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Computing modular degrees using $L$-functions

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par Christophe Delaunay

Résumé. Nous donnons un algorithme pour calculer le degré modulaire d'une courbe elliptique définie sur $\mathbb{Q}$. Notre méthode est basée sur le calcul de la valeur spéciale en $s = 2$ du carré symétrique de la fonction $L$ attachée à la courbe elliptique. Cette méthode est assez efficace et facile à implémenter.

Abstract. We give an algorithm to compute the modular degree of an elliptic curve defined over $\mathbb{Q}$. Our method is based on the computation of the special value at $s = 2$ of the symmetric square of the $L$-function attached to the elliptic curve. This method is quite efficient and easy to implement.

1. Introduction

From the recent and difficult work of [14], [11] and [3], it is now known that every elliptic curves $E/\mathbb{Q}$ is modular. If $N$ denotes its conductor, this implies that there exists a covering map $\varphi$ from $X_0(N)$ to $E$. The pull-back by $\varphi$ of the unique (up to multiplication) invariant differential form $\omega$ on $E$ is $2i\pi cf(\tau)d\tau$, where $f(\tau)$ is a normalized newform of level $N$ and weight 2 on $\Gamma_0(N)$ and where the 'Manin's constant' $c$ is rational and can be assumed positive. Furthermore, the $L$-function associated to $f$ coincides with the Hasse-Weil $L$-function of $E$.

The question of computing the degree of $\varphi$ is natural and interesting because of important conjectures related to this number $\deg(\varphi)$. It is well known (cf. [15]) that there exists a simple relation between $\deg(\varphi)$ and $\|f\|_N^2$, where $\|\cdot\|_N$ denotes the Petersson norm. In [15], D. Zagier explains how to compute explicitly $\|f\|_N$ in the general case of a congruence subgroup $\Gamma$. J. Cremona, in [7] interprets Zagier's method in the language of "M-symbols" and computes $\deg(\varphi)$ for many elliptic curves (large tables of elliptic curves are given in [6]). Both methods are geometric and efficient but tend to be quite slow when the conductor is large. The purpose of this paper is to give an alternative way of computing $\|f\|_N$. This is an analytic method based on well-known results which relate the special value of the $L$-function associated to the symmetric square of $E$ with $\|f\|_N$.

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2. The imprimitive symmetric square of $E$

Let $E$ be an elliptic curve defined over $\mathbb{Q}$ of conductor $N$. The normalized newform attached to $E$ is $f(\tau) = \sum n^q a_n q^n$ ($q = e^{2\pi i \tau}$). We have $\varphi^*(\omega) = 2i\pi c f(\tau) d\tau$, and a conjecture of Manin asserts that $c = 1$ whenever $E$ is a strong Weil curve (there is exactly one such curve in an isogeny class). We then have ([15]):

$$\frac{4\pi^2 c^2 \|f\|^2}{\text{vol}(E)} = \deg(\varphi),$$

where $\text{vol}(E)$ is the volume of a minimal period lattice $\Lambda$ with $E \simeq \mathbb{C}/\Lambda$. Now, the Hasse-Weil function $L(E, s)$ is equal to $\sum a_n n^{-s}$ and can be expanded as an Euler product:

$$L(E, s) = \prod_p L_p(E, p^{-s})^{-1},$$

where $L_p(E, X) = (1 - \alpha_p X)(1 - \beta_p X)$, with:

- if $p \nmid N$:
  $$\begin{cases} 
  |\alpha_p| = |\beta_p| = \sqrt{p} \\
  \alpha_p + \beta_p = a_p.
  \end{cases}$$

- If $p \mid N$ then $\beta_p = 0$, $a_p = \alpha_p$ and:
  $$\alpha_p = \begin{cases} 
  -1 & \text{if } E \text{ has non-split multiplicative reduction at } p \ (p \mid |N|); \\
  1 & \text{if } E \text{ has split multiplicative reduction at } p \ (p \mid |N|); \\
  0 & \text{if } E \text{ has additive reduction at } p \ (p^2 \mid N).
  \end{cases}$$

We define the imprimitive symmetric square $L$-function of $f$ to be:

$$L(\text{Sym}^2 f, s) = \frac{\zeta_N(2s - 2)}{\zeta_N(s - 1)} \sum_{n=1}^{\infty} \frac{\alpha_n^2}{n^s}, \quad \Re(s) > 2$$

$$= \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1}.$$

The subscript $N$ means that we have omitted the Euler factors at the primes dividing $N$.

It can be shown that $L(\text{Sym}^2 f, s)$ has a holomorphic continuation to the whole complex plane, and by Rankin’s method that (cf. [10]):

$$\|f\|^2_N = \frac{N}{8\pi^3} L(\text{Sym}^2 f, 2).$$

This formula allows us to study quadratic twists of an elliptic curve. Indeed, assume that $E$ is the quadratic twist of an elliptic curve $E'$ with conductor $N'$ such that $\text{ord}_p(N') \leq \text{ord}_p(N)$ for all prime $p$. We denote by $\chi$ the underlying quadratic character and by $\text{cond}(\chi)$ its conductor. From classical results about twists of newforms (cf. [2]) and from the fact that for an odd prime $p$ the $p$-adic valuation of $\text{cond}(\chi)$ is $\leq 1$, we can obtain
the following. Let \( p \geq 3 \) be a prime number with \( p \mid \text{cond}(\chi) \);
- if \( p^2 \mid N' \) then \( \text{ord}_p(N) = \text{ord}_p(N') \);
- if \( p \mid N' \) or if \( p \nmid N \) then \( \text{ord}_p(N) = 2 \).

Thus, we can write \( N = MD_1^2D_2^{2k} \) and \( N' = MD_22\lambda \) where \( D_1 \) (resp. \( D_2 \))
is the product of the odd primes \( p \) such that \( p \mid \text{cond}(\chi) \) and \( p \nmid N' \) (resp.
\( p \mid \text{cond}(\chi) \) and \( p \mid N' \)), \( \lambda = \text{ord}_2(N') \), \( k = \text{ord}_2(N) \) so that \( \lambda \leq k \) and \( M, D_1, D_2 \) are odd. We can now state:

**Theorem 1.** Assume that \( E \) is the quadratic twist of \( E' \) with conductor \( N' \) such that \( \text{ord}_p(N') \leq \text{ord}_p(N) \) for all \( p \). Write \( f' = \sum_n a'_n n^{-s} = \prod_p (1 - \alpha_p' p^{-s})^{-1}(1 - \beta_p' p^{-s})^{-1} \) for the newform attached to \( E' \). Let \( N = MD_1^2D_2^{2k} \) and \( N' = MD_22\lambda \) as explained above. Then:

$$
\|f\|_N^2 = \|f'\|_{N'}^2 \frac{1}{D_1} \prod_{p \mid D_1} (p - 1)(p + 1 - a'_p)(p + 1 + a'_p)
\times \frac{1}{D_2} \prod_{p \mid D_2} (p - 1)(p + 1)
\times \begin{cases} 
2^{k-3}(3 - a'_2)(3 + a'_2) & \text{if } \lambda = 0, k \geq 4 \\
2^{k-3} \times 3 & \text{if } \lambda = 1, k \neq \lambda \\
2^{k-\lambda} & \text{if } 2 \leq \lambda \leq k \text{ or if } \lambda = k = 1.
\end{cases}
$$

**Remark:** From this theorem, it is easy to relate \( \deg(\varphi) \) with \( \deg(\varphi') \).

**Proof:** We observe that we have \( \alpha_p = \chi(p)\alpha'_p = \pm \alpha'_p \) and that the Euler product (2) for \( f \) and \( f' \) are clearly related since \( \chi^2 \) is the trivial character modulo \( \text{cond}(\chi) \). Furthermore, this Euler product allows us to give a "local" proof of the theorem. So, suppose that \( E \) is the twist of \( E' \) by a character of prime conductor \( p \geq 3 \) with \( \text{ord}_p(N') < \text{ord}_p(N) \) (if \( \text{ord}_p(N') = \text{ord}_p(N) \) then we have \( L(\text{Sym}_2^2 f', s) = L(\text{Sym}_2^2 f, s) \)). We have \( N = \text{lcm}(N', p^2) = N'p^2 \) (resp., \( = N'p \)) if \( (N', p) = 1 \) (resp., \( (N', p) = p \)).

For \( q \neq p \) the Euler factor at \( q \) of both \( L(\text{Sym}_2^2 f, s) \) and \( L(\text{Sym}_2^2 f', s) \) are the same. Since \( p^2 \mid N \) we have \( a_p = 0 \) ([1]), and there is no Euler factor at \( p \) in \( L(\text{Sym}_2^2 f, s) \).

When \( (N', p) = 1 \) (i.e. \( p \mid D_1 \)) we have:

$$
L(\text{Sym}_2^2 f, s) = L(\text{Sym}_2^2 f', s) \times (1 - \alpha_p^2 p^{-s})(1 - pp^{-s})(1 - \beta_p^2 p^{-s}).
$$

A little calculation with \( s = 2 \) shows that:

$$
\|f\|_N^2 = \|f'\|_{N'}^2 \frac{(p - 1)(p + 1 - a'_p)(p + 1 + a'_p)}{p}.
$$

When \( (N', p) = p \) (i.e. \( p \mid D_2 \)) the Euler factor of \( L(\text{Sym}_2^2 f', s) \) is equal to \((1 - p^{-s})^{-1} \) so:

$$
L(\text{Sym}_2^2 f, s) = L(\text{Sym}_2^2 f', s)(1 - p^{-s}),
$$

and \( \|f\|_N^2 = \|f'\|_{N'}^2 (p - 1)(p + 1)/p \).
The case \( p = 2 \) follows by the same argument except that there is no character of conductor 2 and so we have to deal with \( \text{cond}(\chi) = 4 \) or 8. This also explains why some cases cannot (and do not) occur in list of cases relating \( \|f\|_N \) and \( \|f'\|_{N'} \).

This theorem asserts that we only have to consider elliptic curves \( E \) which are not twists of another curve \( E' \) having a lower conductor.

3. The primitive symmetric square of \( E \)

The imprimitive symmetric square \( L(\text{Sym}^2 f, s) \) does not have a "traditional" functional equation and there is no simple method to compute \( L(\text{Sym}^2 f, 2) \) directly. Thus, we consider the primitive symmetric square \( L \)-function of \( E \), \( L(\text{Sym}^2 f, s) \):

\[
L(\text{Sym}^2 f, s) = L(\text{Sym}^1 f, s) \prod_{p \in S} L_p(\text{Sym}^1 f, p^{-s})^{-1},
\]

where the product is over the finite set \( S \) of bad primes where \( E \) has bad but potentially good reduction, in other words primes \( p \) such that \( p \mid N \) and \( \text{ord}_p(j(E)) \geq 0 \), \( j(E) \) being the \( j \)-invariant of \( E \). The properties of the primitive symmetric square function are studied in [4]. In particular, the following is proved:

**Theorem 2** (Coates-Schmidt). The function \( L(\text{Sym}^2 f, s) \) has a holomorphic continuation to the whole complex plane and there exists \( B \in \mathbb{Z} \) such that the completed function:

\[
\Lambda(\text{Sym}^2 f, s) = \left( \frac{B}{2\pi^3/2} \right)^s \Gamma(s) \Gamma\left( \frac{s}{2} \right) L(\text{Sym}^2 f, s),
\]

is entire and admits the functional equation:

\[
\Lambda(\text{Sym}^2 f, s) = \Lambda(\text{Sym}^2 f, 3 - s).
\]

**Remarks:**

1. If \( p^2 \not| N \), the Euler factor at \( p \) of the primitive and imprimitive symmetric square functions of \( E \) are the same and we have \( \text{ord}_p(B) = \text{ord}_p(N) \). In particular, if \( N \) is squarefree, then \( L(\text{Sym}^2 f, s) = L(\text{Sym}^1 f, s) \) and \( B = N \).

2. The function \( L(\text{Sym}^2 f, s) \) is invariant if we twist \( E \) by a quadratic character of \( \mathbb{Q} \). This is not true in general for the imprimitive symmetric square function.

In order to write down the correct Euler factor at \( p \mid N^2 \), we assume that \( E \) is not the quadratic twist of a curve \( E' \) of lower conductor. For the cases \( p = 2 \) and \( p = 3 \), we have the following tables coming from [4] (we should mention that two cases have been initially forgotten in [4] whenever \( 2^8 \mid N \), and that [13] corrects this mistake).
When several possibilities occur in these tables, the correct Euler factor is given by certain properties of the fields \( \mathbb{Q}_p(E_t)/\mathbb{Q}_p \). The cases \( 2^4\|N \) and \( 2^6\|N \) never appear since we assumed that \( E \) is minimal among its quadratic twists. If \( p \neq 2, 3 \) then \( \text{ord}_p(B) = 1 \) and \( L_p(\text{Sym}^2 f, X) = 1 - pX \) or \( 1 + pX \) depending on whether or not \( \mathbb{Q}_p(E_t)/\mathbb{Q}_p \) is abelian. Nevertheless, for each ambiguous case, one can find in [13] the correct Euler factor: first assume that \( p \geq 5 \). Then we have \( L_p(\text{Sym}^2 f, X) = 1 - pX \) if and only if one of the following conditions holds, where \( c_6 \) and \( c_4 \) are the classical invariants attached to \( E \):

- \( p \equiv 1 \pmod{12} \);
- \( p \equiv 5 \pmod{12} \), \( p^2 \nmid c_6 \) and \( p^2 \nmid c_4 \);
- \( p \equiv 7 \pmod{12} \) and either \( p^2 \nmid c_6 \), or \( p^2 \mid c_6 \) and \( p^2 \mid c_4 \).

For \( p = 2, 2^8\|N \) is the only ambiguous case and:

- if \( 2^9 \nmid c_6 \) then \( L_p(\text{Sym}^2 f, X) = 1 \);
- if \( 2^9 \mid c_6 \) and \( c_4 \equiv \varepsilon 32 \pmod{128} \) then \( L_p(\text{Sym}^2 f, X) = 1 + \varepsilon pX \), where \( \varepsilon = \pm 1 \).

For \( p = 3, 3^4\|N \) is the only ambiguous case and we have \( L_p(\text{Sym}^2 f, X) = 1 - pX \) when one of the two following holds:

- \( c_4 \equiv 27 \pmod{81} \);
- \( c_4 \equiv 9 \pmod{27} \) and \( c_6 \equiv \pm 108 \pmod{243} \).

### 4. Computation of \( L(\text{Sym}^2 f, s) \)

For simplicity, we write:

\[
\Lambda(\text{Sym}^2 f, s) = C^s \Gamma(s) \Gamma \left( \frac{s}{2} \right) L(\text{Sym}^2 f, s) = \gamma(s) L(\text{Sym}^2 f, s),
\]

where \( C = \frac{B}{2^{3/2} \pi^{3/2}} \) and \( L(\text{Sym}^2 f, s) = \sum_n b_n n^{-s} \). Note that the coefficients \( b_n \) are easily computable from the definitions. Furthermore, it follows from Deligne's bounds and the Euler product for \( L(\text{Sym}^2 f, s) \) that \( |b_n| \leq n^2 \). Classical estimates coming from the functional equation of

\[\text{[13]}\]

During the preparation of this paper, we were informed of the preprint [13] of M. Watkins where a similar (but not so detailed) method of computing \( \text{deg}(\varphi) \) is described and used to compute several interesting modular degrees.
\( \Lambda(\text{Sym}^2 f, s) \) give:

\[
L(\text{Sym}^2 f, 2) = \sum_{n \leq X} \frac{b_n}{n^2} + O(B^2 X^{-1}).
\]

This formula implies that the series \( \sum_{n} b_n/n^2 \) converges to \( L(\text{Sym}^2 f, 2) \).

Of course, this is not an efficient method to compute \( \|f\|_N^2 \) because the convergence is very slow. However, it easily gives us a first approximation of \( \|f\|_N^2 \).

Fortunately, a classical method for computing Dirichlet series with functional equation can be applied to our case (cf. [5], Chapter 10):

**Proposition 3.** We have:

\[
\Lambda(\text{Sym}^2 f, s) = \sum_{n \geq 1} \frac{b_n}{n^s} F(s, n) + \sum_{n \geq 1} \frac{b_n}{n^{3-s}} F(3-s, n),
\]

where

\[
F(s, x) = \gamma(s) - \int_0^x \frac{1}{2i\pi} \int_{\text{Re}(z)=\delta} t^{-z} \gamma(z) dz \ t^{s-1} dt.
\]

for all \( \delta > 0 \).

This is a rapidly convergent series since we have:

**Proposition 4.** Let \( s = \sigma + it \) and \( A = \frac{x}{2^{1/4} \zeta} \) then:

\[
|F(s, x)| \leq 3.6 \sqrt{\pi} \frac{x^\sigma}{A - \sigma A^{1/3}} e^{-\frac{\sigma}{2} A^{2/3}}.
\]

**PROOF:** We have:

\[
F(s, x) = \gamma(s) - \int_0^x \frac{1}{2i\pi} \int_{\text{Re}(z)=\delta} t^{-z} \gamma(z) dz \ t^{s-1} dt
\]

\[
= \frac{1}{2i\pi} \int_{\text{Re}(z)=\delta} \Gamma(z) \Gamma \left( \frac{z}{2} \right) \frac{dz}{z-s}.
\]

Hence,

\[
|F(s, x)| \leq \frac{1}{2\pi} \frac{C^\sigma}{\delta - \sigma} \int_{\mathbb{R}} \left| \Gamma \left( \frac{\delta + iT}{2} \right) \Gamma(\delta + iT) \right| dT.
\]

We put \( I = \int_0^\infty \left| \Gamma \left( \frac{\delta + iT}{2} \right) \Gamma(\delta + iT) \right| dT = I_1 + I_2 \) where,

\[
I_1 = \int_0^\delta \left| \Gamma \left( \frac{\delta + iT}{2} \right) \Gamma(\delta + iT) \right| dT,
\]

\[
I_2 = \int_\delta^\infty \left| \Gamma \left( \frac{\delta + iT}{2} \right) \Gamma(\delta + iT) \right| dT.
\]
The formula $\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} e^{R(s)}$, $|R(s)| \leq 1/(6|s|)$ gives the estimates:
\[
\left| \Gamma \left( \frac{\delta + iT}{2} \right) \Gamma \left( \frac{\delta}{2} + iT \right) \right| \leq \pi 2^{\frac{\delta}{2} + 1} T^{\frac{3\delta}{2} - 1} e^{-\frac{3\pi T}{4} \frac{1}{\sqrt{3\delta + \frac{1}{2}}} e^{\frac{1}{\delta}} } \quad \text{for } T > \delta,
\]
\[
\left| \Gamma \left( \frac{\delta + iT}{2} \right) \Gamma(\delta + iT) \right| \leq \pi 2^{\frac{\delta}{2} + 1} \delta^{\frac{3\delta}{2} - 1} e^{-\frac{3\delta}{2} \frac{1}{\delta^2}} e^{-\frac{1}{2} \left( \frac{T^2}{\delta} - 3 \right)} \quad \text{for } T \leq \delta.
\]

With an easy but tedious calculation, we obtain:
\[
I \leq 3.6\pi^{3/2} \delta^{\frac{3\delta - 1}{2}} e^{-\frac{3\delta}{2} 2^\delta}.
\]

Thus, we have:
\[
|F(s, x)| \leq 3.6\sqrt{\pi} \frac{x^{\sigma}}{\delta - \sigma} \left( \frac{x}{2^{1/4} C} \right)^{-\delta} \delta^{\frac{3\delta - 1}{2}} e^{-\frac{3\delta}{2} 2^\delta}.
\]

The proposition is then proved by taking $\delta = \left( \frac{x}{2^{1/4} C} \right)^{2/3}$.

This proposition allows us to estimate the tail of the series in (4) (we have $|b_n| \leq n^2$). In order to compute $F(s, x)$, we push the line of integration to the left catching all the residues of $t^{-\sigma} \gamma(z)$:

**Proposition 5.**

\[
F(s, x) = \gamma(s) - \sum_{q=0}^{\infty} x^{s+2q} \left( \frac{v_{2q} - \log(x) u_{2q}}{s + 2q} + \frac{u_{2q}}{(s + 2q)^2} + \frac{x u_{2q+1}}{s + 2q + 1} \right),
\]

with
\[
u_{2q} = \frac{2(-1)^q}{C^{2q} q!(2q)!},
\]
\[
u_{2q+1} = \frac{(-1)^q \sqrt{\pi} 2^{2q+1} q!}{(2q + 1)!^2 C^{2q+1}},
\]
\[
v_{2q} = \frac{2(-1)^q}{C^{2q} q!(2q)!} \left( \log(C) - \frac{3}{2} \gamma + \frac{1}{2} \sum_{j=1}^{q} j^{-1} + \sum_{j=1}^{2q} j^{-1} \right).
\]

It is clear that the terms in this expression can be recursively computed. In practice, we compute $N_0$ such that:
\[
\left| \sum_{n=N_0+1}^{\infty} \frac{b(n)}{n^2} F(2, n) \right| < \varepsilon \quad \text{and}
\]
\[
\left| \sum_{n=N_0+1}^{\infty} \frac{b(n)}{n} F(1, n) \right| < \varepsilon.
\]
We then compute $i_0$ terms in the series of proposition 5, where $i_0$ is the smallest integer such that (cf. [12]):

$$C^2 N_0^{-i_0-1/2} \left\lfloor \frac{i_0}{2} \right\rfloor ! i_0! > \frac{10N_0}{\pi \varepsilon}.$$ 

We thus obtain $\Lambda(\text{Sym}^2 f, 2)$ with a sufficiently high accuracy and we deduce from it the value of $\|f\|_N^2$, hence of $\deg(\varphi)$. Using this method, we can quickly compute modular degrees of strong Weil curves. As a check on the computations, we use the fact that $\deg(\varphi)$ is an integer.

**REMARK.** In fact, what is really obtained here is an algorithm to compute $L(\text{Sym}^2 f, 2)$ from which $\|f\|$ and then $\deg(\varphi)$ can be easily recovered. Nevertheless, the quantity $\|f\|$ makes sense and is also interesting in greater generality, namely for any holomorphic form $f$ of integral weight $k \geq 2$ and level $N$, not necessarily related to an elliptic curve. In this general case, one can also define $L(\text{Sym}^2 f, s)$ the primitive symmetric square $L$-function related to the $L$-function of $f$ and we have:

$$\|f\|^2 = \frac{N}{2^{k-1} \pi^k} L(\text{Sym}^2 f, k).$$

This $L$-function does have a traditional functional equation and the adaptation of the method above is possible. However, in our case ($f$ is related to an elliptic curve), computing the Euler factors involves looking at the elliptic curve whenever the reduction is additive ($p^2 \mid N$); in general, such a study is not possible and the case of non-squarefree $N$ seems not to be easy. When $N$ is squarefree the adaptation of the method is very simple since the Euler factors of $L(\text{Sym}^2 f, s)$ are all given by (2) and the functional equation is:

$$\Lambda(\text{Sym}^2 f, s) = \Lambda(\text{Sym}^2 f, 2k - 1 - s),$$

where,

$$\Lambda(\text{Sym}^2 f, s) = \left( \frac{N^s}{2^s \pi^{3s/2}} \right) \Gamma(s) \Gamma \left( \frac{s}{2} - \left\lfloor \frac{k - 1}{2} \right\rfloor \right) L(\text{Sym}^2 f, s).$$

Furthermore, when $N$ is squarefree, one can adapt the method to compute (conjecturally) special values of general symmetric powers $L(\text{Sym}^n f, k)$ since they also satisfy a traditional (and conjectural) functional equation.

5. Some estimates

From the functional equation of $L(\text{Sym}^2 f, s)$, one can show that $\|f\|_N^2 \ll \varepsilon N^{1+\varepsilon}$. In fact, $N^2$ can be replaced by a suitable power of $\log(N)$. Thus, estimates for $\text{vol}(E)$ provide upper bounds on $\deg(\varphi)$ (modulo Manin’s conjecture).
**Proposition 6.** Let $C$ be a nonnegative real number. There exist $a \in \mathbb{R}$ and $A \in \mathbb{R}$ depending on $C$ such that:

$$|j(E)| \leq C \implies a\Delta_{\text{min}}^{-1/6} < \text{vol}(E) < A\Delta_{\text{min}}^{-1/6},$$

where $\Delta_{\text{min}}$ is the discriminant of the minimal model of $E$.

**Proof:** This proposition comes from a straightforward estimate for the fundamental periods $\omega_1$ and $\omega_2$ of $E$, since we have $\text{vol}(E) = |\Im(w_1\bar{w}_2)|$.

Assuming Manin’s conjecture, we see that proposition 6 gives the upper bound $\deg(\varphi) \ll N^{1+\varepsilon}\Delta^{1/6}$ for elliptic curves with bounded $j$-invariant.

**Proposition 7.** Let $\mathcal{E}$ be an infinite family of elliptic curves defined over $\mathbb{Q}$ such that:

- $j(E)$ is bounded for $E \in \mathcal{E}$.
- $\Delta_{\text{min}}(E)$ is squarefree.

Then:

- $\deg(\varphi) \ll N^{7/6}\log(N)^3 \quad (N \to +\infty)$,
- $\deg(\varphi) \gg N^{7/6}/\log(N) \quad (N \to +\infty)$.

**Proof:** The upper bound comes from the classical estimate $L(\text{Sym}^2 f, 2) \ll \log(N)^3$ and from the fact that the Manin’s constant is bounded whenever the conductor is squarefree. The last estimate comes from the lower bound $L(\text{Sym}^2 f, 2) \gg 1/\log(N)$ for $N$ squarefree (cf. [8]).

The curves $E_k$ defined by $y^2 + xy = x^3 + k$ (with $432k^2 + k$ squarefree) give a infinite family of elliptic curves for which the conditions in the proposition hold.

The lower bound of the proposition also holds in the more general setting where the condition “$\Delta_{\text{min}}(E)$ is squarefree” is replaced by “$E$ is semi-stable (i.e. $N$ is squarefree)”.

We wrote a GP-PARI ([9]) program for computing the modular degrees using the method explained above. In the following table we give three examples for which the modular degree is very large. In each cases, $\deg(\varphi)$ was computed in a few minutes. The column $\#(a_n)$ indicates the number of coefficients $a_n$ needed (for an accuracy of $\deg(\varphi) \approx 10^{-4}$).
<table>
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<th>(a_1, a_2, a_3, a_4, a_6)</th>
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<th>(\deg(\varphi))</th>
<th>(#{a_n})</th>
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</tbody>
</table>

The last curve is in fact the quadratic twist of the curve \(E'\) with coefficients \([1, 0, 1, 120229952, -3351306510322]\) of conductor 1290. We need 5000 coefficients \(a_n\) to compute \(\deg(\varphi(E')) = 1068480\).

References


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