$. For any integer $m \geq 1$ we set

$$A_m(x) = h_m(\alpha) - \frac{1}{1!} h_m^{(1)}(\alpha) x + \ldots + \frac{(-1)^{m-1}}{(m-1)!} h_m^{(m-1)}(\alpha) x^{m-1}$$

where $h_m^{(k)}(\alpha)$ denotes the formal $k$-th derivative of the polynomial $h_m(\alpha)$ with respect to $\alpha$.

Lemma 3.1. For any $x, y$ and any $m \geq 1$ one has:

$$A_m(x) = \sum_{r=1}^{m} a_r \binom{m-1}{r-1} (\alpha - x)^{m-r}$$

and

$$A_m(y) = \sum_{r=1}^{m} \binom{m-1}{r-1} A_r(x)(x - y)^{m-r}.$$
Proof. Equality (3) states that $A_m(x) = h_m(\alpha - x)$, which follows directly from the Taylor expansion (2). As for the second equality, by applying (3) and using the identity
\[
\binom{m - 1}{r - 1} \binom{r - 1}{j - 1} = \binom{m - 1}{j - 1} \binom{m - j}{r - j}
\]
one contains
\[
\sum_{r=1}^{m} \binom{m - 1}{r - 1} A_r(x)(x - y)^{m-r}
\]
\[
= \sum_{r=1}^{m} \binom{m - 1}{r - 1} (x - y)^{m-r} \sum_{j=1}^{r} a_j \binom{r - 1}{j - 1} (\alpha - x)^{r-j}
\]
\[
= \sum_{j=1}^{m} a_j \binom{m - 1}{j - 1} \sum_{r=j}^{m} \binom{m - j}{r - j} (x - y)^{m-r}(\alpha - x)^{r-j}
\]
\[
= \sum_{j=1}^{m} a_j \binom{m - 1}{j - 1} (\alpha - y)^{m-j}.
\]
This equals $A_m(y)$ by (3) and so the lemma is proved. \qed

4. Equivariant rigid analytic functions on $E(t, \varepsilon)$

4.1. Let $t$ be an element of $\mathbb{C}_p$, let $\varepsilon > 0$ be a real number and denote by $B(C(t), \varepsilon)$ the union of all disjoint open balls $B(x, \varepsilon)$ which have a nonempty intersection with $C(t)$. Choose $\alpha \in \bar{\mathbb{Q}}_p$ such that $|t - \alpha| < \varepsilon$. Then one has $H(t, \varepsilon) = H(\alpha, \varepsilon)$. Let $S$ be a system of right representatives of $G$ with respect to $H(t, \varepsilon)$ and assume $e \in S$. One has $B(C(t), \varepsilon) = \bigcup_{\sigma \in S} B(\sigma(\alpha), \varepsilon)$. Consider the affinoid $E(t, \varepsilon) = \mathbb{C}_p \cup \{\infty\} \setminus B(C(t), \varepsilon)$ and let $A(E(t, \varepsilon))$ be the set of equivariant rigid analytic functions on $E(t, \varepsilon)$. If $t$ is Lipschitzian then the functions $F_{m,n}(t, z)$ defined in Section 2 are elements of $A(E(t, \varepsilon))$. In this section we shall prove that all the elements of $A(E(t, \varepsilon))$ can be expressed in terms of the functions $F_{m,n}(t, z)$, $m, n \geq 0$.

4.2. We have the following proposition.

Proposition 4.1. Let $t$ be an element of $\mathbb{C}_p$. Denote $K_t = \mathbb{Q}_p[t] \cap \bar{\mathbb{Q}}_p$ and let $\varepsilon > 0$ and $\alpha \in K_t$ such that $|\alpha - t| < \varepsilon$. There exists a sequence $\{\alpha_n\}_{n \geq 1}$ of elements of $K_t$ and a sequence $\{\varepsilon_n\}_{n \geq 1}$ of positive real numbers such that:

(i) $\varepsilon_1 = \varepsilon$, $\alpha_1 = \alpha$,

(ii) For any $n \geq 1$ one has $\varepsilon_n+1 \leq \inf\{\varepsilon_n/2, |t - \alpha_n|\}$,

(iii) $|t - \alpha_n| < \varepsilon_n$, $n \geq 1$, and deg $\alpha_n$ is smallest with this property.
The proof easily follows by induction on \( n \) since any ball \( B(t, \varepsilon) \) contains elements of \( K_t \) (see [APZ1]).

In what follows we work with sequences \( \{\alpha_n\}_n \) and \( \{\varepsilon_n\}_n \) as in Proposition 4.1. It is clear that \( \lim_{n \to \infty} \varepsilon_n = 0 \), and \( t = \lim_{n \to \infty} \alpha_n \). Note also that the ball \( B(\alpha_{n+1}, \varepsilon_{n+1}) \) is contained in \( B(\alpha_n, \varepsilon_n) \) for all \( n \geq 1 \). Let us consider the subgroup \( H(t, \varepsilon_n) = H(\alpha_n, \varepsilon_n) \) defined in Section 2. Denote \( d_n = |G : H(T, \varepsilon_n)| = N(t, \varepsilon_n) \), and let \( S_n \) be a fixed system of representatives of right cosets of \( G \) with respect to \( H(t, \varepsilon_n) \). We shall assume that the identity element \( e \) of \( G \) belongs to each \( S_n \). We remark that \( S_1 = S \) and \( d_n \) divides \( d_{n+1} \) for all \( n \geq 1 \).

4.3. Let \( f \in A(E(t, \varepsilon)) \). Then (see [FP], Ch I) \( f \) admits a Mittag-Leffler decomposition: \( f(z) = \sum_{\sigma \in S} f_\sigma(z) + f(\infty) \) where \( f(\infty) \) is the value of \( f \) at infinity and

\[
(4) \quad f_\sigma(z) = \sum_{m \geq 1} \frac{a_{\sigma,m}}{(z - \sigma(\alpha))^m}, \quad \lim_{m \to \infty} \frac{|a_{\sigma,m}|}{\varepsilon^m} = 0, \quad \sigma \in S.
\]

Since \( f \) is equivariant then for any \( z \in E(t, \varepsilon) \) and any \( \tau \in G \) one has \( \sum_{\sigma \in S} \tau(f_\sigma(z)) = \sum_{\sigma \in S} f_\sigma(\tau(z)) \) and \( \tau(f(\infty)) = f(\infty) \). Hence \( f(\infty) \in \mathbb{Q}_p \) and for any \( \sigma \in S \) one can write:

\[
(5) \quad f_\sigma(\sigma(z)) = \sigma(f_\sigma(z)), \quad a_{\sigma,m} = \sigma(a_{e,m}), \quad m \geq 1.
\]

Next we remark that for any \( \tau \in H(t, \varepsilon) \) and any \( \sigma \in S \) the element \( \sigma(\tau(\alpha)) \) belongs to \( B(\sigma(\alpha), \varepsilon) \), and so the function \( f_\sigma(z) = \sum_{m \geq 1} \frac{a_{\sigma,m}}{(z - \sigma(\alpha))^m} \) can also be written as

\[
f_\sigma(z) = \sum_{n \geq 1} \frac{a_{\sigma,\tau,m}}{(z - \sigma\tau(\alpha))^m} = f_{\sigma\tau}(z)
\]

where

\[
a_{\sigma,\tau,m} = \sum_{i=1}^{m} \binom{m-1}{m-1} a_{\sigma,i}(\sigma(\alpha) - \sigma\tau(\alpha))^{m-i}.
\]

In what follows we shall assume that \( f(\infty) = 0 \).

4.4. At this point we derive another convenient expression for \( f(z) \), using the above elements \( \alpha_n \). Fix \( n \geq 1 \). Then \( d = d_1 \) divides \( d_n = |G : H(t, \varepsilon_n)| \). Denote \( q_n = d_n/d \) and let \( B(\alpha_n^{(j)}, \varepsilon_n) \), \( 1 \leq j \leq q_n \) be all the balls of radius \( \varepsilon_n \) centered at suitable conjugates of \( \alpha_n \) and such that these balls cover \( C_{H(t, \varepsilon)}(T) = C_{e}(t, \varepsilon) \). Then

\[
(6) \quad f_e(z) = \frac{d}{d_n \sum_{1 \leq j \leq q_n} \sum_{m \geq 1} \frac{A_{e,m}^{(j)}}{(z - \alpha_n^{(j)})^m}}
\]
where

\[ A_{e,m}^{(j)} = \sum_{i=1}^{m} \left( \begin{array}{c} m - 1 \\ i - 1 \end{array} \right) a_{e,i}(\alpha - \alpha_{n}^{(j)})^{m-i}. \]

According to (5) for all \( \sigma \in S \) one has

\[ f_{\sigma}(z) = \frac{d}{d_{n}} \sum_{1 \leq j \leq q_{n}} \sum_{m \geq 1} \frac{A_{\sigma,m}^{(j)}}{(z - \sigma(\alpha_{n}^{(j)}))^{m}}, \]

where \( A_{\sigma,m}^{(j)} = \sigma(A_{e,m}^{(j)}). \) As a consequence of (4) there exists a positive real number \( M \) such that for any \( m \geq 1 \) one has:

\[ |a_{\sigma,m}| \leq M \varepsilon^{m}. \]

It follows from (7) that

\[ |A_{e,m}^{(j)}| \leq M \varepsilon^{m} \]

for any \( n \geq 2 \) and \( 1 \leq j \leq q_{n}. \)

4.5. At this point we assume that \( t \) is a Lipschitzian element of \( C_{p}, \varepsilon > 0 \) and \( \alpha \in B(t, \varepsilon), \alpha \in K_{t}. \) Let \( f \in A(E(t, \varepsilon)), f = \sum_{\sigma \in S} f_{\sigma}(z) \) with \( f_{\sigma}(z) \) given by (4). For any \( m \geq 1 \) denote

\[ h_{m}(\alpha) = a_{e,1}\alpha^{m-1} + \left( \begin{array}{c} m - 1 \\ 1 \end{array} \right) a_{e,2}\alpha^{m-2} + \ldots + \left( \begin{array}{c} m - 1 \\ m - 1 \end{array} \right) a_{e,m}. \]

Also, for \( \sigma \in S \) consider the function \( F_{m,n}^{\sigma}(t, z) \) defined in Section 2.

**Theorem 4.2.** Let \( t \) be a Lipschitzian element of \( C_{p}, \varepsilon > 0, \alpha \in B(t, \varepsilon) \cap K_{t} \) and \( f \in A(E(t, \varepsilon)). \) Then for any \( z \in E(t, \varepsilon) \) one has

\[ f(z) = \sum_{\sigma \in S} \sum_{m \geq 1} \sum_{0 \leq j < m} \frac{(-1)^{j}}{j!} \sigma(h_{m}^{(j)}(\alpha)) F_{j,m}^{\sigma}(t, z). \]

**Proof.** For any \( m \geq 1 \) let \( A_{m}(x) = \sum_{1 \leq i \leq m} a_{e,i} \left( \begin{array}{c} m - 1 \\ i - 1 \end{array} \right) (\alpha - x)^{m-i} \) and

\[ A(x, z) = \sum_{m \geq 1} \frac{A_{m}(x)}{(z - x)^{m}}. \]

Step 1. Fix \( z_{0} \in E(t, \varepsilon). \) We assert that for any \( z \in B(z_{0}, \varepsilon), \) the function \( x \mapsto A(x, z) \) is defined and is Lipschitzian on \( B(t, \varepsilon). \) Firstly we remark that for any \( x \in B(t, \varepsilon) \) one has (see (8)):

\[ \left| \frac{A_{m}(x)}{(z - x)^{m}} \right| \leq \frac{\sum a_{e,i} \left( \begin{array}{c} m - 1 \\ i - 1 \end{array} \right) (\alpha - x)^{m-i}}{\varepsilon^{m}}. \]
and

\[
\varepsilon^{-m} \left| \sum_{i=1}^{m/2} a_{e,i} \left( \frac{m-1}{i-1} \right) (\alpha - x)^{m-i} \right| \leq \max_{1 \leq i \leq [m/2]} \left( M \left( \frac{|\alpha - x|}{\varepsilon} \right)^{m-i} \right) = M \left( \frac{|\alpha - x|}{\varepsilon} \right)^{m-[m/2]}
\]

Since \( \frac{|\alpha - x|}{\varepsilon} < 1 \), by (4) and the above considerations it follows that

\[
\left| \frac{A_m(x)}{(z-x)^m} \right| \to 0 \text{ when } m \to \infty.
\]

Then the function \( A(x, z) \) is defined on \( B(t, \varepsilon) \), as claimed. Now let \( x, y \in B(t, \varepsilon) \). For any \( m \geq 1 \) we have

\[
\frac{A_m(y)}{(z-y)^m} = \frac{A_m(y)}{(z-x)^m} \left( 1 + \sum_{i \geq 1} D_i \left( \frac{y-x}{z-x} \right)^i \right)
\]

where \( D_i \) are suitable natural numbers. Then one can write

\[
\left| \frac{A_m(x)}{(z-x)^m} - \frac{A_m(y)}{(z-y)^m} \right| \leq \max_{i \geq 1} \left( \left| \frac{A_m(x) - A_m(y)}{(z-x)^m} \right|, \left| \frac{A_m(y)(y-x)^i}{(z-x)^{m+i}} \right| \right).
\]

But (see (8)) for any \( i \geq 1 \) and \( z \in B(z_0, \varepsilon) \) one has

\[
\left| \frac{A_m(y)(y-x)^i}{(z-x)^{m+i}} \right| \leq \left| A_m(y) \right| \frac{|y-x|^i}{|z-x|^m} \cdot \frac{|y-x|^i}{|z-x|^i} \leq M \frac{|y-x|}{\varepsilon}.
\]

Also by an easy computation one sees that:

\[
\left| \frac{A_m(x) - A_m(y)}{(z-x)^m} \right| \leq M \frac{|y-x|}{\varepsilon}.
\]

Finally, one has \( |A(x, y) - A(y, z)| \leq \frac{M}{\varepsilon} |x - y| \) i.e. \( A(x, z) \) is Lipschitzian on \( B(t, \varepsilon) \). The above considerations also show that for any \( \delta > 0 \) we have

\[
\left| \frac{A_m(x)}{(z-x)^m} - \frac{A_m(y)}{(z-y)^m} \right| \leq \delta |x - y|
\]

for all \( m \) large enough in terms of \( z \) and \( \delta \), uniformly for \( x, y \in C_e(t, \varepsilon) \).
Step 2. Let us denote \( D = C_e(t, \varepsilon) = B(t, \varepsilon) \cap C(t) \). Then \( D \) is a compact Lipschitzian subset of \( \mathbb{C}_p \) and we consider the integral

\[
F(z) = \int_D A(x, z) d\pi_t(x), \quad z \in B(z_0, \varepsilon).
\]

Here we use the definition of the integral with respect the p-adic measure \( \pi_t \) as in [APZ2]. We assert that

\[
f_e(z) = F(z), \quad z \in B(z_0, \varepsilon),
\]

where \( e \) is the identity element of \( G \).

To see this, consider the sequences \( \{\varepsilon_n\}_n \) and \( \{\alpha_n\}_n \) from Proposition 4.1. Let \( H(t, \varepsilon_n), \ d_n, \ S_n \) be as above. In particular \( \varepsilon_1 = \varepsilon, \ \alpha_1 = \alpha, \ d_1 = d \). For any \( n \geq 1 \) let \( B(\alpha_n^{(i)}, \varepsilon_n), \ 1 \leq i \leq q_n \) be the open balls of radius \( \varepsilon_n \) which cover \( D \). Then one has:

\[
F(z) = \int_D A(x, z) d\pi_t(x) = \lim_n \Phi[A(x, z), \alpha_n^{(i)}, \varepsilon_n]
\]

where

\[
\Phi[A(x, z), \alpha_n^{(i)}, \varepsilon_n] = \frac{d}{d_n} \sum_{1 \leq i \leq q_n} A(\alpha_n^{(i)}, z)
\]

is the Riemann sum associated to \( (A, \alpha_n^{(i)}, \varepsilon_n) \) (see[APZ2]). We have

\[
\frac{d}{d_n} A(\alpha_n^{(i)}, z) = \frac{d}{d_n} \sum_{m \geq 1} A_m(\alpha_n^{(i)})(z - \alpha_n^{(i)})^m.
\]

From (6) it now follows that

\[
\Phi[A, \alpha_n^{(i)}, \varepsilon_n] = f_e(z).
\]

Since this equality is valid for any \( n \) we conclude that

\[
F(z) = \int_D A(x, z) d\pi_t(x) = f_e(z).
\]

Step 3. We now apply formula (2) to obtain another expression for \( A_m(x) \). One has:

\[
A_m(x) = h_m(\alpha) - \frac{1}{1!} h_m^{(1)}(\alpha)x + \ldots + \frac{(-1)^{m-1}}{(m-1)!} h_m^{(m-1)}(\alpha)x^{m-1}
\]

\[
= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) \frac{x^j}{(z-x)^m}.
\]
Therefore

\[
\int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x) = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_j^m(\alpha) \int_D \frac{x^j}{(z-x)^m} d\pi_t(x)
\]

(13)

\[
= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_j^m(\alpha) F_{j,m}^e(t, z).
\]

We claim that

\[
F(x) = f_\alpha(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_j^m(\alpha) F_{j,m}^e(t, z).
\]

(14)

In order to prove this formula we need the following result:

**Lemma 4.3.** Let \( t \) be a Lipschitzian element of \( \mathbb{C}_p, \varepsilon > 0 \) a real number, \( g : B(C(t), \varepsilon) \to \mathbb{C}_p \) a Lipschitzian function, and let \( c \) be a real number such that \( |g(x) - g(y)| \leq c|x - y| \) for all \( x, y \in C(t) \). Then there exists a real number \( k \) independent of \( g \) such that:

\[
\left| \int_{C(t)} g(x) d\pi_t \right| \leq \max(||g||, ck)
\]

when \( ||g|| = \sup_{x \in C(t)} |g(x)| \).

**Proof.** Let \( \{\varepsilon_n\}_{n \geq 1} \) be a decreasing sequence of positive real numbers such that \( \lim_{n} \varepsilon_n = 0, \varepsilon_n/\varepsilon_{n+1} \leq 2 \) and \( C(t) \subseteq B(t, \varepsilon_1) \). Then one has:

\[
\int_{C(t)} g(x) d\pi_t = \lim_{n} \Phi(g, \tau(t), \varepsilon_n),
\]

where (see Section 2) \( d_n = [G : H(t, \varepsilon_n)] \), \( S_n \) is a system of right cosets of \( G \) with respect \( H(t, \varepsilon_n) \) and \( \Phi(g, \tau(t), \varepsilon_n) = \frac{1}{d_n} \sum_{\tau \in S_n} g(\tau(t)) \) is the Riemann sum associated to \( \varepsilon_n, S_n \) and \( g \) (see [APZ2]).

In particular \( \Phi(g, \tau(t), \varepsilon_1) = g(t) \).

Let \( n \geq 1 \). Then \( d_n \) divides \( d_{n+1} \) and for any \( \tau \in S_{n+1} \) there exists exactly one element \( \sigma \in S_n \) such that \( \tau(t) \in B(\sigma(t), \varepsilon_n) \). Then we have

\[
|g(\sigma(t)) - g(\tau(t))| \leq c\varepsilon_n,
\]

and so

\[
\left| \frac{1}{d_n} \sum_{\sigma \in S_n} g(\sigma(t)) - \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) \right|
\]

\[
\leq \frac{c\varepsilon_n}{|d_{n+1}|}.
\]

Let \( n \) be large enough such that

\[
|\int_{C(t)} g(x) d\pi_t| = \left| \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) \right|.
\]
Then by the above considerations one has:

\[
\left| \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) \right| = \left| \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) - \frac{1}{d_{n}} \sum_{\tau \in S_{n}} g(\sigma(t)) + \frac{1}{d_{n}} \sum_{\sigma \in S_{n}} g(\sigma(t)) + \ldots \right|
\]

\[
\ldots + \frac{1}{d_{2}} \sum_{\chi \in S_{2}} g(\chi(t)) - g(t) + g(t) \leq \max_{1 \leq i \leq n} |g|, c \frac{\epsilon_{i}}{|d_{i+1}|}.
\]

Now let us take \( k = \sup_{n} \frac{\epsilon_{n}}{|d_{n+1}|} = \sup_{n} \frac{\epsilon_{n+1}}{|d_{n+1}|} \cdot \frac{\epsilon_{n}}{\epsilon_{n+1}} < \infty \) since \( \lim_{n} \frac{\epsilon_{n}}{|d_{n}|} = 0 \), \( t \) being Lipschitzian by hypothesis.

Let \( \delta > 0 \) be a real number. Then by (4), (11) and (12) it follows that for \( m \) large enough one has: \( \left| \frac{A_{m}(x)}{(z-x)^{m}} \right| < \delta \) and \( \left| \frac{A_{m}(x)}{(z-x)^{m}} - \frac{A_{m}(y)}{(z-y)^{m}} \right| < \delta|x - y| \) for any \( x, y \in D \). Lemma 4.3 implies that \( \left| \int_{D} \frac{A_{m}(x)}{(z-x)^{m}} d\pi_{t}(x) \right| \to 0 \) as \( m \to \infty \).

Therefore

\[
F(z) = \int_{D} \sum_{m \geq 1} \frac{A_{m}(x)}{(z-x)^{m}} d\pi_{t}(x) = \sum_{m \geq 1} \int_{D} \frac{A_{m}(x)}{(z-x)^{m}} d\pi_{t}(x)
\]

and using (13) one obtains (14).

Step 4. Let \( \sigma \in S \) and denote \( D^{\sigma} = B(\sigma(\alpha), \epsilon) \cap C(t) = C_{\sigma}(t, \epsilon) \). Working as above, one gets:

\[
f_{\sigma}(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} \frac{(-1)^{j}}{j!} h_{m}^{(j)}(\sigma(\alpha)) F_{j,m}^{\sigma}(t, z).
\]

Finally by adding these equalities for \( \sigma \in S \) one obtains the expression of \( f(z) \) stated in Theorem 4.2.

**Corollary 4.4.** The notations and hypothesis are as in Theorem 4.2 Assume \( \alpha \in Q_{p} \). Then \( S = \{e\} \) and one has:

\[
f(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} h_{m}^{(j)}(\alpha) F_{j,m}(t, z).
\]
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