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## GENERALIZED FUNCTIONAL EQUATION OF TWO VARIABLES IN INFORMATION THEORY\*

BHU DEV SHARMA et RAM AUTAR (1)

**Abstract.** — *Daróczy* [1] generalized the functional equation given by *Kendall* [4] as

$$f(x) + (1-x)^\beta f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^\beta f\left(\frac{x}{1-y}\right), \beta > 0, \beta \neq 1 \quad (0.1)$$

and introduced the concept of information functions of type- $\beta$  which are the solutions of the functional equation (0.1) under suitable boundary conditions. Authors [9] have formed the functional equation of two variables

$$f(x_1; y_1) + (1-x_1)f\left(\frac{x_2}{1-x_1}; \frac{y_2}{1-y_1}\right) = f(x_2; y_2) + (1-x_2)f\left(\frac{x_1}{1-x_2}; \frac{y_1}{1-y_2}\right) \quad (0.2)$$

This is a generalization of *Kendall's* functional equation in two variables. The solutions of (0.2) are inaccuracy functions under the conditions

$$f(0; 0) = f(1; 1); f(1/2; 1/2) = 1. \quad (0.3)$$

In this paper we have generalized the functional equation (0.2) by introducing two parameters  $\beta$  and  $\gamma$  such that  $\beta, \gamma > 0$  and  $\beta \neq 1$  when  $\gamma = 1$  as

$$\begin{aligned} f(x_1; y_1) + (1-x_1)^\gamma (1-y_1)^{\beta-\gamma} f\left(\frac{x_2}{1-x_1}; \frac{y_2}{1-y_1}\right) \\ = f(x_2; y_2) + (1-x_2)^\gamma (1-y_2)^{\beta-\gamma} f\left(\frac{x_1}{1-x_2}; \frac{y_1}{1-y_2}\right). \end{aligned} \quad (0.4)$$

This equation is a generalization of *Daróczy* equation (0.1) in more than one way because it is a functional equation in two variables and involves two parameters  $\beta$  and  $\gamma$ . The solutions of this functional equation under boundary conditions (0.3) are defined as the inaccuracy functions of type  $(\beta, \gamma)$ . These inaccuracy functions of type  $(\beta, \gamma)$  give rise to new measure of inaccuracy  $H_n^{(\beta, \gamma)}(P; Q)$  of type  $(\beta, \gamma)$ . Also when  $\gamma = 1$  and  $\beta \rightarrow 1$  then (0.4) is same as (0.2). Some properties of  $H_n^{(\beta, \gamma)}(P; Q)$  are studied.

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## I. INTRODUCTION

Authors [9] have earlier formed a functional equation of two variables given by

$$\begin{aligned} f(x_1; y_1) + (1 - x_1)f\left(\frac{x_2}{1 - x_1}; \frac{y_2}{1 - y_1}\right) \\ = f(x_2; y_2) + (1 - x_2)f\left(\frac{x_1}{1 - x_2}; \frac{y_1}{1 - y_2}\right) \end{aligned} \quad (1.1)$$

where  $x_1, y_1, x_2, y_2, \in [0,1)$ ,  $x_1 + x_2 \leq 1$  and  $y_1 + y_2 \leq 1$ .

Solutions of this functional equation under the boundary conditions

$$f(0; 0) = f(1; 1) \quad , \quad f\left(\frac{1}{2}; \frac{1}{2}\right) = 1 \quad (1.2)$$

are defined as inaccuracy functions.

The functional equation (1.1) generalizes the functional equation

$$f(x) + (1 - x)f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)f\left(\frac{x}{1 - y}\right) \quad (1.3)$$

for all  $(x, y)$  such that  $x, y \in [0,1)$  and  $x + y \leq 1$ , given by Kendall [4] and we note that  $f(x; x) = f(x)$  is the Kendall's information function.

Daróczy [1] generalized the functional equation (1.3) as

$$f(x) + (1 - x)^\beta f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)^\beta f\left(\frac{x}{1 - y}\right) \quad , \quad \beta > 0 \quad (1.4)$$

and obtained that under the boundary conditions

$$f(1) = f(0); f\left(\frac{1}{2}\right) = 1 \quad (1.5)$$

$$f(x) = [x^\beta + (1 - x)^\beta - 1](2^{1-\beta} - 1)^{-1} \quad , \quad \beta \neq 1 \quad (1.6)$$

which according to Daróczy is information function of type- $\beta$ .

When  $\beta \rightarrow 1$ , under the regularity condition

$$f(x) \rightarrow -x \log_2 x - (1 - x) \log_2 (1 - x) \quad (1.7)$$

which is Shannon's [7] information function [4].

For a probability distribution  $P = (p_1, \dots, p_n)$ ,  $\sum_{i=1}^n p_i = 1$ , the entropy of type- $\beta$  is defined as

$$H_n^\beta(p_1, \dots, p_n) = \sum_{i=2}^n s_i^\beta f\left(\frac{p_i}{s_i}\right) \quad (1.8)$$

where  $s_i = p_1 + p_2 + \dots + p_i$ ,  $i = 2, 3, \dots, n$  and  $f$  is an information function (1.6).

A generalization of (1.4) in two variables studied by Sharma [8] is

$$\begin{aligned} f(x_1; y_1) + (1 - x_1)(1 - y_1)^{\beta-1} f\left(\frac{x_2}{1 - x_1}; \frac{y_2}{1 - y_1}\right) \\ = f(x_2; y_2) + (1 - x_2)(1 - y_2)^{\beta-1} f\left(\frac{x_1}{1 - x_2}; \frac{y_1}{1 - y_2}\right) \end{aligned} \quad (1.9)$$

where  $0 \leq x_1, y_1, x_2, y_2 < 1$ ,  $x_1 + x_2 \leq 1$  and  $y_1 + y_2 \leq 1$ .

This contains only one parameter  $\beta (> 0)$ . In the present paper we introduce a generalization including two parameters  $\beta > 0$  and  $\gamma > 0$ . This generalization contains all earlier ones as particular cases.

In section 2 of this paper we shall give generalized functional equation in two variables and define both inaccuracy function and measure of inaccuracy of type  $(\beta, \gamma)$ .

We shall study some properties of new measure of inaccuracy of type  $(\beta, \gamma)$  in section 3. In section 4, we shall define the joint and marginal measures of inaccuracy of type  $(\beta, \gamma)$  and discuss their properties.

## 2. GENERALIZED FUNCTIONAL EQUATION IN TWO VARIABLES

In what follows, a generalized functional equation would mean,

$$\begin{aligned} f(x_1; y_1) + (1 - x_1)^\gamma (1 - y_1)^{\beta-\gamma} f\left(\frac{x_2}{1 - x_1}; \frac{y_2}{1 - y_1}\right) \\ = f(x_2; y_2) + (1 - x_2)^\gamma (1 - y_2)^{\beta-\gamma} f\left(\frac{x_1}{1 - x_2}; \frac{y_1}{1 - y_2}\right) \end{aligned} \quad (2.1)$$

where  $\beta, \gamma > 0$ ,  $x_1, y_1, x_2, y_2 \in [0, 1)$ ,  $x_1 + x_2 \leq 1$  and  $y_1 + y_2 \leq 1$ . Here  $\beta$  and  $\gamma$  are two parameters. It would be noted that if  $x_1 = y_1$ ,  $x_2 = y_2$  and  $f(x; x) = f(x)$  then (2.1) gives rise to (1.4) for all values of  $\gamma$ .

### Definitions

I. An Inaccuracy function  $f(x; y)$  of two real variables  $x$  and  $y$  where  $x, y \in (0,1)$  of type  $(\beta, \gamma)$  is defined to be a function satisfying the functional equation (2.1) and the boundary conditions (1.2).

II. Measure of Inaccuracy of type  $(\beta, \gamma)$  of the discrete probability distribution  $Q = (q_1, \dots, q_n)$  with respect to probability distribution  $P = (p_1, \dots, p_n)$  where  $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$ , obtained from the inaccuracy function  $f$  of type  $(\beta, \gamma)$  is defined by the quantity

$$H_n^{(\beta, \gamma)}(P; Q) = \sum_{i=2}^n s_i^\gamma t_i^{\beta - \gamma} f\left(\frac{p_i}{s_i}; \frac{q_i}{t_i}\right) \quad (2.2)$$

where  $s_i = p_1 + p_2 + \dots + p_i$ ,  $t_i = q_1 + q_2 + \dots + q_i$  and  $i = 2, 3, \dots, n$ .

Elsewhere we have given two characterizations of inaccuracy function of type  $(\beta, \gamma)$  [10] under the specified conditions.

The forms of the functions obtained are

$$f(x; y) = K^{(\beta, \gamma)}(x; y) = [x^\gamma y^{\beta - \gamma} + (1 - x)^\gamma (1 - y)^{\beta - \gamma} - 1](2^{1 - \beta} - 1)^{-1} \quad (2.3)$$

for all  $x, y \in [0, 1]$ ,  $\beta > 0$ ,  $\gamma > 0$  and  $\beta \neq 1$  when  $\gamma = 1$ .

Also when  $\beta = \gamma = 1$ , we have

$$f(x; y) = K(x; y) = \lim_{\beta \rightarrow 1} K^{(\beta, 1)}(x; y) = -x \log_2 y - (1 - x) \log_2 (1 - y).$$

This is Kerridge's [5] Inaccuracy function.

### 3. NEW MEASURES OF INACCURACY

Let  $X$  be a random variate assuming the values  $x_1, \dots, x_n$ . If an experimenter asserts that the probability of the  $i$ th event is  $q_i$  while  $p_i$  is its true probability, then by definition II, the Measure of Inaccuracy of type  $(\beta, \gamma)$  is obtained by the quantity

$$H_n^{(\beta, \gamma)}(P; Q) = (2^{1 - \beta} - 1)^{-1} \left( \sum_{i=1}^n p_i^\gamma q_i^{\beta - \gamma} - 1 \right) \quad (3.1)$$

where  $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1$ ,  $\beta, \gamma > 0$  and  $\beta \neq 1$ .

If  $\gamma \neq 1$ ,  $\gamma > 0$  and  $\beta \rightarrow 1$ , then  $H_n^{(\beta, \gamma)}(P; Q)$  is in general an infinite quantity.

Further if  $\gamma = 1$  and  $\beta \rightarrow 1$  then

$$H(P; Q) = \lim_{\beta \rightarrow 1} H_n^{(\beta, 1)}(P; Q) = - \sum_{i=1}^n p_i \log_2 q_i \tag{3.2}$$

which is Measure of Inaccuracy defined by Kerridge.

If  $p_i = q_i$  for each  $i$ , then we have

$$H_n^\beta(P) = (2^{1-\beta} - 1)^{-1} \left( \sum_{i=1}^n p_i^\beta - 1 \right) \tag{3.3}$$

which is entropy of type- $\beta$  defined by Daróczy [1] and Havrada and Charuat [3]. This, in the limiting case when  $\beta \rightarrow 1$ , gives Shannon's entropy for a complete probability distribution  $P = (p_1, \dots, p_n)$ .

It would be observed that

$$H_n^{(\beta, \gamma)}(P; Q) = 0 \tag{3.4}$$

if  $p_i = q_i = 1$  for one value of  $i = k$  and consequently  $p_i = q_i = 0$  for  $i \neq k$  provided  $\beta > 0$  and  $\beta \neq 1$ , which implies that in the absence of inaccuracy there is a correct statement made with complete certainty.

In the following theorem we prove some properties of  $H_n^{(\beta, \gamma)}(P; Q)$ .

**Theorem 1.** The measure of inaccuracy  $H_n^{(\beta, \gamma)}(P; Q)$  has the following properties :

(i) *Symmetric*:  $H_n^{(\beta, \gamma)}(P; Q)$  is a symmetric function of its arguments provided the same probabilities  $p_i$  and  $q_i$  correspond, that is

$$\begin{aligned} H_n^{(\beta, \gamma)}(p_1, \dots, p_{n-1}, p_n; q_1, \dots, q_{n-1}, q_n) \\ = H_n^{(\beta, \gamma)}(p_n, p_1, \dots, p_{n-1}; q_n, q_1, \dots, q_{n-1}). \end{aligned}$$

(ii) *Normalised*:  $H_2^{(\beta, \gamma)}\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\right) = 1$ .

(iii) *Expansible* :

$$H_{n+1}^{(\beta, \gamma)}(p_1, \dots, p_n, 0; q_1, \dots, q_n, 0) = H_n^{(\beta, \gamma)}(p_1, \dots, p_n; q_1, \dots, q_n).$$

(iv) *Recursive type* ( $\beta, \gamma$ ) :

$$\begin{aligned} H_n^{(\beta, \gamma)}(p_1, \dots, p_n; q_1, \dots, q_n) - H_{n-1}^{(\beta, \gamma)}(p_1 + p_2, p_3, \dots, p_n; q_1 + q_2, q_3, \dots, q_n) \\ = (p_1 + p_2)^\gamma (q_1 + q_2)^{\beta-\gamma} H_2^{(\beta, \gamma)}\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) \end{aligned} \tag{3.5}$$

(v) *Strongly additive type*  $(\beta, \gamma)$ :

$$\begin{aligned}
 & H_{mn}^{(\beta, \gamma)}(p_1 p_{11}, \dots, p_1 p_{1n}, \dots, p_m p_{m1}, \dots, p_m p_{mn}; \\
 & \qquad \qquad \qquad q_1 q_{11}, \dots, q_1 q_{1n}, \dots, q_m q_{m1}, \dots, q_m q_{mn}) \\
 & = H_m^{(\beta, \gamma)}(p_1, \dots, p_m; q_1, \dots, q_m) \\
 & \qquad \qquad \qquad + \sum_{j=1}^m p_j^\gamma q_j^{\beta-\gamma} H_n^{(\beta, \gamma)}(p_{j1}, \dots, p_{jn}; q_{j1}, \dots, q_{jn}) \quad (3.6)
 \end{aligned}$$

for all  $(p_1, \dots, p_m)$  and  $(q_1, \dots, q_m) \in \Delta_m$ ;  $(p_{j1}, \dots, p_{jn})$  and  $(q_{j1}, \dots, q_{jn}) \in \Delta_n$  where  $\Delta_m = \left\{ (p_1, \dots, p_m) : p_j \geq 0, \sum_{j=1}^m p_j = 1 \right\}$ .

*Proof.* The properties (i) to (iii) are obvious and can be verified easily. We, however prove (iv) and (v) by direct computation.

$$\begin{aligned}
 \text{(iv)} \quad & H_n^{(\beta, \gamma)}(p_1, p_2, \dots, p_n; q_1, q_2, \dots, q_n) \\
 & \quad - H_{n-1}^{(\beta, \gamma)}(p_1 + p_2, p_3, \dots, p_n; q_1 + q_2, q_3, \dots, q_n) \\
 & = (2^{1-\beta} - 1)^{-1} \left[ \left( \sum_{i=1}^n p_i^\gamma q_i^{\beta-\gamma} - 1 \right) - (p_1 + p_2)^\gamma (q_1 + q_2)^{\beta-\gamma} \right. \\
 & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. - \sum_{i=3}^n p_i^\gamma q_i^{\beta-\gamma} + 1 \right] \\
 & = (2^{1-\beta} - 1)^{-1} (p_1 + p_2)^\gamma (q_1 + q_2)^{\beta-\gamma} \\
 & \quad \left[ \left( \frac{p_1}{p_1 + p_2} \right)^\gamma \left( \frac{q_1}{q_1 + q_2} \right)^{\beta-\gamma} + \left( \frac{p_2}{p_1 + p_2} \right)^\gamma \left( \frac{q_2}{q_1 + q_2} \right)^{\beta-\gamma} - 1 \right] \\
 & = (p_1 + p_2)^\gamma (q_1 + q_2)^{\beta-\gamma} H_2^{(\beta, \gamma)} \\
 & \qquad \qquad \qquad \left( \frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2} \right).
 \end{aligned}$$

(v) We have

$$\begin{aligned}
 & H_m^{(\beta, \gamma)}(p_1, \dots, p_m; q_1, \dots, q_m) + \sum_{j=1}^m p_j^\gamma q_j^{\beta-\gamma} H_n^{(\beta, \gamma)}(p_{j1}, \dots, p_{jn}; q_{j1}, \dots, q_{jn}) \\
 & = (2^{1-\beta} - 1)^{-1} \left[ \left( \sum_{i=1}^m p_i^\gamma q_i^{\beta-\gamma} - 1 \right) + \sum_{j=1}^m p_j^\gamma q_j^{\beta-\gamma} \left( \sum_{i=1}^n p_{ji}^\gamma q_{ji}^{\beta-\gamma} - 1 \right) \right] \\
 & = (2^{1-\beta} - 1)^{-1} \left[ \sum_{j=1}^m \sum_{i=1}^n (p_j p_{ji})^\gamma (q_j q_{ji})^{\beta-\gamma} - 1 \right] \\
 & = H_{mn}^{(\beta, \gamma)}(p_1 p_{11}, \dots, p_1 p_{1n}, \dots, p_m p_{m1}, \dots, p_m p_{mn}; \\
 & \qquad \qquad \qquad q_1 q_{11}, \dots, q_1 q_{1n}, \dots, q_m q_{m1}, \dots, q_m q_{mn}).
 \end{aligned}$$

**Theorem 2.** Let  $P_1 = (p_{11}, p_{12}, \dots, p_{1n}) \in \Delta_n$  and  $P_2 = (p_{21}, p_{22}, \dots, p_{2m}) \in \Delta_m$  and similar notations for  $Q_1$  and  $Q_2$ . Define

$$P_1 * P_2 = (p_{11}p_{21}, \dots, p_{11}p_{2m}; \dots; p_{1n}p_{21}, \dots, p_{1n}p_{2m}) \tag{3.7}$$

then for  $\beta > 0, \gamma > 0$  and  $\beta \neq 1$ , we have

$$H_{mn}^{(\beta, \gamma)}(P_1 * P_2; Q_1 * Q_2) = H_m^{(\beta, \gamma)}(P_1; Q_1) + H_m^{(\beta, \gamma)}(P_2; Q_2) + (2^{1-\beta} - 1)H_n^{(\beta, \gamma)}(P_1; Q_1)H_m^{(\beta, \gamma)}(P_2; Q_2) \tag{3.8}$$

$$\text{and } H_{mn}^{(\beta, \gamma)}(P_1 * P_2; Q_1 * Q_2) = H_n^{(\beta, \gamma)}(P_1; Q_1) + H_m^{(\beta, \gamma)}(P_2; Q_2) \tag{3.9}$$

if  $\beta \rightarrow 1$  while  $\gamma > 0$ .

*Proof.*

$$\begin{aligned} H_{mn}^{(\beta, \gamma)}(P_1 * P_2; Q_1 * Q_2) &= \left[ \sum_{j=1}^m \sum_{i=1}^n (p_{1i}p_{2j})^\gamma (q_{1i}q_{2j})^{\beta-\gamma} - 1 \right] (2^{1-\beta} - 1)^{-1} \\ &= \left[ \left( \sum_{i=1}^n p_{1i}^\gamma q_{1i}^{\beta-\gamma} \right) \left( \sum_{j=1}^m p_{2j}^\gamma q_{2j}^{\beta-\gamma} \right) - 1 \right] (2^{1-\beta} - 1)^{-1} \\ &= [ \{ (2^{1-\beta} - 1)H_n^{(\beta, \gamma)}(P_1; Q_1) + 1 \} \\ &\quad \{ (2^{1-\beta} - 1)H_m^{(\beta, \gamma)}(P_2; Q_2) + 1 \} ] \times (2^{1-\beta} - 1)^{-1} \\ &= H_n^{(\beta, \gamma)}(P_1; Q_1) + H_m^{(\beta, \gamma)}(P_2; Q_2) \\ &\quad + (2^{1-\beta} - 1)H_n^{(\beta, \gamma)}(P_1; Q_1)H_m^{(\beta, \gamma)}(P_2; Q_2). \end{aligned}$$

When  $\beta \rightarrow 1$ , then last term vanishes and we have

$$H_{mn}^{(\beta, \gamma)}(P_1 * P_2; Q_1 * Q_2) = H_n^{(\beta, \gamma)}(P_1; Q_1) + H_m^{(\beta, \gamma)}(P_2; Q_2)$$

and when  $\beta \neq 1$ , then

$$H_{mn}^{(\beta, \gamma)}(P_1 * P_2; Q_1 * Q_2) \geq H_n^{(\beta, \gamma)}(P_1; Q_1) + H_m^{(\beta, \gamma)}(P_2; Q_2)$$

according as

$$(2^{1-\beta} - 1)H_n^{(\beta, \gamma)}(P_1; Q_1)H_m^{(\beta, \gamma)}(P_2; Q_2) \geq 0. \tag{3.10}$$

**Theorem 3.** For  $(p_1, \dots, p_n) \in \Delta_n$ ,  $(q_{1i}, \dots, q_{mi})$  and  $(q_1, \dots, q_m) \in \Delta_m$ ,  $i = 1, 2, \dots, n$ , we have

$$H_m^{(\beta, \gamma)} \left( \sum_{i=1}^n p_i q_{1i}, \dots, \sum_{i=1}^n p_i q_{mi} ; q_1, \dots, q_m \right) \geq \sum_{i=1}^n p_i H_m^{(\beta, \gamma)}(q_{1i}, \dots, q_{mi} ; q_1, \dots, q_m) \quad (3.11)$$

where  $\beta$  and  $\gamma$  are positive numbers such that  $\beta \neq 1$  and  $\gamma < 1$ .

The inequality is reversed if  $\gamma > 1$ .

*Proof.* We have

$$\begin{aligned} &H_m^{(\beta, \gamma)} \left( \sum_{i=1}^n p_i q_{1i}, \dots, \sum_{i=1}^n p_i q_{mi} ; q_1, \dots, q_m \right) \\ &= (2^{1-\beta} - 1)^{-1} \left[ \sum_{k=1}^m \left( \sum_{i=1}^n p_i q_{ki} \right)^\gamma q_k^{\beta-\gamma} - 1 \right] \\ &\geq (2^{1-\beta} - 1)^{-1} \left[ \sum_{k=1}^m \sum_{i=1}^n p_i q_{ki}^\gamma q_k^{\beta-\gamma} - 1 \right] \text{ if } \gamma < 1 \text{ [2].} \\ &= (2^{1-\beta} - 1)^{-1} \left( \sum_{i=1}^n p_i \right) \left[ \sum_{k=1}^m q_{ki}^\gamma q_k^{\beta-\gamma} - 1 \right] \\ &= \sum_{i=1}^n p_i H_m^{(\beta, \gamma)}(q_{1i}, \dots, q_{mi} ; q_1, \dots, q_m). \end{aligned}$$

Since  $\left( \sum_{i=1}^n p_i q_{ki} \right)^\gamma \leq \sum_{i=1}^n p_i q_{ki}^\gamma$  if  $\gamma > 1$  [2], therefore, the inequality in (3.11) is reversed if  $\gamma > 1$ .

**Theorem 4.** Let  $\beta$  and  $\gamma$  be the positive numbers such that  $\beta \neq 1$ . For  $n \geq N + 1$  where  $N \geq 2$ ,  $(p_1, \dots, p_n)$  and  $(q_1, \dots, q_n) \in \Delta_n$ , we have

$$\begin{aligned} &H_n^{(\beta, \gamma)}(p_1, \dots, p_n ; q_1, \dots, q_n) - H_{n-N+1}^{(\beta, \gamma)} \left( \sum_{i=1}^N p_i, p_{N+1}, \dots, p_n ; \sum_{i=1}^N q_i, q_{N+1}, \dots, q_n \right) \\ &= \left( \sum_{i=1}^N p_i \right)^\gamma \left( \sum_{i=1}^N q_i \right)^{\beta-\gamma} H_N^{(\beta, \gamma)} \left( \frac{p_1}{\sum_{i=1}^N p_i}, \dots, \frac{p_N}{\sum_{i=1}^N p_i} ; \frac{q_1}{\sum_{i=1}^N q_i}, \dots, \frac{q_N}{\sum_{i=1}^N q_i} \right) \end{aligned} \quad (3.12)$$

$$\geq \left( \sum_{i=1}^N p_i^\gamma \right) \left( \sum_{i=1}^N q_i^{\beta-\gamma} \right) H_N^{(\beta,\gamma)} \left( \frac{p_1}{\sum_{i=1}^N p_i}, \dots, \frac{p_N}{\sum_{i=1}^N p_i}; \frac{q_1}{\sum_{i=1}^N q_i}, \dots, \frac{q_N}{\sum_{i=1}^N q_i} \right) \quad (3.13)$$

according as

$$\gamma > 1, \beta - \gamma > 1 \text{ or } \gamma < 1, \beta - \gamma < 1. \quad (3.14)$$

*Proof.* The theorem is a generalization of property (iv) of the theorem I. We have

$$\begin{aligned} & H_n^{(\beta,\gamma)}(p_1, \dots, p_n; q_1, \dots, q_n) \\ & \quad - H_{n-N+1}^{(\beta,\gamma)} \left( \sum_{i=1}^N p_i; p_{N+1}, \dots, p_n; \sum_{i=1}^N q_i, q_{N+1}, \dots, q_n \right) \\ & = (2^{1-\beta} - 1)^{-1} \left[ \left( \sum_{i=1}^n p_i^\gamma q_i^{\beta-\gamma} - 1 \right) \right. \\ & \quad \left. - \left\{ \left( \sum_{i=1}^N p_i \right)^\gamma \left( \sum_{i=1}^N q_i \right)^{\beta-\gamma} + \sum_{i=N+1}^n p_i^\gamma q_i^{\beta-\gamma} - 1 \right\} \right] \\ & = (2^{1-\beta} - 1)^{-1} \left[ \sum_{i=1}^n p_i^\gamma q_i^{\beta-\gamma} - \left( \sum_{i=1}^N p_i \right)^\gamma \left( \sum_{i=1}^N q_i \right)^{\beta-\gamma} \right] \\ & = (2^{1-\beta} - 1)^{-1} \left( \sum_{i=1}^N p_i \right)^\gamma \left( \sum_{i=1}^N q_i \right)^{\beta-\gamma} \left[ \sum_{i=1}^N \left( \frac{p_i}{\sum_{i=1}^N p_i} \right)^\gamma \left( \frac{q_i}{\sum_{i=1}^N q_i} \right)^{\beta-\gamma} - 1 \right] \\ & = \left( \sum_{i=1}^N p_i \right)^\gamma \left( \sum_{i=1}^N q_i \right)^{\beta-\gamma} H_N^{(\beta,\gamma)} \left( \frac{p_1}{\sum_{i=1}^N p_i}, \dots, \frac{p_N}{\sum_{i=1}^N p_i}; \frac{q_1}{\sum_{i=1}^N q_i}, \dots, \frac{q_N}{\sum_{i=1}^N q_i} \right) \end{aligned}$$

Now to prove the inequalities we recall [2] that

$$\left( \sum_{i=1}^N a_i \right)^\gamma \geq \left( \sum_{i=1}^N a_i^\gamma \right) \text{ according to as } \gamma \geq 1.$$

From this inequality, we have

$$\left( \sum_{i=1}^N p_i \right)^\gamma \geq \left( \sum_{i=1}^N p_i^\gamma \right) \text{ according to as } \gamma \geq 1 \quad (3.15)$$

and

$$\left( \sum_{i=1}^N q_i \right)^{\beta-\gamma} \geq \left( \sum_{i=1}^N q_i^{\beta-\gamma} \right) \text{ according to as } \beta - \gamma \geq 1. \quad (3.16)$$

Combining (3.15) and (3.16) with (3.12) we get the inequalities (3.13). It may be seen that inequalities can also be obtained by combining either (3.15) or (3.16) with (3.12).

**Corollaries :**

$$(i) \quad H_n^{(\beta, \gamma)}(p_1, \dots, p_n; q_1, \dots, q_n) - H_2^{(\beta, \gamma)}\left(\sum_{i=1}^{n-1} p_i, p_n; \sum_{i=1}^{n-1} q_i, q_n\right) \\ = \left(\sum_{i=1}^{n-1} p_i\right)^\gamma \left(\sum_{i=1}^{n-1} q_i\right)^{\beta-\gamma} H_{n-1}^{(\beta, \gamma)}\left(\frac{p_1}{\sum_{i=1}^{n-1} p_i}, \dots, \frac{p_{n-1}}{\sum_{i=1}^{n-1} p_i}; \frac{q_1}{\sum_{i=1}^{n-1} q_i}, \dots, \frac{q_{n-1}}{\sum_{i=1}^{n-1} q_i}\right) \quad (3.17)$$

$$\geq \left(\sum_{i=1}^{n-1} p_i^\gamma\right) \left(\sum_{i=1}^{n-1} q_i^{\beta-\gamma}\right) H_{n-1}^{(\beta, \gamma)}\left(\frac{p_1}{\sum_{i=1}^{n-1} p_i}, \dots, \frac{p_{n-1}}{\sum_{i=1}^{n-1} p_i}; \frac{q_1}{\sum_{i=1}^{n-1} q_i}, \dots, \frac{q_{n-1}}{\sum_{i=1}^{n-1} q_i}\right) \quad (3.18)$$

according as (3.14).

$$(ii) \quad H_n^{(\beta, \gamma)}\left(\frac{p_{11}}{p_{21}}, \dots, \frac{p_{1n}}{p_{2n}}; p_{11}, \dots, p_{1n}\right) - H_2^{(\beta, \gamma)}\left(\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}, \frac{p_{1n}}{p_{2n}}; \sum_{i=1}^{n-1} p_{1i}, p_{1n}\right) \\ = \left(\sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}\right)^\gamma \left(\sum_{i=1}^{n-1} p_{1i}\right)^{\beta-\gamma} H_{n-1}^{(\beta, \gamma)}\left(\frac{p_{11}}{p_{21}} \mid \sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}, \dots, \frac{p_{1, n-1}}{p_{2, n-1}} \mid \sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}; \right. \\ \left. \frac{p_{11}}{\sum_{i=1}^{n-1} p_{1i}}, \dots, \frac{p_{1, n-1}}{\sum_{i=1}^{n-1} p_{1i}}\right) \quad (3.19)$$

$$\geq \left[\sum_{i=1}^{n-1} \left(\frac{p_{1i}}{p_{2i}}\right)^\gamma\right] \left[\sum_{i=1}^{n-1} p_{1i}^{\beta-\gamma}\right] H_{n-1}^{(\beta, \gamma)}\left(\frac{p_{11}}{p_{21}} \mid \sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}, \dots, \frac{p_{1, n-1}}{p_{2, n-1}} \mid \sum_{i=1}^{n-1} \frac{p_{1i}}{p_{2i}}; \right. \\ \left. \frac{p_{11}}{\sum_{i=1}^{n-1} p_{1i}}, \dots, \frac{p_{1, n-1}}{\sum_{i=1}^{n-1} p_{1i}}\right) \quad (3.20)$$

according as (3.14).

**4. MEASURES OF INACCURACY OF TYPE  $(\beta, \gamma)$   
FOR BIVARIATE DISTRIBUTIONS**

Let  $X$  and  $Y$  be two discrete finite random variables assuming the values  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_m)$  respectively. We define the measure of Inaccuracy of type  $(\beta, \gamma)$  of the probability distribution  $Q(X)$  with respect to probability distribution  $P(X)$  by

$$H_n^{(\beta, \gamma)}[P(X); Q(X)] = H_n^{(\beta, \gamma)}[(p_1, \dots, p_n; q_1, \dots, q_n)] \tag{4.1}$$

where  $p_i = P(X = x_i)$ ,  $q_i = Q(X = x_i)$ ,  $i = 1, 2, \dots, n$  and

$$\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 1.$$

Similarly for  $Y$ , we define

$$H_m^{(\beta, \gamma)}[P(Y); Q(Y)] = H_m^{(\beta, \gamma)}(p'_1, \dots, p'_m; q'_1, \dots, q'_m) \tag{4.2}$$

where

$$p'_j = P(Y = y_j) \quad , \quad q'_j = Q(Y = y_j) \quad , \quad j = 1, 2, \dots, m,$$

and  $\sum_{j=1}^m p'_j = \sum_{j=1}^m q'_j = 1.$

Next if  $p(x_i, y_j)$  and  $q(x_i, y_j)$  are the joint probability of  $(x_i, y_j)$  for the distribution  $P$  and  $Q$  of  $(X, Y)$  respectively, then the joint Inaccuracy of type  $(\beta, \gamma)$  of  $Q(X, Y)$  with respect to  $P(X, Y)$  is defined as

$$H_{mn}^{(\beta, \gamma)} \left[ \begin{matrix} P(X, Y) \\ Q(X, Y) \end{matrix} \right] = H_{mn}^{(\beta, \gamma)} [p(x_1, y_1), \dots, p(x_1, y_m), \dots, p(x_n, y_1), \dots, p(x_n, y_m); q(x_1, y_1), \dots, q(x_1, y_m), \dots, q(x_n, y_1), \dots, q(x_n, y_m)] \tag{4.3}$$

such that  $\sum_{i=1}^n \sum_{j=1}^m p(x_i, y_j) = \sum_{i=1}^n \sum_{j=1}^m q(x_i, y_j) = 1.$

Further since  $P(Y/x_i)$  and  $Q(Y/x_i)$  are complete probability distributions of  $Y$ , given  $x_i$ , we may define

$$H_m^{(\beta, \gamma)}[P(Y/x_i); Q(Y/x_i)] = (2^{1-\beta} - 1)^{-1} \left[ \sum_{j=1}^m p^\gamma(y_j/x_i) q^{\beta-\gamma}(y_j/x_i) - 1 \right] \tag{4.4}$$

And now the marginal inaccuracy of  $Y$  given  $X$  can be defined as

$$H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] = \sum_{i=1}^n p^\gamma(x_i) q^{\beta-\gamma}(x_i) H_m^{\beta-\gamma}[P(Y/x_i); Q(Y/x_i)] \tag{4.5}$$

Similarly the marginal inaccuracy of  $X$  given  $Y$  is defined by

$$H_n^{(\beta, \gamma)}[P(X/Y); Q(X/Y)] = \sum_{j=1}^m p^\gamma(y_j) q^{\beta-\gamma}(y_j) H_n^{(\beta, \gamma)}[P(X/y_j); Q(X/y_j)] \quad (4.6)$$

It may be noted that if  $p_i = q_i$  for each  $i$ , then we get the corresponding entropies of type  $\beta$  [1] and in the limiting case  $\beta \rightarrow 1$ , these give corresponding Shannon's entropies for the bivariate case.

**Theorem 5.** Let  $\beta$  and  $\gamma$  be positive numbers with  $\beta \neq 1$ , then

$$\begin{aligned} H_{mn}^{(\beta, \gamma)}[P(X, Y); Q(X, Y)] &= H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] \\ &= H_m^{(\beta, \gamma)}[P(Y); Q(Y)] + H_n^{(\beta, \gamma)}[P(X/Y); Q(X/Y)] \end{aligned} \quad (4.7)$$

$$(4.8)$$

*Proof.* By the definitions given above, we have

$$\begin{aligned} &H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] \\ &= (2^{1-\beta} - 1)^{-1} \left[ \left\{ \sum_{i=1}^n p^\gamma(x_i) q^{\beta-\gamma}(x_i) - 1 \right\} \right. \\ &\quad \left. + \sum_{i=1}^n p^\gamma(x_i) q^{\beta-\gamma}(x_i) \left\{ \sum_{j=1}^m p^\gamma(y_j/x_i) q^{\beta-\gamma}(y_j/x_i) - 1 \right\} \right] \\ &= (2^{1-\beta} - 1)^{-1} \left[ \sum_{i=1}^n \sum_{j=1}^m p^\gamma(x_i, y_j) q^{\beta-\gamma}(x_i, y_j) - 1 \right] \end{aligned}$$

as  $p(x_i, y_j) = p(x_i)p(y_j/x_i)$  and  $q(x_i, y_j) = q(x_i)q(y_j/x_i)$

$$= H_{mn}^{(\beta, \gamma)}[P(X, Y); Q(X, Y)].$$

The other result can similarly be proved.

**Corollary 1.** When  $X$  and  $Y$  are statistically independent discrete random variables, then

$$\begin{aligned} H_{mn}^{(\beta, \gamma)}[P(X, Y); Q(X, Y)] &= H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \\ &\quad + (2^{1-\beta} - 1) H_n^{(\beta, \gamma)}[P(X); Q(X)] H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \end{aligned} \quad (4.9)$$

*Proof.* From above theorem,

$$\begin{aligned} H_{mn}^{(\beta, \gamma)}[P(X, Y); Q(X, Y)] &= H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] \\ &= H_n^{(\beta, \gamma)}[P(X); Q(X)] \\ &\quad + \sum_{i=1}^n p^\gamma(x_i) q^{\beta-\gamma}(x_i) \left[ \sum_{j=1}^m p^\gamma(y_j) q^{\beta-\gamma}(y_j) - 1 \right] (2^{1-\beta} - 1)^{-1} \end{aligned}$$

as  $X$  and  $Y$  are statistically independent.

$$\begin{aligned}
 &= H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \left[ \sum_{i=1}^n p^\gamma(x_i) q^{\beta-\gamma}(x_i) \right] \\
 &= H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \\
 &\quad + (2^{1-\beta} - 1) H_n^{(\beta, \gamma)}[P(X); Q(X)] H_m^{(\beta, \gamma)}[P(Y); Q(Y)].
 \end{aligned}$$

When  $\beta \rightarrow 1$ , the last term in (4.9) vanishes and we have

$$H_{mn}^{(\beta, \gamma)}[P(X, Y); Q(X, Y)] = H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y); Q(Y)]. \quad (4.10)$$

In case  $\beta \neq 1$ , we have the inequalities,

$$H_{mn}^{(\beta, \gamma)}[P(X, Y); Q(X, Y)] \geq H_n^{(\beta, \gamma)}[P(X); Q(X)] + H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \quad (4.11)$$

according as

$$(2^{1-\beta} - 1) H_n^{(\beta, \gamma)}[P(X); Q(X)] H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \geq 0 \quad (4.12)$$

REMARKS : These results correspond to Shannon's result  $H(X; Y) \leq H(X) + H(Y)$ , etc.

**Corollary 2.** We have

$$\begin{aligned}
 H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] &= H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \\
 &\quad + (2^{1-\beta} - 1) H_n^{(\beta, \gamma)}[P(X); Q(X)] H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \quad (4.13)
 \end{aligned}$$

and  $H_n^{(\beta, \gamma)}[P(X/Y); Q(X/Y)] = H_n^{(\beta, \gamma)}[P(X); P(X)]$

$$+ (2^{1-\beta} - 1) H_n^{(\beta, \gamma)}[P(X); Q(X)] H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \quad (4.14)$$

if  $X$  and  $Y$  are statistically independent.

*Proof.* By definitions we have

$$\begin{aligned}
 &H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] \\
 &= \sum_{i=1}^n p^\gamma(x_i) q^{\beta-\gamma}(x_i) \left[ \sum_{j=1}^m p^\gamma(y_j/x_i) q^{\beta-\gamma}(y_j/x_i) - 1 \right] (2^{1-\beta} - 1)^{-1} \\
 &= \sum_{i=1}^n p^\gamma(x_i) q^{\beta-\gamma}(x_i) \left[ \sum_{j=1}^m p^\gamma(y_j) q^{\beta-\gamma}(y_j) - 1 \right] (2^{1-\beta} - 1)^{-1}
 \end{aligned}$$

as  $X$  and  $Y$  are statistically independent.

$$\begin{aligned}
&= H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \{ (2^{1-\beta} - 1)H_n^{(\beta, \gamma)}[P(X); Q(X)] + 1 \} \\
&= H_m^{(\beta, \gamma)}[P(Y); Q(Y)] + (2^{1-\beta} - 1)H_n^{(\beta, \gamma)}[P(X); Q(X)]H_m^{(\beta, \gamma)}[P(Y); Q(Y)]
\end{aligned}$$

and similarly we can prove (4.14).

When  $\beta \rightarrow 1$ , the last terms in (4.13) and (4.14) vanish and we have

$$H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] = H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \quad (4.15)$$

$$\text{and} \quad H_n^{(\beta, \gamma)}[P(X/Y); Q(X/Y)] = H_n^{(\beta, \gamma)}[P(X); Q(X)] \quad (4.16)$$

Furthermore if  $\beta \neq 1$ , then

$$H_m^{(\beta, \gamma)}[P(Y/X); Q(Y/X)] \geq H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \quad (4.17)$$

$$\text{and} \quad H_n^{(\beta, \gamma)}[P(X/Y); Q(X/Y)] \geq H_n^{(\beta, \gamma)}[P(X); Q(X)] \quad (4.18)$$

$$\text{according as } (2^{1-\beta} - 1)H_n^{(\beta, \gamma)}[P(X); Q(X)]H_m^{(\beta, \gamma)}[P(Y); Q(Y)] \geq 0. \quad (4.19)$$

REMARKS. These results correspond to Shannon's results

$$H(X) \geq H(X/Y) \text{ and } H(Y) \geq H(Y/X).$$

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