Equitable colorations of graphs

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Abstract. — An edge coloration of a graph is a coloration of its edges in such a way that no two edges of the same colour are adjacent. We generalize this concept by introducing the notion of equitable coloration, i.e., coloration of the edges of a graph such that if \( f_i(x) \) denotes the number of edges with colour \( i \) which are adjacent to vertex \( x \), we have \( |f_i(x) - f_j(x)| \leq 1 \) for every vertex \( x \) and every pair of colours \( i, j \). Equitable colorations are also defined for hypergraphs.

Finally some results on edge colorations are generalized to the case of equitable colorations.

1. Coloration of Hypergraphs

A hypergraph \( H = (X, U) \) consists of a finite set \( X \) of vertices \( x_1, \ldots, x_n \) and a family \( U \) of nonempty edges \( U_j (j = 1, \ldots, m) \) satisfying \( \bigcup_{j=1}^{m} U_j = X \).

A hypergraph \( H \) is unimodular if its edge incidence matrix \( A \) (\( a_{ij} = 1 \) if \( x_i \in U_j \) or 0 otherwise) is totally unimodular. The subhypergraph of \( H = (X, U) \) spanned by a subset \( Y \subset X \) is the hypergraph \( H(Y) = (Y, U(Y)) \) where \( U(Y) = \{ U_j \cap Y | U_j \cap Y \neq \emptyset \} \). An equitable \( k \)-coloration \( E \) of \( H = (X, U) \) is a partition of \( X \) into \( k \) subsets \( F_1, \ldots, F_k \) such that for every edge \( U_j \)

\[
\sum p \neq q (| U_j \cap F_p | - | U_j \cap F_q |) \leq 1 \quad \forall p, q \in \{1, \ldots, k\}
\]

The result of Camion [1] and Ghouila-Houri [2] about totally unimodular matrices may be formulated in terms of hypergraphs as follows [3][4] :

**Lemma**: A hypergraph \( H \) is unimodular if and only if all its subhypergraphs have an equitable bicoloration.

We have the following :

**Theorem 1**: A unimodular hypergraph \( H \) has an equitable \( k \)-coloration for any \( k \).

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Proof: Given a coloration \( E \) of the vertices of \( H \) with \( k \) colours (\( E \) is not necessarily an equitable \( k \)-coloration), for each edge \( U_j \) we define a vector \( E(j) = (f^j_1, f^j_2, ..., f^j_k) \) where \( f^j_p \) is the number of vertices of \( U_j \) which have colour \( p \). Let \( F_p \subset X \) be the subset of vertices which have colour \( p \). For every edge let \( e(j) = \max (f^j_p - f^j_q) \geq 0 \); let \( e^* = \max e(j) \). If \( e^* < 2 \), \( E \) is an equitable \( k \)-coloration of \( H \). Otherwise, let \( U_j \) be an edge such that \( e(j) = e^* = f^j_p - f^j_q \). We consider the subgraph \( H' \) spanned by \( F_p \cup F_q \). It follows from the lemma that \( H' \) has an equitable 2-coloration \( E' \); we colour its vertices with 2 colours \( p \) and \( q \) in such a way that \( |f^j_p - f^j_q| \leq 1 \) for every \( U_j \). The values \( f^j_i \) are unchanged for \( r \neq p, q \) and for every \( U_j \). Thus at least one value \( e(j) \) is such that the number of pairs \( p, q \) with \( |f^j_p - f^j_q| \leq e(j) - 1 \) has increased by at least one unit and the other \( e(j) \) have not increased. This procedure can be repeated until \( e^* < 2 \). We get thus an equitable \( k \)-coloration of \( H \). End of proof.

A transversal of a hypergraph \( H = (X, U) \) is a subset of vertices \( T \) such that \( T \cap U_j \neq \emptyset \) for \( j = 1, ..., m \). The following corollary is a slight generalization of a theorem in Berge [3].

Corollary 1: Let \( H = (X, U) \) be a unimodular hypergraph and \( k = \min \sum_j |U_j| \) the minimal cardinality of its edges. The set \( X \) of vertices of \( H \) may be partitioned into \( k \) transversals.

Proof: Consider an equitable \( k \)-coloration of \( H \) where \( k = \min \sum_j |U_j| \); such a \( k \)-coloration exists from theorem 1. Clearly in each edge there will be at least one vertex of each colour. Hence the subsets \( F_1, ..., F_k \) defined by the \( k \)-coloration are transversals.

Following Berge [3], we call strong chromatic number of \( H = (X, U) \) the smallest integer \( k \) such that there exists a partition of \( X \) into subsets \( F_1, ..., F_k \) with \( |F_i \cap U_j| \leq 1 \) \( i = 1, ..., k \). The next corollary is due to Berge [5].

Corollary 2: The strong chromatic number of a unimodular hypergraph is equal to the maximal cardinality of its edges.

Proof: Let \( k = \max \sum_j |U_j| \) and consider an equitable \( k \)-coloration of \( H \); let \( F_i \) be the set of vertices with colour \( i \) for \( i = 1, ..., k \). Obviously

\[
|F_i \cap U_j| \leq 1 \quad i = 1, ..., k
\]

\[
|F_i \cap U_j| \leq 1 \quad j = 1, ..., m
\]

We can also apply theorem 1 to graphs; an equitable \( k \)-coloration of a graph is then a coloration of its edges with \( k \) colours such that for each vertex \( x \), we have:

\[
|f_p(x) - f_q(x)| \leq 1 \forall p, q \in \{1, ..., k\}
\]
where \( f_p(x) \) denotes the number of edges with colour \( p \) which are adjacent to \( x \).

**Corollary 3** : A bipartite graph \( G = (X, U) \) has an equitable \( k \)-coloration for any \( k \).

**Proof** : This result is obtained by applying theorem 1 to the hypergraph \( H \) obtained as follows : its vertices are the edges of \( G \) and its edges are the sets of edges which are adjacent to the same vertex of \( G \). \( H \) is unimodular since its edge incidence matrix is the transposed matrix of the edge incidence matrix of \( G \).

When applied to the case of graphs, corollary 1 becomes the theorem of Gupta [3] : If \( G = (X, U) \) is a bipartite graph with minimum degree \( k \), then there exists a partition of \( U \) into \( k \) spanning subsets of edges \( H_1, \ldots, H_k \). (\( H_i \) is a spanning subset if the edges in \( H_i \) meet all vertices of \( G \).)

Moreover corollary 2 gives the well-known result : the chromatic index of a bipartite graph is equal to the maximum degree of the vertices in \( G \) (the chromatic index of \( G \) is by definition the smallest \( k \) such that the edges of \( G \) may be partitioned into \( k \) subsets of nonadjacent edges).

2. \( p \)-bounded colorations

We will now generalize some results about edge colorations. A \( p \)-bounded \( k \)-coloration \( E \) of a graph \( G \) is a partition of its edges into \( k \) nonempty subsets \( F_1, \ldots, F_k \) such that for any vertex \( x \) : \( |f_j(x) - f_i(x)| \leq p \) for \( i, j = 1, \ldots, k \) where \( f_j(x) \) is the number of edges of \( F_j \) which are adjacent to \( x \). An equitable \( k \)-coloration is thus a 1-bounded \( k \)-coloration. Let \( E = \{ F_1, \ldots, F_k \} \) be a \( p \)-bounded \( k \)-coloration and \( f_1 \geq \ldots \geq f_k \) the cardinalities of \( F_1, \ldots, F_k \) respectively.

**Theorem 2** : If the sequence \( (f_1, \ldots, f_k) \) corresponds to a \( p \)-bounded \( k \)-coloration of \( G \), then any sequence \( f'_1, \ldots, f'_k \) with :

\[
\begin{align*}
  a) & \quad f'_1 \geq \ldots \geq f'_k \\
  b) & \quad \sum_{i=1}^l f'_i \leq \sum_{i=1}^l f_i \quad l = 1, \ldots, k - 1 \\
  c) & \quad \sum_{i=1}^k f'_i = \sum_{i=1}^k f_i
\end{align*}
\]

corresponds to a \( p \)-bounded \( k \)-coloration of \( G \).

**Proof** : A) We first prove that any couple of subsets \( F_i, F_j \) in \( E \) with \( f_i - f_j = K \geq 2 \) may be replaced by two subsets \( F'_i, F'_j \) with \( f'_i - f'_j = K - 2 \). \( E_{ij} = (F_i, F_j) \) is a \( p \)-bounded bicolouration of \( G_{ij} = (X, F_i \cup F_j) \); we consider any edge \( u \) in \( G_{ij} \) and construct an alternating path \( P \) containing \( u \) (i.e., the
edges of which belong alternately to $F_i$ and $F_j$; we extend the path $P$ as far as possible; we obtain thus either an alternating circuit (with even length) or an alternating open path. We remove $P$ from $G_{ij}$ and repeat the same construction with another edge $u$, until all edges in $G_{ij}$ are removed.

Since $f_i - f_j = K \geq 2$, there is at least one alternating path $P$ in which the first edge and last edge belong to $F_i$; we interchange the edges of $P \cap F_i$ and $P \cap F_j$.

Let $x$ and $y$ be the endpoints of $P$. Since $P$ terminates at $x$ with an edge in $F_i$ we have $f_i(x) \geq f_j(x) + 1$; by interchanging the edges of $P$ we get

$$f_j(x) \leq f_i'(x) = f_i(x) - 1 \leq f_i(x)$$

$$f_j(x) \leq f_j'(x) = f_j(x) + 1 \leq f_i(x)$$

The same inequalities hold for $y$. Furhtermore, for all vertices $z \neq x, y$ we have $f_i'(z) = f_i(z)$ and $f_j'(z) = f_j(z)$. So we obtain a new $p$-bounded bicoloration $(F'_i, F'_j)$ with $f'_i - f'_j = K - 2$.

B) By successive applications of the above described procedure we can obtain $p$-bounded $k$-colorations corresponding to any sequence $(f'_1, ..., f'_k)$ satisfying $a), b)$ and $c)$. This ends the proof.

Theorem 2 is a generalization of a result which appears in Folkman and Fulkerson [6]. (Their theorem corresponds to the case where $p = 1$ and $k$ is at least equal to the chromatic index of $G$.) We raise now and answer the following question: given a graph $G$, what is the smallest value $p$ such that $G$ has a $p$-bounded $k$-coloration for any $k$? From corollary 3, we know that if $G$ is bipartite, then the minimum value of $p$ is $p = 1$. If $G$ is not bipartite, it is not the case: a triangle has for instance no equitable 2-coloration. (Clearly for any $k$ not less than the chromatic index of $G$, there is a 1-bounded $k$-coloration of $G$.)

**Théorème 3:** Let $G$ be any graph; for any $k$, $G$ has a $2$-bounded $k$-coloration.

**Proof:** The theorem is true for a graph $G$ with one edge. Suppose that it is true for graphs with at most $m - 1$ edges; we will show that it is also true for graphs with $m$ edges. Let $G$ be a graph with $m$ edges; let us remove from $G$ an edge $u$ joining vertices $x$ and $y$. By our induction hypothesis, $G' = G - u$ has a $2$-bounded $k$-coloration for any $k$. Given some integer $k$, let $F_1, ..., F_k$ be the subsets of edges defined by such a $k$-coloration of $G'$.

There exist 2 integers $a, b \geq 0$ such that

$$a \leq f_i(x) \leq a + 2 \quad \text{for } i = 1, ..., k$$

$$b \leq f_i(y) \leq b + 2 \quad \text{for } i = 1, ..., k$$

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We can assume that there is at least one colour, say \( q \), such that \( f_q(x) = a \) (otherwise \( a \) is replaced by \( a + 1 \)); similarly there is one colour \( r \) such that \( f_r(y) = b \). We have to examine the following cases:

A) There is a colour \( s \) with \( f_s(x) < a + 2 \) and \( f_s(y) < b + 2 \). Then \( u \) may be introduced into \( F_s \) and \( F_1, ..., F_k \) is a 2-bounded \( k \)-coloration of \( G \).

B) For every colour \( s \) with \( f_s(x) < a + 2 \) we have \( f_s(y) = b + 2 \) and for every colour \( t \) with \( f_t(y) < b + 2 \) we have \( f_t(x) = a + 2 \). Let us consider colours \( q \) and \( r \); we have \( q \neq r \) (otherwise we are in case \( A \)).

We determine an alternating chain \( C \) starting at \( x \) with an \( r \)-edge (i.e., an edge in \( F_r \)) and whose edges are alternately \( r \)-edges and \( q \)-edges. We extend chain \( C \) as far as possible. Then 2 cases may occur:

B1) The last vertex in \( C \) is \( y \); so the last edge in \( C \) is a \( q \)-edge (because if we arrive at \( y \) with an \( r \)-edge we can introduce one more \( q \)-edge into \( C \) since \( f_q(y) = b + 2 > f_r(y) = b \)). By interchanging the \( q \)-edges and the \( r \)-edges in \( C \) we obtain a 2-bounded \( k \)-coloration of \( G' \) with \( f_q(x) = f_r(x) = a + 1 \) and \( f_q(y) = f_r(y) = b + 1 \). So \( u \) may be introduced into \( F_q \) (or \( F_r \)) and \( F_1, ..., F_k \) is a 2-bounded \( k \)-coloration of \( G \).

B2) The last vertex in \( C \) is \( z \neq y \). Again by interchanging the \( q \)-edges and the \( r \)-edges in \( C \) we obtain a 2-bounded \( k \)-coloration of \( G' \) with \( f_r(x) = a + 1 \), \( f_q(y) = b \) (if \( C \) ends for instance with a \( q \)-edge we have \( f_q(z) + 2 \geq f_q(z) > f_r(z) \) and after having interchanged the \( r \)-edges and the \( q \)-edges, we still have \( |f_r(z) - f_q(z)| \leq 2 \)).

We can now introduce edge \( u \) into \( F_r \) and we still obtain a 2-bounded \( k \)-coloration of \( G \).

We have examined all possible cases and the proof is completed.

We now define an odd cycle as a connected graph containing an odd number of edges and such that all degrees are even.

**Theorem 4**: A connected graph \( G \) has an equitable bicoloration if and only if it is not an odd cycle.

**Proof**: A) Suppose \( G \) is an odd cycle; for any equitable bicoloration \( \{ F_1, F_2 \} \) we must have \( f_1(x) = f_2(x) \) at each vertex \( x \). Hence, \( F_1 \) and \( F_2 \) have the same cardinality; but this is not possible since \( G \) contains an odd number of edges.

B) Conversely if \( G \) is not an odd cycle, then from Euler's theorem, the edges of \( G \) may be partitioned into a unique even cycle (if all degrees are even) or into one or more chains joining 2 vertices with odd degrees. By coloring the edges in each chain (or in the unique cycle if all degrees are even) alternately with colours 1 and 2 we obtain an equitable bicoloration of \( G \).

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Necessary and sufficient conditions for a graph $G$ to have an equitable $k$-coloration ($k > 2$) are much more difficult to obtain (this would in fact solve the four color problem). However we can formulate:

**Proposition**: If in a connected graph $G$ all degrees are multiples of $k$ and if the number of edges is not a multiple of $k$, then $G$ has no equitable $k$-coloration.

Proof as in theorem 4, A.

However even if all degrees and the number of edges in a connected graph $G$ are multiples of $k$, $G$ may not have an equitable $k$-coloration for $k > 2$.

**REFERENCES**


