VICTOR ALEXANDRU
NICOLAE POPESCU
ALEXANDRU ZAHARESCU

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par Victor Alexandru, Nicolae Popescu et Alexandru Zaharescu

1. Introduction

Let \( p \) be a prime number, \( \mathbb{Q}_p \) the field of \( p \)-adic numbers, \( \overline{\mathbb{Q}_p} \) a fixed algebraic closure of \( \mathbb{Q}_p \) and \( \mathbb{C}_p \) the completion of \( \overline{\mathbb{Q}_p} \) with respect to the \( p \)-adic absolute value. Let \( t \in \mathbb{C}_p \) and set \( E(t) = \mathbb{P}^1(\mathbb{C}_p) \setminus C(t) = \mathbb{C}_p \cup \{\infty\} \setminus C(t) \) where \( C(t) \) denotes the orbit of \( t \) with respect to the group \( G \) of all continuous automorphisms of \( \mathbb{C}_p \) over \( \mathbb{Q}_p \). In this paper we are interested in the \( G \)-equivariant rigid analytic functions on \( E(t) \) and their restrictions to affinoids of the form \( E(t, \varepsilon) = \mathbb{C}_p \cup \{\infty\} \setminus C(t, \varepsilon) \) where \( C(t, \varepsilon) \) stands for the \( \varepsilon \)-neighborhood of \( C(t) \).

These functions are easily described in case \( t \) is algebraic over \( \mathbb{Q}_p \). For instance, if \( t \in \mathbb{Q}_p \) then one can use the equivariant transformation \( z \mapsto \frac{1}{z-t} \) to send \( t \) to the point at infinity. Then the equivariant rigid analytic functions on \( E(t) \) will correspond to the entire functions which are equivariant...
and these are simply power series $f(x) = \sum_{n \geq 0} a_n x^n$ with $a_n \in \mathbb{Q}_p$ for any $n$ and such that $\lim_{n \to \infty} |a_n|^{1/n} = 0$.

If $t$ is transcendental over $\mathbb{Q}_p$ it is not obvious that there are any nonconstant equivariant rigid analytic functions on $E(t)$. For certain elements $t$ (called Lipschitzian) such a function $z \mapsto F(t, z)$ is constructed in [APZ2]. In this paper we define for any Lipschitzian element $t$ of $E$ and any natural numbers $m, n$ an equivariant rigid analytic function $F_{m,n}(t, z)$ on $E(t)$, which is related to our basic trace series $F(t, z)$. Then in Theorem 4.2 below we express any equivariant rigid analytic function on an affinoid $E(t, \varepsilon)$ in terms of the above functions $F_{m,n}(t, z)$.

2. Background material

2.1. Let $p$ be a prime number and $\mathbb{Q}_p$ the field of $p$-adic numbers endowed with the $p$-adic absolute value $| |$, normalized such that $|p| = 1/p$. Let $\overline{\mathbb{Q}_p}$ be a fixed algebraic closure of $\mathbb{Q}_p$ and denote by the same symbol $| |$ the unique extension of $| |$ to $\overline{\mathbb{Q}_p}$. Further, denote by $(\overline{\mathbb{Q}_p}, | |)$ the completion of $(\overline{\mathbb{Q}_p}, | |)$ (see [Am], [Ar]). Let $G = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ endowed with the Krull topology. The group $G$ is canonically isomorphic with the group $\text{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$ of all continuous automorphisms of $\mathbb{C}_p$ over $\mathbb{Q}_p$. For any $x \in \mathbb{C}_p$ denote $C(x) = \{ \sigma(x) | \sigma \in G \}$ the orbit of $x$ and let $\overline{\mathbb{Q}_p}[x]$ be the closure of the ring $\mathbb{Q}_p[x]$ in $\mathbb{C}_p$. For any $x \in \overline{\mathbb{Q}_p}$ denote $\deg(x) = \dim_{\mathbb{Q}_p}(x : \mathbb{Q}_p)$.

2.2. Let $x \in \mathbb{C}_p$. Given a real number $\varepsilon > 0$ let $B(x, \varepsilon) = \{y \in \mathbb{C}_p, |x - y| < \varepsilon \}$ the open ball of radius $\varepsilon$ centered at $x$. If $M$ is a compact subset of $\mathbb{C}_p$ and $\varepsilon > 0$ is a real number, denote by $N(M, \varepsilon)$ the number of all disjoint balls of radius $\varepsilon$ which have a non-empty intersection with $M$. We say that $M$ is Lipschitzian if $\lim_{\varepsilon \to 0} \frac{N(M, \varepsilon)}{|N(M, \varepsilon)|} = 0$. We call an element $x \in \mathbb{C}_p$ Lipschitzian if $C(x)$ is Lipschitzian.

According to [APZ2] if $x$ is Lipschitzian then one can integrate Lipschitzian functions (see definition in 2.3 below) with respect to the $p$-adic Haar measure $\pi_t$ induced by $G$ on the set $C(x)$.

Let $G_x = \{ \sigma \in G : \sigma(x) = x \}$ and $P$ a closed subgroup of $G$ which contains $G_x$. Then $C_P(x) = \{ \sigma(x) : \sigma \in P \}$, the orbit of $x$ with respect to $P$, is a compact subset of $C(x) = C_G(x)$. If $x$ is Lipschitzian then $C_P(x)$ is a Lipschitzian compact set for any $P$ with $G_x \subset P$. This follows from the fact that for any $\varepsilon > 0$, $N(C_P(x), \varepsilon)$ divides $N(C(x), \varepsilon)$.

Let $x \in \mathbb{C}_p$ and $P$ a closed subgroup of $G$ which contains $G_x$. For any $\varepsilon > 0$ let $H_P(x, \varepsilon) = \{ \sigma \in P : |x - \sigma(x)| < \varepsilon \}$ and $N_P(x, \varepsilon) = N(C_P(x), \varepsilon)$. Then $H_P(x, \varepsilon)$ is an open subgroup of $P$ and $N_P(x, \varepsilon) = [P : H_P(x, \varepsilon)]$. In particular $N(x, \varepsilon) = [G : H(x, \varepsilon)]$, where $H(x, \varepsilon) = H_G(x, \varepsilon)$.
2.3. The notion of rigid analytic function is defined in [FP] (see also [Am]). According to [APZ2], a rigid analytic function defined on a subset $D$ of $\mathbb{C}_p$ is said to be equivariant if for any $z \in D$ one has $C(z) \subset D$ and $f(\sigma(z)) = \sigma(f(z))$ for all $\sigma \in G$. A function $f : C(t) \rightarrow \mathbb{C}_p$, $t \in \mathbb{C}_p$ is Lipschitzian if there exists a real number $c > 0$ such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in C(t)$.

Let $t$ be a Lipschitzian element of $\mathbb{C}_p$ and $f : C(t) \rightarrow \mathbb{C}_p$ a Lipschitzian function. Then the integral

$$\int_{C(t)} f(x) d\pi_t(x)$$

is well defined (see [APZ2]). In particular for any polynomial $P(X) \in \mathbb{C}_p[X]$, any $z \in \mathbb{C}_p \cup \{\infty\} \setminus C(t)$ and any natural number $n$ the function $z \mapsto f(x, z) = \frac{P(x)}{(z-x)^n}$ is Lipschitzian on $C(t)$ and we consider the integral $\int_{C(t)} f(x, z) d\pi_t(x)$. Let us denote

$$F_{m,n}(t, z) = \int_{C(t)} \frac{x^m}{(z-x)^n} d\pi_t(x), \quad m \geq 0, \quad n \geq 0.$$

According to [APZ2] for any $m \geq 0$ one has

$$\int_{C(t)} x^m d\pi_t(x) = Tr(t^m).$$

This shows that $F_{m,0}(t, z) = Tr(t^m) \in \mathbb{Q}_p$, $F_{0,0}(t, z) = 1$ and $1 + F_{1,1}(t, \frac{1}{z}) = F(t, z)$, the trace function associated to $t$. Also by the equality

$$\frac{1}{(1-u)^m} = (1 + u + u^2 + \ldots + u^n + \ldots)^m = \sum_{s=0}^{\infty} \left( \begin{array}{c} m+s-1 \\ s \end{array} \right) u^s$$

valid for any positive integer $m$ and any $u$ with $|u| < 1$, it follows that for $|z| > |x|$ one has

$$\frac{x^m}{(z-x)^n} = \sum_{s \geq 0} \left( \begin{array}{c} n+s-1 \\ s \end{array} \right) \frac{x^{m+s}}{z^{n+s}}.$$ 

Then one may write:

$$F_{m,n}(t, z) = \sum_{s \geq 0} \left( \begin{array}{c} m+s-1 \\ s \end{array} \right) \frac{Tr(t^{m+s})}{z^{n+s}}.$$

This formula represents the expansion of $F_{m,n}(t, z)$ in a suitable neighborhood of infinity. As in Theorem 6.1 of [APZ2] one shows that for all $m \geq 0, n \geq 0, F_{m,n}(t, z)$ is an equivariant rigid analytic function defined on $\mathbb{C}_p \cup \{\infty\} \setminus C(t)$.
Remark 2.1. \( F'_{m,n}(t,z) = -nF_{m,n+1}(t,z) \) for any \( m \geq 0, n \geq 1 \). As a consequence one has \( F_{m,n+1}(t,z) = \frac{(-1)^n}{n!} F_{m,1}^{(n)}(t,z) \), where the derivative is taken with respect to \( z \).

2.4. The above considerations can be generalized as follows: Let \( \varepsilon > 0 \) be a real number and \( S \) a system of right representatives of \( G \) with respect to the subgroup \( H(t,\varepsilon) \). Assume that the identity element \( e \) of \( G \) belongs to \( S \) and that \( t \) is Lipschitzian. For any \( \sigma \in S \) the subset \( C_\sigma(t,\varepsilon) = \{ \tau(t) : \tau \in \sigma H(t,\varepsilon) \} \) is a compact subset of \( C(t) \). For \( m \geq 0, n \geq 0 \) denote

\[
F_{m,n}^{\sigma}(t,z) = \int_{C_\sigma(t,\varepsilon)} \frac{x^m}{(z-x)^n} \, d\pi_t(x).
\]

It is clear that

\[
F_{m,n}(t,z) = \sum_{\sigma \in S} F_{m,n}^{\sigma}(t,z).
\]

In fact this formula represents the Mittag-Leffler decomposition of \( F_{m,n}(t,z) \) viewed as a rigid analytic function in the connected affinoid \( \mathbb{C}_p \cup \{ \infty \} \setminus \cup_{\sigma \in S} B(\sigma(t),\varepsilon) = E(t,\varepsilon) \). In what follows we try to obtain a similar decomposition for any element of the set \( A(E(t,\varepsilon)) \) of equivariant rigid analytic functions on \( E(t,\varepsilon) \).

3. A combinatorial Lemma

Let \( \alpha, x, y, \{ a_m \}_{m \geq 1} \) be variables. For any \( m \geq 1 \), let us denote

\[
h_m(\alpha) = a_1 \alpha^{m-1} + a_2 \binom{m-1}{1} \alpha^{m-2} + \ldots + \binom{m-1}{m-1} a_m
\]

where as usually \( \binom{m}{k} = \frac{m(m-1)\ldots(m-k+1)}{k!} \). For any integer \( m \geq 1 \) we set

\[
A_m(x) = h_m(\alpha) - \frac{1}{1!} h_m^{(1)}(\alpha)x + \ldots + \frac{(-1)^{m-1}}{(m-1)!} h_m^{(m-1)}(\alpha)x^{m-1}
\]

where \( h_m^{(k)}(\alpha) \) denotes the formal \( k \)-th derivative of the polynomial \( h_m(\alpha) \) with respect to \( \alpha \).

Lemma 3.1. For any \( x, y \) and any \( m \geq 1 \) one has:

\[
A_m(x) = \sum_{r=1}^{m} a_r \binom{m-1}{r-1} (\alpha - x)^{m-r}
\]

and

\[
A_m(y) = \sum_{r=1}^{m} \binom{m-1}{r-1} A_r(x)(x - y)^{m-r}.
\]
Proof. Equality (3) states that $A_m(x) = h_m(\alpha - x)$, which follows directly from the Taylor expansion (2). As for the second equality, by applying (3) and using the identity
\[
\binom{m-1}{r-1} \binom{r-1}{j-1} = \binom{m-1}{j-1} \binom{m-j}{r-j}
\]
one contains
\[
\sum_{r=1}^{m} \binom{m-1}{r-1} A_r(x)(x-y)^{m-r}
= \sum_{r=1}^{m} \binom{m-1}{r-1} (x-y)^{m-r} \sum_{j=1}^{r} a_j \binom{r-1}{j-1} (\alpha - x)^{r-j}
= \sum_{j=1}^{m} a_j \sum_{r=j}^{m} \binom{m-1}{j-1} \binom{m-j}{r-j} (x-y)^{m-r} (\alpha - x)^{r-j}
= \sum_{j=1}^{m} a_j \binom{m-1}{j-1} (\alpha - y)^{m-j}.
\]
This equals $A_m(y)$ by (3) and so the lemma is proved. \(\square\)

4. Equivariant rigid analytic functions on $E(t, \varepsilon)$

4.1. Let $t$ be an element of $\mathbb{C}_p$, let $\varepsilon > 0$ be a real number and denote by $B(C(t), \varepsilon)$ the union of all disjoint open balls $B(x, \varepsilon)$ which have a nonempty intersection with $C(t)$. Choose $\alpha \in \overline{\mathbb{Q}}_p$ such that $|t - \alpha| < \varepsilon$. Then one has $H(t, \varepsilon) = H(\alpha, \varepsilon)$. Let $S$ be a system of right representatives of $G$ with respect to $H(t, \varepsilon)$ and assume $\varepsilon \in S$. One has $B(C(t), \varepsilon) = \bigcup_{\sigma \in S} B(\sigma(\alpha), \varepsilon)$. Consider the affinoid $E(t, \varepsilon) = \mathbb{C}_p \cup \{\infty\} \setminus B(C(t), \varepsilon)$ and let $A(E(t, \varepsilon))$ be the set of equivariant rigid analytic functions on $E(t, \varepsilon)$. If $t$ is Lipschitzian then the functions $F_{m,n}(t, z)$ defined in Section 2 are elements of $A(E(t, \varepsilon))$. In this section we shall prove that all the elements of $A(E(t, \varepsilon))$ can be expressed in terms of the functions $F_{m,n}(t, z)$, $m, n \geq 0$.

4.2. We have the following proposition.

Proposition 4.1. Let $t$ be an element of $\mathbb{C}_p$. Denote $K_t = \overline{\mathbb{Q}}_p[t] \cap \overline{\mathbb{Q}}_p$ and let $\varepsilon > 0$ and $\alpha \in K_t$ such that $|\alpha - t| < \varepsilon$. There exists a sequence $\{\alpha_n\}_{n \geq 1}$ of elements of $K_t$ and a sequence $\{\varepsilon_n\}_{n \geq 1}$ of positive real numbers such that:

(i) $\varepsilon_1 = \varepsilon$, $\alpha_1 = \alpha$,
(ii) For any $n \geq 1$ one has $\varepsilon_{n+1} \leq \inf\{\varepsilon_n/2, |t - \alpha_n|\}$,
(iii) $|t - \alpha_n| < \varepsilon_n$, $n \geq 1$, and $\deg \alpha_n$ is smallest with this property.
The proof easily follows by induction on $n$ since any ball $B(t, \varepsilon)$ contains elements of $K_t$ (see [APZ1]).

In what follows we work with sequences $\{\alpha_n\}_n$ and $\{\varepsilon_n\}_n$ as in Proposition 4.1. It is clear that $\lim_{n \to \infty} \varepsilon_n = 0$, and $t = \lim_{n \to \infty} \alpha_n$. Note also that the ball $B(\alpha_{n+1}, \varepsilon_{n+1})$ is contained in $B(\alpha_n, \varepsilon_n)$ for all $n \geq 1$. Let us consider the subgroup $H(t, \varepsilon_n) = H(\alpha_n, \varepsilon_n)$ defined in Section 2. Denote $d_n = [G : H(T, \varepsilon_n)] = N(t, \varepsilon_n)$, and let $S_n$ be a fixed system of representatives of right cosets of $G$ with respect to $H(t, \varepsilon_n)$. We shall assume that the identity element $e$ of $G$ belongs to each $S_n$. We remark that $S_1 = S$ and $d_n$ divides $d_{n+1}$ for all $n \geq 1$.

4.3. Let $f \in A(E(t, \varepsilon))$. Then (see [FP], Ch I) $f$ admits a Mittag-Leffler decomposition: $f(z) = \sum_{\sigma \in S} f_{\sigma}(z) + f(\infty)$ where $f(\infty)$ is the value of $f$ at infinity and

$$f_{\sigma}(z) = \sum_{m \geq 1} \frac{a_{\sigma,m}}{(z - \sigma(\alpha))^m}, \quad \lim_{m \to \infty} \frac{|a_{\sigma,m}|}{\varepsilon^m} = 0, \quad \sigma \in S.$$  

Since $f$ is equivariant then for any $z \in E(t, \varepsilon)$ and any $\tau \in G$ one has $\sum_{\sigma \in S} \tau(f_{\sigma}(z)) = \sum_{\sigma \in S} f_{\sigma}(\tau(z))$ and $\tau(f(\infty)) = f(\infty)$. Hence $f(\infty) \in \mathbb{Q}_p$ and for any $\sigma \in S$ one can write:

$$f_{\sigma}(\sigma(z)) = \sigma(f_{\sigma}(z)), \quad a_{\sigma,m} = \sigma(a_{\varepsilon,m}), \quad m \geq 1.$$  

Next we remark that for any $\tau \in H(t, \varepsilon)$ and any $\sigma \in S$ the element $\sigma(\tau(\alpha))$ belongs to $B(\sigma(\alpha), \varepsilon)$, and so the function $f_{\sigma}(z) = \sum_{m \geq 1} \frac{a_{\sigma,m}}{(z - \sigma(\alpha))^m}$ can also be written as

$$f_{\sigma}(z) = \sum_{m \geq 1} \frac{a_{\sigma\tau,m}}{(z - \sigma\tau(\alpha))^m} = f_{\sigma\tau}(z)$$  

where

$$a_{\sigma\tau,m} = \sum_{i=1}^{m} \binom{m-1}{i-1} a_{\sigma,i}(\sigma(\alpha) - \sigma\tau(\alpha))^{m-i}.$$  

In what follows we shall assume that $f(\infty) = 0$.

4.4. At this point we derive another convenient expression for $f(z)$, using the above elements $\alpha_n$. Fix $n \geq 1$. Then $d = d_1$ divides $d_n = [G : H(t, \varepsilon_n)]$. Denote $q_n = d_n/d$ and let $B(\alpha_n^{(j)}, \varepsilon_n), \; 1 \leq j \leq q_n$ be all the balls of radius $\varepsilon_n$ centered at suitable conjugates of $\alpha_n$ and such that these balls cover $C_{H(t, \varepsilon)}(T) = C_{\varepsilon}(t, \varepsilon)$. Then

$$f_{\varepsilon}(z) = \frac{d}{d_n} \sum_{1 \leq j \leq q_n} \sum_{m \geq 1} \frac{A_{\varepsilon,j,m}}{(z - \alpha_n^{(j)})^m}$$  

where $A_{\varepsilon,j,m}$ is a function of $\varepsilon_n$ and $\alpha_n^{(j)}$. 

$$A_{\varepsilon,j,m} = \sum_{i=1}^{m} \binom{m-1}{i-1} A_{\varepsilon,i}(\alpha_n^{(j)})^{m-i}.$$  

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$$f_{\varepsilon}(z) = \frac{d}{d_n} \sum_{1 \leq j \leq q_n} \sum_{m \geq 1} \frac{A_{\varepsilon,j,m}}{(z - \alpha_n^{(j)})^m}$$  

where $A_{\varepsilon,j,m}$ is a function of $\varepsilon_n$ and $\alpha_n^{(j)}$. 

$$A_{\varepsilon,j,m} = \sum_{i=1}^{m} \binom{m-1}{i-1} A_{\varepsilon,i}(\alpha_n^{(j)})^{m-i}.$$  

In what follows we shall assume that $f(\infty) = 0$. 

4.7. At this point we derive another convenient expression for $f(z)$, using the above elements $\alpha_n$. Fix $n \geq 1$. Then $d = d_1$ divides $d_n = [G : H(t, \varepsilon_n)]$. Denote $q_n = d_n/d$ and let $B(\alpha_n^{(j)}, \varepsilon_n), \; 1 \leq j \leq q_n$ be all the balls of radius $\varepsilon_n$ centered at suitable conjugates of $\alpha_n$ and such that these balls cover $C_{H(t, \varepsilon)}(T) = C_{\varepsilon}(t, \varepsilon)$. Then

$$f_{\varepsilon}(z) = \frac{d}{d_n} \sum_{1 \leq j \leq q_n} \sum_{m \geq 1} \frac{A_{\varepsilon,j,m}}{(z - \alpha_n^{(j)})^m}$$  

where $A_{\varepsilon,j,m}$ is a function of $\varepsilon_n$ and $\alpha_n^{(j)}$. 

$$A_{\varepsilon,j,m} = \sum_{i=1}^{m} \binom{m-1}{i-1} A_{\varepsilon,i}(\alpha_n^{(j)})^{m-i}.$$  

In what follows we shall assume that $f(\infty) = 0$.
where

\[ A_{\varepsilon,n}^{(j)} (z) = \sum_{i=1}^{m} \binom{m - 1}{i - 1} a_{\varepsilon,i} (\alpha - \alpha_n^{(j)})^{m-i}. \]

According to (5) for all \( \sigma \in S \) one has

\[ f_{\sigma}(z) = \frac{d}{dn} \sum_{1 \leq j \leq q_n} \sum_{m \geq 1} \frac{A_{\sigma,n}^{(j)}}{(z - \sigma(\alpha_n^{(j)}))^m}, \]

where \( A_{\sigma,n}^{(j)} = \sigma(A_{\varepsilon,n}^{(j)}). \) As a consequence of (4) there exists a positive real number \( M \) such that for any \( m \geq 1 \) one has:

\[ |a_{\sigma,m}| \leq M \varepsilon^m. \]

It follows from (7) that

\[ |A_{\sigma,n}^{(j)}| \leq M \varepsilon^m \]

for any \( n \geq 2 \) and \( 1 \leq j \leq q_n \).

4.5. At this point we assume that \( t \) is a Lipschitzian element of \( C_p, \varepsilon > 0 \) and \( \alpha \in B(t, \varepsilon), \alpha \in K_t \). Let \( f \in A(E(t, \varepsilon)) \), \( f = \sum_{\sigma \in S} f_{\sigma}(z) \) with \( f_{\sigma}(z) \) given by (4). For any \( m \geq 1 \) denote

\[ h_m(\alpha) = a_{\varepsilon,1} \alpha^{m-1} + \binom{m - 1}{1} \alpha^{m-2} + \ldots + \binom{m - 1}{m-1} a_{\varepsilon,m}. \]

Also, for \( \sigma \in S \) consider the function \( F_{\sigma,m}(t, z) \) defined in Section 2.

**Theorem 4.2.** Let \( t \) be a Lipschitzian element of \( C_p, \varepsilon > 0, \alpha \in B(t, \varepsilon) \cap K_t \) and \( f \in A(E(t, \varepsilon)) \). Then for any \( z \in E(t, \varepsilon) \) one has

\[ f(z) = \sum_{\sigma \in S} \sum_{m \geq 1} \sum_{0 \leq j \leq m} \frac{(-1)^j}{j!} \sigma(h_m^{(j)}(\alpha)) F_{\sigma,m}^{(j)}(t, z). \]

**Proof.** For any \( m \geq 1 \) let \( A_m(x) = \sum_{1 \leq i \leq m} a_{\varepsilon,i} \binom{m - 1}{i - 1} (\alpha - x)^{m-i} \) and

\[ A(x, z) = \sum_{m \geq 1} \frac{A_m(x)}{(z-x)^m}. \]

Step 1. Fix \( z_0 \in E(t, \varepsilon) \). We assert that for any \( z \in B(z_0, \varepsilon) \), the function \( x \mapsto A(x, z) \) is defined and is Lipschitzian on \( B(t, \varepsilon) \). Firstly we remark that for any \( x \in B(t, \varepsilon) \) one has (see (8)):

\[ \left| \frac{A_m(x)}{(z-x)^m} \right| \leq \frac{\sum_{1 \leq i \leq m} a_{\varepsilon,i} \binom{m - 1}{i - 1} (\alpha - x)^{m-i}}{e^m}. \]
and

$$
e^{-m} \left| \sum_{i=1}^{[m/2]} a_{e,i} \left( \frac{m-1}{i-1} \right) (\alpha - x)^{m-i} \right| \leq \max_{1 \leq i \leq [m/2]} \left( M \left( \frac{|\alpha - x|}{\varepsilon} \right)^{m-i} \right)$$

Notice that

$$\varepsilon^{-m} \left| \sum_{i=[m/2]+1}^{m} a_{e,i} \left( \frac{m-1}{i-1} \right) (\alpha - x)^{m-i} \right| \leq \max_{[m/2]+1 \leq i \leq m} \left( \frac{|a_{e,i}|}{\varepsilon^i} \right).$$

Since $\frac{|\alpha - x|}{\varepsilon} < 1$, by (4) and the above considerations it follows that

$$\left| \frac{A_m(x)}{(x - y)^m} \right| \to 0 \text{ when } m \to \infty.$$ Then the function $A(x, z)$ is defined on $B(t, \varepsilon)$, as claimed. Now let $x, y \in B(t, \varepsilon)$. For any $m \geq 1$ we have

$$A_m(y) = A_m(y) \left( 1 + \sum_{i \geq 1} D_i \left( \frac{y-x}{z-x} \right)^i \right)$$

where $D_i$ are suitable natural numbers. Then one can write

$$\left| \frac{A_m(x)}{(z - x)^m} - \frac{A_m(y)}{(z - y)^m} \right| \leq \max_{i \geq 1} \left( \left| \frac{A_m(x) - A_m(y)}{(z - x)^m} \right|, \left| \frac{A_m(y)(y-x)^i}{(z - x)^{m+i}} \right| \right).$$

But (see (8)) for any $i \geq 1$ and $z \in B(z_0, \varepsilon)$ one has

$$\left| \frac{A_m(y)(y-x)^i}{(z - x)^{m+i}} \right| \leq \left| \frac{A_m(y)}{z - x} \right| \left| \frac{|y-x|^i}{|z-x|^m \cdot |z-x|^i} \right| \leq M \frac{|y-x|}{\varepsilon}.$$ 

Also by an easy computation one sees that:

$$\left| \frac{A_m(x) - A_m(y)}{(z - x)^m} \right| \leq M \frac{|y-x|}{\varepsilon}.$$ 

Finally, one has $|A(x, y) - A(y, z)| \leq \frac{M}{\varepsilon} |x - y|$ i.e. $A(x, z)$ is Lipschitzian on $B(t, \varepsilon)$. The above considerations also show that for any $\delta > 0$ we have

$$(12) \quad \left| \frac{A_m(x)}{(z - x)^m} - \frac{A_m(y)}{(z - y)^m} \right| \leq \delta |x - y|$$

for all $m$ large enough in terms of $z$ and $\delta$, uniformly for $x, y \in C_\varepsilon(t, \varepsilon)$. 
Step 2. Let us denote $D = C_\varepsilon(t, e) = B(t, e) \cap C(t)$. Then $D$ is a compact Lipschitzian subset of $\mathbb{C}_p$ and we consider the integral

$$F(z) = \int_D A(x, z) d\pi_t(x), \ z \in B(z_0, \varepsilon).$$

Here we use the definition of the integral with respect the $p$-adic measure $\pi_t$ as in [APZ2]. We assert that

$$f_e(z) = F(z), \ z \in B(z_0, \varepsilon),$$

where $e$ is the identity element of $G$.

To see this, consider the sequences $\{\varepsilon_n\}_n$ and $\{\alpha_n\}_n$ from Proposition 4.1. Let $H(t, \varepsilon_n), \ d_n, \ S_n$ be as above. In particular $\varepsilon_1 = \varepsilon, \ \alpha_1 = \alpha, \ d_1 = d$. For any $n \geq 1$ let $B(\alpha_n^{(i)}, \varepsilon_n)$, $1 \leq i \leq q_n$ be the open balls of radius $\varepsilon_n$ which cover $D$. Then one has:

$$F(z) = \int_D A(x, z) d\pi_t(x) = \lim_n \Phi[A(x, z), \alpha_n^{(i)}, \varepsilon_n]$$

where

$$\Phi[A(x, z), \alpha_n^{(i)}, \varepsilon_n] = \frac{d}{d_n} \sum_{1 \leq i \leq q_n} A(\alpha_n^{(i)}, z)$$

is the Riemann sum associated to $(A, \alpha_n^{(i)}, \varepsilon_n)$ (see[APZ2]). We have

$$\frac{d}{d_n} A(\alpha_n^{(i)}, z) = \frac{d}{d_n} \sum_{m \geq 1} A_m(\alpha_n^{(i)})(z - \alpha_n^{(i)})^m.$$

From (6) it now follows that

$$\Phi[A, \alpha_n^{(i)}, \varepsilon_n] = f_e(z).$$

Since this equality is valid for any $n$ we conclude that

$$F(z) = \int_D A(x, z) d\pi_t(x) = f_e(z).$$

Step 3. We now apply formula (2) to obtain another expression for $A_m(x)$. One has:

$$A_m(x) = h_m(\alpha) - \frac{1}{1!} h_m^{(1)}(\alpha)x + \ldots + \frac{(-1)^{m-1}}{(m-1)!} h_m^{(m-1)}(\alpha)x^{m-1}$$

$$= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) \frac{x^j}{(z-x)^m}.$$
Therefore
\[
\int_D \frac{A_m(x)}{(z-x)^m} d\pi_t(x) = \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) \int_D \frac{x^j}{(z-x)^m} d\pi_t(x)
\]
\[
= \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) F_{j,m}^e(t, z).
\]

We claim that
\[
F(z) = f_0(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h_m^{(j)}(\alpha) F_{j,m}^e(t, z).
\]

In order to prove this formula we need the following result:

**Lemma 4.3.** Let \( t \) be a Lipschitzian element of \( C_p, \varepsilon > 0 \) a real number, \( g : B(C(t), \varepsilon) \to C_p \) a Lipschitzian function, and let \( c \) be a real number such that \( |g(x) - g(y)| \leq c|x - y| \) for all \( x, y \in C(t) \). Then there exists a real number \( k \) independent of \( g \) such that:

\[
\left| \int_{C(t)} g(x) d\pi_t \right| \leq \max(||g||, ck)
\]

when \( ||g|| = \sup_{x \in C(t)} |g(x)| \).

**Proof.** Let \( \{\varepsilon_n\}_{n \geq 1} \) be a decreasing sequence of positive real numbers such that \( \lim_n \varepsilon_n = 0, \varepsilon_n/\varepsilon_{n+1} \leq 2 \) and \( C(t) \subseteq B(t, \varepsilon_1) \). Then one has:

\[
\int_{C(t)} g(x) d\pi_t = \lim_n \int_{C(t)} g(x) d\pi_t = \lim_n \frac{1}{d_n} \sum_{\tau \in S_n} g(\tau(t))\text{ is the Riemann sum associated to } \varepsilon_n, S_n \text{ and } g \text{ (see [APZ2]).}
\]

In particular \( \Phi(g, \tau(t), \varepsilon_1) = g(t) \).

Let \( n \geq 1 \). Then \( d_n \) divides \( d_{n+1} \) and for any \( \tau \in S_{n+1} \) there exists exactly one element \( \sigma \in S_n \) such that \( \tau(t) \in B(\sigma(t), \varepsilon_n) \). Then we have

\[
|g(\sigma(t)) - g(\tau(t))| \leq c\varepsilon_n, \text{ and so } \frac{1}{d_n} \sum_{\sigma \in S_n} g(\sigma(t)) - \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t))
\]

\[
\leq \frac{c\varepsilon_n}{|d_{n+1}|}. \text{ Let } n \text{ be large enough such that}
\]

\[
\left| \int_{C(t)} g(x) d\pi_t \right| = \left| \frac{1}{d_{n+1}} \sum_{\tau \in S_{n+1}} g(\tau(t)) \right|
\]
Then by the above considerations one has:

\[ \left| \frac{1}{d_{n+1}} \sum_{\tau \in \mathcal{S}_{n+1}} g(\tau(t)) \right| = \]

\[ \left| \frac{1}{d_{n+1}} \sum_{\tau \in \mathcal{S}_{n+1}} g(\tau(t)) - \frac{1}{d_n} \sum_{\sigma \in \mathcal{S}_n} g(\sigma(t)) + \frac{1}{d_n} \sum_{\sigma \in \mathcal{S}_n} g(\sigma(t)) + \ldots \right. \]

\[ \ldots + \frac{1}{d_2} \sum_{\chi \in \mathcal{S}_2} g(\chi(t)) - g(t) + g(t) \right| \leq \max_{1 \leq i \leq n} \left| g_i \right| \epsilon_i \left| \frac{\epsilon_n}{d_{n+1}} \right|. \]

Now let us take \( k = \sup_{n} \frac{\epsilon_n}{|d_{n+1}|} = \sup_{n} \frac{\epsilon_{n+1}}{|d_{n+1}|} \cdot \frac{\epsilon_n}{\epsilon_{n+1}} \) < \( \infty \) since \( \lim_{n} \frac{\epsilon_n}{|d_n|} = 0 \), \( t \) being Lipschitzian by hypothesis. \( \square \)

Let \( \delta > 0 \) be a real number. Then by (4), (11) and (12) it follows that for \( m \) large enough one has: \( \left| \frac{A_m(z)}{(z-x)^m} \right| < \delta \) and \( \left| \frac{A_m(z)}{(z-x)^m} - \frac{A_m(y)}{(z-y)^m} \right| < \delta |x - y| \) for any \( x, y \in D \). Lemma 4.3 implies that \( \int_{D} \frac{A_m(z)}{(z-x)^m} d\pi_t(x) \rightarrow 0 \) as \( m \rightarrow \infty \).

Therefore

\[ F(z) = \int_{D} \sum_{m \geq 1} \frac{A_m(z)}{(z-x)^m} d\pi_t(x) = \sum_{m \geq 1} \int_{D} \frac{A_m(z)}{(z-x)^m} d\pi_t(x) \]

and using (13) one obtains (14).

Step 4. Let \( \sigma \in S \) and denote \( D^\sigma = B(\sigma(\alpha), \epsilon) \cap C(t) = C^\sigma(t, \epsilon) \). Working as above, one gets:

\[ f_\sigma(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} \frac{(-1)^j}{j!} h^{(j)}_m(\sigma(\alpha)) F^\sigma_{j,m}(t, z). \]

Finally by adding these equalities for \( \sigma \in S \) one obtains the expression of \( f(z) \) stated in Theorem 4.2. \( \square \)

**Corollary 4.4.** The notations and hypothesis are as in Theorem 4.2 Assume \( \alpha \in \mathbb{Q}_p \). Then \( S = \{e\} \) and one has:

\[ f(z) = \sum_{m \geq 1} \sum_{j=0}^{m-1} h^{(j)}_m(\alpha) F_{j,m}(t, z). \]
References


Victor ALEXANDRU
Department of Mathematics
University of Bucharest
Romania

Nicolae POPESCU
Institute of Mathematics of the Romanian Academy
P.O. Box 1-764
70700 Bucharest, Romania
E-mail: nipopesc@stoilow.imar.ro

Alexandru ZAHARESCU
Department of Mathematics
University of Illinois at Urbana-Champaign
Altgeld Hall, 1409 W. Green Street
Urbana, IL, 61801, USA
E-mail: zaharesc@math.uiuc.edu