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LOCAL H^{-1} GALERKIN AND ADJOINT LOCAL H^{-1} GALERKIN PROCEDURES FOR ELLIPTIC EQUATIONS (*)

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Abstract. — *Two essentially dual, finite element methods for approximating the solution of the boundary value problem* $Lu = \nabla \cdot (a \nabla u) + b \cdot \nabla u + cu = f$ *on* Ω, *a rectangle, with* $u = 0$ *on* $\partial\Omega$ *are shown to give optimal order convergence. The local* H^{-1} *method is based on the inner product* $(u, L^* v)$ *and the adjoint method on* (Lu, v). *Discontinuous spaces can be employed for the approximate solution in the local* H^{-1} *procedure and for the test space in the adjoint method.*

1. INTRODUCTION

Consider the elliptic boundary value boundary problem

$$\left.\begin{aligned}(Lu)(p) = \nabla \cdot (a(p)\nabla u) + b(p) \cdot \nabla u + c(p)u = f(p), \qquad & p \in \Omega,\\ u(p) = 0, \qquad p \in \partial\Omega, \qquad & \end{aligned}\right\} \quad (1)$$

where Ω is the square $I \times I$ and $I = (0, 1)$. We assume that a, $(\nabla a)_i$, b_i, $c \in C^1(\overline{\Omega})$, that $f \in L_2(\Omega)$, and that $0 < a_0 \leqq a(p) \leqq a_1$, $p \in \overline{\Omega}$, where a_0 and a_1 are constants. We further assume that, given $g \in L_2(\Omega)$, there exists a unique function $\varphi \in H^2(\Omega)$ satisfying $L\varphi = g$ in Ω and $\varphi = 0$ on $\partial\Omega$.

We shall use the following notation. Let $\delta : 0 = x_0 < x_1 < \ldots < x_N = 1$ be a partition of $[0, 1]$. Set $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, and $h = \max_{1 \leqq j \leqq N} h_j$. For $E \subset I$ let $P_r(E)$ denote the functions defined on I whose restrictions to E coincide with polynomials of degree at most r. Let

$$\mathscr{M}(-1, r, \delta) = \bigcap_{j=1}^{N} P_r(I_j)$$

and, for k a non-negative integer,

$$\mathscr{M}(k, r, \delta) = \mathscr{M}(-1, r, \delta) \cap C^k(I),$$

$$\mathscr{M}^0(k, r, \delta) = \mathscr{M}(k, r, \delta) \cap \{v \mid v(0) = v(1) = 0\},$$

$$\tilde{\mathscr{M}}(k-1, r-1, \delta) = \{v' : v \in \mathscr{M}^0(k, r, \delta)\}.$$

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We assume that δ is quasi-uniform and that $r \geqq 1$. For brevity, we set

$$\mathcal{N} = \mathcal{M}^0(k+2, r+2, \delta) \otimes \mathcal{M}^0(k+2, r+2, \delta),$$
$$\mathcal{Q} = \tilde{\mathcal{M}}(k+1, r+1, \delta) \otimes \tilde{\mathcal{M}}(k+1, r+1, \delta),$$

and

$$\mathcal{M} = \mathcal{M}(k, r, \delta) \otimes \mathcal{M}(k, r, \delta).$$

Note that $\mathcal{Q}$ and $\mathcal{M}$ are the images of $\mathcal{N}$ under the maps given by $\partial^2/\partial x\, \partial y$ and $\partial^4/\partial x^2\, \partial y^2$, respectively.

The local H^{-1} Galerkin approximation is defined as the solution $U \in \mathcal{M}$ of the equations

$$(U, L^* \varphi) = (f, \varphi), \qquad \varphi \in \mathcal{N}, \tag{2}$$

where the inner product is the standard $L_2(\Omega)$ one. The adjoint local H^{-1} Galerkin approximation is given by $W \in \mathcal{N}$ satisfying

$$(LW, \varphi) = (f, \varphi), \qquad \varphi \in \mathcal{M}. \tag{3}$$

We first show that there exists a unique U and a unique W satisfying (2) and (3), respectively, for $L = \Delta$. Optimal L_2 error estimates are also obtained for the operator Δ. We then generalize our results to obtain optimal L_2 results for operators of the form given in (1).

Let $H^k(\Omega)$ be the Sobolev space of functions having $L_2(\Omega)$-derivatives through order k. Denote the usual norm on $H^s(\Omega)$ by $\|\cdot\|_s$; for $s = 0$ the subscript will be omitted. We also use the norm

$$\|w\|_{-1} = \sup_{z \in H^1(\Omega)} \frac{(w, z)}{\|z\|_1}.$$

If the reader wishes to use any of the results derived below for non-integral indices, then standard interpolation theory [3] should be applied.

2. ERROR ESTIMATES FOR $L = \Delta$

First note that, since $\dim \mathcal{M} = \dim \mathcal{N}$, uniqueness implies existence.

LEMMA 1 : *Suppose that $V \in \mathcal{M}$ satisfies*

$$(V, \Delta\varphi) = 0, \qquad \varphi \in \mathcal{N}.$$

Then, $V \equiv 0$.

Proof : Note that there exists a unique $Q \in \mathcal{N}$ such that $Q_{xxyy} = V$. Integrating by parts, we have

$$(\nabla Q_{xy}, \nabla w) = 0, \qquad w \in \mathcal{Q}.$$

Since $Q_{xy} \in \mathcal{Q}$, we note that $Q_{xxy} = 0$ and $Q_{yyx} = 0$. Thus, $V = 0$.

Since the matrix arising in (3) is the adjoint of that of (2), there exists a unique W satisfying (3) for $L = \Delta$.

We now derive L_2 and negative norm error estimates for $U-u$ when $L = \Delta$. Let $Z \in \mathcal{N}$ satisfy $Z_{xxyy} = U$. Also let $z_{xxyy} = u$ in Ω and $z = 0$ on $\partial\Omega$. We observe from (1) and (2) with $\xi = Z-z$ that

$$(\nabla \xi_{xy}, \nabla w) = 0, \qquad w \in \mathcal{Q}. \tag{4}$$

THEOREM 1 : *Let z and Z be as defined above, and let $z_{xy} \in H^s(\Omega)$ for some s such that* $1 \leqq s \leqq r+2$. Then,

$$||(z-Z)_{xy}|| + h\,||(z-Z)_{xy}||_1 \leqq C\,||z_{xy}||_s h^s.$$

Proof : It follows from (4) that

$$||\nabla \xi_{xy}|| = \inf_{\chi \in \mathcal{Q}} ||\nabla(z_{xy}-\chi)||. \tag{5}$$

Let $T : H^1(I) \to \mathcal{M}(k+1, r+1, \delta)$ be determined by the relations

$$\int_0^1 (g-Tg)'\, v\, dx = \int_0^1 (g-Tg)\, dx = 0, \qquad v \in \mathcal{M}(k, r, \delta).$$

It is easy to see that $(g-Tg)(0) = (g-Tg)(1) = 0$, by taking $v = x$ or $1-x$. Since $(Tg)'$ is the $L_2(I)$-projection of g' into $\mathcal{M}(k, r, \delta)$,

$$||(g-Tg)'||_{L_2(I)} \leqq C\,||g^{(s)}||_{L_2(I)} h^{s-1}, \qquad 1 \leqq s \leqq r+2.$$

Let

$$-\varphi'' = \zeta = g-Tg, \qquad x \in I,$$
$$\varphi'(0) = \varphi'(1) = 0,$$
$$\int_0^1 \varphi\, dx = 0.$$

Then for $v \in \mathcal{M}(k, r, \delta)$ appropriately chosen

$$||\zeta||^2 = (\zeta', \varphi'-v) \leqq C\,||\zeta'||_{L_2(I)}||\zeta||_{L_2(I)} h,$$

and

$$||g-Tg||_{L_2(I)} \leqq C\,||g^{(s)}||_{L_2(I)} h^{(s)}, \qquad 1 \leqq s \leqq r+2.$$

Consider $(T \otimes T)\, z_{xy} \in \mathcal{M}(k+1, r+1, \delta) \otimes \mathcal{M}(k+1, r+1, \delta)$. It is easy to see that $(T \otimes T)\, z_{xy} \in \mathcal{Q}$ and that

$$||z_{xy}-(T \otimes T)\, z_{xy}||_q \leqq C\,||z_{xy}||_s h^{s-q}, \qquad 2 \leqq s \leqq r+2, \quad 0 \leqq q \leqq 1, \tag{6}$$

since $T \otimes T - I \otimes I = (T-I) \otimes I + I \otimes (T-I) + (T-I) \otimes (T-I)$. Thus, from (5) and (6),

$$||\nabla \xi_{xy}|| \leqq C\,||z_{xy}||_s h^{s-1}, \qquad 2 \leqq s \leqq r+2.$$

The inequality

$$||\nabla \xi_{xy}|| \leqq C\,||\nabla z_{xy}|| \leqq C\,||z_{xy}||_1,$$

is obvious, and the desired result follows:

$$\| \nabla \xi_{xy} \| \leqq C \| z_{xy} \|_s h^{s-1}, \qquad 1 \leqq s \leqq r+2.$$

Since ξ_{xy} has average value zero,

$$\| \xi_{xy} \|_1 \leqq C \| z_{xy} \|_s h^{s-1}, \qquad 1 \leqq s \leqq r+2.$$

To obtain the $L_2(\Omega)$ estimate, first let

$$-\Delta\varphi = \xi_{xy}, \qquad (x, y) \in \Omega,$$

$$\frac{\partial\varphi}{\partial n} = 0, \qquad (x, y) \in \partial\Omega.$$

Since $(\xi_{xy}, 1) = 0$, there exists φ such that $(\varphi, 1) = 0$ and $\| \varphi \|_2 \leqq C \| \xi_{xy} \|$. Then,

$$\| \xi_{xy} \|^2 = (\nabla \xi_{xy}, \nabla(\varphi - \chi)), \qquad \chi \in \mathscr{Q},$$

and

$$\| \xi_{xy} \|^2 \leqq C \| \nabla \xi_{xy} \| \inf_{\chi \in \mathscr{Q}} \| \nabla(\varphi - \chi) \|.$$

The function ξ_{xy} can be expanded in a double cosine series:

$$\xi_{xy} = \sum_{p,q=1}^{\infty} c_{pq} \cos \pi px \cos \pi qy.$$

Thus,

$$\varphi = \frac{1}{\pi^2} \sum_{p,q=1}^{\infty} \frac{c_{pq}}{p^2+q^2} \cos \pi px \cos \pi qy.$$

It then follows by approximating each product of cosines in $\mathscr{Q}$ that

$$\inf_{\chi \in \mathscr{Q}} \| \nabla(\varphi - \chi) \| \leqq Ch \| \xi_{xy} \|,$$

and the theorem has been proved.

Denote by P the restriction of the projection T to the subclass of $H^1(I)$ consisting of functions having zero average value. Let $\mathscr{P} = P \otimes P$.

We wish to obtain a better H^1 estimate of $v = \mathscr{P} z_{xy} - Z_{xy}$ than would follow from (6) and theorem 1. We deduce from (4) that

$$(\nabla v, \nabla w) = (\nabla(\mathscr{P} z_{xy} - z_{xy}), \nabla w) = \tau_x + \tau_y, \qquad w \in \mathscr{Q}. \tag{7}$$

Using the definition of P and integration by parts, we see that, for $w \in \mathscr{Q}$,

$$\begin{aligned} \tau_x &= (((I \otimes P)(P \otimes I) z_{xy} - z_{xy})_x, w_x) \\ &= (I \otimes (P - I) z_{xxy}, w_x) \\ &= -(I \otimes (P-I) z_{xxxy}, w) \\ &\quad + \int_0^1 I \otimes (P - I) z_{xxy}(., y) w(., y) \big|_0^1 dy. \end{aligned} \tag{8}$$

Note that z has the representation

$$
\begin{aligned}
z(x, y) = & \int_0^y \int_0^x (x-\alpha)(y-\beta) u(\alpha, \beta)\, d\alpha\, d\beta \\
& - x \int_0^y \int_0^1 (1-\alpha)(y-\beta) u(\alpha, \beta)\, d\alpha\, d\beta \\
& - y \int_0^1 \int_0^x (x-\alpha)(1-\beta) u(\alpha, \beta)\, d\alpha\, d\beta \\
& + xy \int_0^1 \int_0^1 (1-\alpha)(1-\beta) u(\alpha, \beta)\, d\alpha\, d\beta. \qquad (9)
\end{aligned}
$$

One can easily verify from (9) that the boundary terms in (8) are zero since $z_{xxy}(0, y) = 0$ and $z_{xxy}(1, y) = 0$. We also observe that

$$\int_0^1 z_{xxxy}\, dy = z_{xxx}(x, 1) - z_{xxx}(x, 0) = 0,$$

since z vanishes on the boundary. Similarly, $\int_0^1 z_{yyyx}\, dx = 0$. Thus, we see that

$$\|v\|_1 \leqq C \|\psi\|_{-1}, \qquad (10)$$

where

$$\psi = I \otimes (I-P)(z_{xxxy}) + (I-P) \otimes I(z_{xyyy}). \qquad (11)$$

It follows that

$$
\begin{aligned}
\|\psi\|_{-1} \leqq & \left(\int_0^1 \left\| I \otimes (I-P) \frac{\partial^4 z}{\partial x^3 \partial y}(x, .) \right\|^2_{H^{-1}(I)} dx \right)^{1/2} \\
& + \left(\int_0^1 \left\| (I-P) \otimes I \frac{\partial^4 z}{\partial x \partial y^3}(., y) \right\|^2_{H^{-1}(I)} dy \right)^{1/2}.
\end{aligned}
$$

It is easy to show that

$$\|(I-P) f\|_{H^{-1}(I)} \leqq C \|f^{(s)}\|_{L^2(I)} h^{s+1},$$

provided that

$$\int_0^1 f\, dx = 0,$$

by using the auxiliary problem

$$-\varphi'' = g - \int_0^1 g\, dx, \qquad x \in I,$$

$$\varphi'(0) = \varphi'(1) = 0,$$

where $g \in H^1(I)$. Thus,

$$\|\psi\|_{-1} \leqq C\left\{\left\|\frac{\partial^{s+1} u}{\partial x^s \partial y}\right\| + \left\|\frac{\partial^{s+1} u}{\partial x \partial y^s}\right\|\right\} h^{s+2} \tag{12}$$

for $0 \leqq s \leqq r+1$.

THEOREM 2 : *Let u be the solution to* (1) *with* $L = \Delta$, *and let* $U \in \mathcal{M}$ *satisfy* (2). *Let* $\hat{U}$ *be the* L_2 *projection of u into* $\mathcal{M}$. *Then,*

$$\|U - \hat{U}\| \leqq C\left\{\left\|\frac{\partial^{s+1} u}{\partial x^s \partial y}\right\| + \left\|\frac{\partial^{s+1} u}{\partial x \partial y^s}\right\|\right\} h^{s+1} \tag{13}$$

for $0 \leqq s \leqq r+1$.

Proof : Since $\hat{U}$ satisfies

$$(\hat{U} - u, v) = 0, \qquad v \in \mathcal{M},$$

one can easily verify that

$$\hat{U} = (\mathcal{P} z_{xy})_{xy}.$$

Thus, (13) follows from (10), (12), and the quasi-uniformity hypothesis on the partition δ.

COROLLARY : *The error* $U-u$ *satisfies the following bounds:*

$$\|U - u\| \leqq C\|u\|_s h^s, \qquad 1 \leqq s \leqq r+1,$$

$$\|U - u\|_{L_\infty(\Omega)} \leqq C\left\{\|u\|_{W^s_\infty(\Omega)} + \left\|\frac{\partial^{s+1} u}{\partial x^s \partial y}\right\| + \left\|\frac{\partial^{s+1} u}{\partial x \partial y^s}\right\|\right\} h^s.$$

$$0 \leqq s \leqq r+1.$$

Proof: The $L_2(\Omega)$-estimate is a trivial consequence of (13). To obtain the $L_\infty(\Omega)$-estimate, note first that (13) and the quasi-uniformity of δ imply that, for $0 \leqq s \leqq r+1$,

$$\|U - \hat{U}\|_{L_\infty(\Omega)} \leqq \|v\|_{W^2_\infty(\Omega)} \leqq C\left\{\left\|\frac{\partial^{s+1} u}{\partial x^s \partial y}\right\| + \left\|\frac{\partial^{s+1} u}{\partial x \partial y^s}\right\|\right\} h^s.$$

It follows from inequality (28) of [2] or from [1] that

$$\|u - \hat{U}\|_{L_\infty(\Omega)} \leqq C\|u\|_{W^s_\infty(\Omega)} h^s, \qquad 0 \leqq s \leqq r+1.$$

We now wish to consider the adjoint local H^{-1} Galerkin procedure for $L = \Delta$. As noted earlier, there exists a unique $W \in \mathcal{N}$ satisfying

$$(\Delta W, v) = (f, v), \qquad v \in \mathcal{M}. \tag{14}$$

THEOREM 3: *Let u be the solution to* (1) *with* $L = \Delta$ *and assume that* $u_{xy} \in H^s(\Omega)$, $1 \leqq s \leqq r+2$. *Let* $W \in \mathcal{N}$ *be defined by* (14). *Then,*

$$\|(W-u)_{xy}\| + h\|(W-u)_{xy}\|_1 \leqq C\|u_{xy}\|_s h^s.$$

Proof: Just as in (4),

$$(\nabla (W-u)_{xy}, \nabla w_{xy}) = 0, \qquad w \in \mathcal{N}$$

Since w_{xy} represents an arbitrary element of $\mathcal{Q}$, the theorem follows from the analysis of (4) given in the proof of theorem 1.

Next, we shall derive an $H^1(\Omega)$-estimate of the error $W-u$. Note that

$$\begin{aligned} \|\nabla (W-u)\|^2 &= -(\Delta(W-u),\ W-u) \\ &= -(\Delta(W-u),\ W-u-\chi), \qquad \chi \in \mathcal{M}. \end{aligned} \tag{15}$$

We choose $\chi \in \mathcal{M}$ as the local H^{-1} Galerkin approximation to $W-u$; i. e.,

$$(W-u-\chi,\ \Delta\varphi) = 0, \qquad \varphi \in \mathcal{N}. \tag{16}$$

By the corollary to theorem 2,

$$\| W-u-\chi \| \leqq C \| W-u \|_1 h.$$

From (15) and (16), we see that

$$\|\nabla (W-u)\|^2 = -(W-u-\chi,\ \Delta(W-u-\mu)), \qquad \mu \in \mathcal{N}.$$

Hence,

$$\begin{aligned} \|\nabla (W-u)\|^2 &\leqq Ch \| W-u \|_1 \inf_{\mu \in \mathcal{N}} \| u-\mu \|_2 \\ &\leqq Ch^{s+1} \| W-u \|_1 \| u \|_{s+2}, \qquad 0 \leqq s \leqq r+1. \end{aligned}$$

Since the boundary values of u were imposed strongly on the elements of $\mathcal{N}$, the $L_2(\Omega)$-norm of the $\nabla(W-u)$ is equivalent to the $H^1(\Omega)$-norm of $W-u$; thus,

$$\| W-u \|_1 \leqq C \| u \|_{s+2} h^{s+1}, \qquad 0 \leqq s \leqq r+1.$$

As a result of the quasi-uniformity of δ, it follows easily that

$$\| W-u \|_2 \leqq C \| u \|_{s+2} h^{s}, \qquad 0 \leqq s \leqq r+1. \tag{17}$$

Now, we shall seek an estimate of the error in $L_2(\Omega)$. Consider

$$\begin{aligned} \Delta\varphi &= W-u \qquad \text{on } \Omega, \\ \varphi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Then,

$$\begin{aligned} \| W-u \|^2 &= (W-u,\ \Delta\varphi) \\ &= (\varphi,\ \Delta(W-u)) \\ &= (\varphi-\varphi^*.\ \Delta(W-u)), \qquad \varphi^* \in \mathcal{M}. \end{aligned}$$

Thus, choosing an appropriate φ^*, we obtain the inequality

$$\| W-u \|^2 \leqq C \| \varphi \|_2 h^2 \| \Delta(W-u) \| \\ \leqq C \| W-u \| \| \Delta(W-u) \| h^2;$$

therefore,

$$\| W-u \| \leqq C \| u \|_{s+2} h^{s+2}, \qquad 0 \leqq s \leqq r+1.$$

Summarizing the above results, we have proved the following theorem.

THEOREM 4: *Let u be the solution to* (1) *with $L = \Delta$ and assume that $u \in H^s(\Omega)$,* $2 \leqq s \leqq r+3$. *Then, if W is defined by* (14),

$$\| W-u \|_q \leqq C \| u \|_s h^{s-q}, \qquad 0 \leqq q \leqq 2.$$

If $k \geqq 0$, then the range on q in theorem 4 can be extended to $0 \leqq q \leqq \min(k+3, s)$ by repeated use of quasi-uniformity to obtain the analogue of (17) in $H^{k+3}(\Omega)$.

3. THE GENERAL CASE

Let $U \in \mathcal{M}$ be determined as the solution of (2), and introduce an auxiliary function $U_1 \in \mathcal{M}$ as the solution of

$$(U_1 - u, \Delta v) = 0, \qquad v \in \mathcal{N}.$$

Let $\xi = U - U_1$, and let ψ be given by the Dirichlet problem

$$L^* \psi = \xi \quad \text{on } \Omega, \\ \psi = 0 \quad \text{on } \partial\Omega.$$

Then, if $\psi^* \in \mathcal{N}$,

$$\| \xi \|^2 = (\xi, L^* \psi) \\ = (\xi, L^*(\psi - \psi^*)) + (\xi, L^* \psi^*) \\ = (\xi, L^*(\psi - \psi^*)) + (\eta, L^* \psi^*),$$

where $\eta = u - U_1$. We choose $\psi^* \in M$ to satisfy

$$(\Delta(\psi - \psi^*), v) = 0, \qquad v \in \mathcal{M}.$$

Thus, with $\tilde{b}$ and $\tilde{c}$ indicating the lower order coefficients of L^*,

$$\| \xi \|^2 = (a\xi, \Delta(\psi - \psi^*)) + (\xi, \tilde{b} \cdot \nabla(\psi - \psi^*)) \\ + (\xi, \tilde{c}(\psi - \psi^*)) + (\eta, L^* \psi^*) \\ = (a\xi - \chi, \Delta(\psi - \psi^*)) + (\xi, \tilde{b} \cdot \nabla(\psi - \psi^*)) \\ + (\xi, \tilde{c}(\psi - \psi^*)) + (\eta, L^* \psi^*), \qquad \chi \in \mathcal{M}.$$

It is well-known that, since $a \in C^1(\bar{\Omega})$,

$$\inf_{\chi \in \mathcal{M}} \| a\xi - \chi \| \leqq C \| \xi \| h.$$

Replacing u by ψ and W by ψ^* in theorem 4, we observe that

$$\| \psi - \psi^* \|_q \leqq C \| \psi \|_2 h^{2-q}, \qquad 0 \leqq q \leqq 2.$$

Since $\| \psi \|_2 \leqq C \| \xi \|$,

$$\| \xi \|^2 \leqq C \{ h \| \xi \|^2 + \| \eta \| \| \xi \| \}.$$

Hence, for h sufficiently small,

$$\| \xi \| \leqq C \| \eta \|.$$

Consequently, we have the following theorem.

THEOREM 5: *There exists $h_0 = h_0(L) > 0$ such that a unique solution $U \in \mathcal{M}$ of* (2) *exists for $h \leqq h_0$; moreover, if $1 \leqq s \leqq r+1$ and if $u \in H^s(\Omega)$ is the solution of* (1), *then*

$$\| U - u \| \leqq C \| u \|_s h^s.$$

We shall now consider error estimates for the adjoint local H^{-1} Galerkin procedure. Note that the ellipticity of L implies a Gårding inequality of the form

$$C_0 \| \varphi \|_1^2 \leqq -(L\varphi, \varphi) + C_1 \| \varphi \|^2$$

for $\varphi \in H^2(\Omega)$ such that $\varphi = 0$ on $\partial\Omega$, where C_0 is some positive constant. Since (1) and (3) imply that $(L(W-u), \psi) = 0$ for $\psi \in \mathcal{M}$,

$$C_0 \| W-u \|_1^2 - C_1 \| W-u \|^2 \leqq -(L(W-u), W-u-\psi), \qquad \psi \in \mathcal{M}.$$

For h sufficiently small, theorem 5 when applied to the operator L^* instead of L implies the existence of $\psi \in \mathcal{M}$ such that

$$(Lv, W-u-\psi) = 0, \qquad v \in \mathcal{N},$$

and

$$\| W-u-\psi \| \leqq C \| W-u \|_1 h.$$

Thus, for any $\theta \in \mathcal{N}$:

$$\begin{aligned} C_0 \| W-u \|_1^2 - C_1 \| W-u \|^2 &\leqq -(L(\theta-u), W-u-\psi) \\ &\leqq C \| u-\theta \|_2 \| W-u \|_1 h. \end{aligned}$$

By noting that $\| W-u \|^2 \leqq \| W-u \|_1 \| W-u \|$, we see that

$$\| W-u \|_1 \leqq C(\| u \|_{s+2} h^{s+1} + \| W-u \|), \qquad 0 \leqq s \leqq r+1.$$

Again by the quasi-uniformity of δ,

$$\| W-u \|_2 \leqq C(\| u \|_{s+2} h^s + h^{-1} \| W-u \|), \qquad 0 \leqq s \leqq r+1.$$

In order to obtain an $L_2(\Omega)$-estimate, we now consider the auxiliary Dirichlet problem given by

$$L^* \varphi = W-u \quad \text{on} \quad \Omega,$$
$$\varphi = 0 \quad \text{on} \quad \partial\Omega.$$

Then,

$$\| W-u \|^2 = (W-u,\ L^* \varphi) = (L(W-u),\ \varphi)$$
$$= (L(W-u),\ \varphi - \varphi^*), \qquad \varphi^* \in \mathscr{M}.$$

Thus, choosing an appropriate φ^*, we obtain the inequality

$$\| W-u \|^2 \leqq C \| W-u \|_2 \| \varphi \|_2 h^2$$
$$\leqq C \| W-u \|_2 \| W-u \| h^2,$$

and

$$\| W-u \| \leqq C \| W-u \|_2 h^2$$
$$\leqq C(\| u \|_{s+2} h^{s+2} + \| W-u \| h), \qquad 0 \leqq s \leqq r+1.$$

Hence, we have proved the following theorem.

THEOREM 6: *There exists $h_0 = h_0(L) > 0$ such that there exists a unique solution $W \in \mathscr{N}$ of* (3), *and if $2 \leqq s \leqq r+3$ and if the solution u of* (1) *belongs to $H^s(\Omega)$, then*

$$\| W-u \|_q \leqq C \| u \|_s h^{s-q}, \qquad 0 \leqq q \leqq 2.$$

The range on q can be extended just as for theorem 4.

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