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FINITE ELEMENT METHODS FOR NONLINEAR PARABOLIC EQUATIONS (*)

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Communiqué par P.-A. RAVIART

Summary. — *Linear two-step A-stable methods of the second order introduced in* [15] *together with finite element discretizations in space are applied for the solution of nonlinear parabolic initial-boundary value problems. These include linear problems with time dependent coefficients as a special case. The resulting schemes are algebraically linear and unconditionally stable.* A priori *error estimates in the L_2-norm of optimal order of accuracy are derived. Similar error estimates hold for linear one-step A-stable methods.*

1. INTRODUCTION

We shall consider the approximate solution of the initial-boundary value problem

$$\alpha(x, t)\frac{\partial u}{\partial t} = Pu, \qquad (x, t)\in\Omega\times(0, T], \tag{1.1}$$

$$u(x, t) = 0, \qquad (x, t)\in\Gamma\times(0, T], \tag{1.2}$$

$$u(x, 0) = u^0(x), \qquad x\in\Omega. \tag{1.3}$$

Here $x = (x_1, \ldots, x_N)$ is a point of a bounded domain Ω lying in the N-dimensional Euclidean space, Γ is its boundary and

$$\left.\begin{array}{l} Pu = \displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[k_{ij}(x, t, u)\frac{\partial u}{\partial x_j}\right] + \operatorname{div}\mathbf{f}(x, t, u) + g(x, t, u), \\ \mathbf{f}(x, t, u) = (f_1(x, t, u), \ldots, f_N(x, t, u))^T \end{array}\right\} \tag{1.4}$$

(T written as a superscript means transposition of a vector or of a matrix). Concerning the coefficients and the right-hand side of (1.1), all assumptions are summed up in:

A_1: (i) $\alpha(x, t)$ is bounded from below and above by a positive constant and is uniformly Lipschitz continuous as a function of t, i. e.

$$\left.\begin{array}{l} 0 < m_1 \leqq \alpha(x, t) \leqq m_2,\ (x, t)\in\Omega\times(0, T]; \\ |\alpha(x, t_1) - \alpha(x, t_2)| \leqq L|t_1 - t_2|, \\ t_1, t_2\in(0, T], \qquad x\in\Omega, \end{array}\right\} \tag{1.5}$$

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(ii) the matrix $\{k_{ij}(x, t, u)\}_{i,j=1}^{N}$ is uniformly positive definite and bounded, i. e.

$$\left.\begin{array}{c} c^{-1}\sum_{i=1}^{N}\xi_i^2 \leqq \sum_{i,j=1}^{N} k_{ij}(x, t, u)\xi_i\xi_j \leqq c\sum_{i=1}^{N}\xi_i^2, \\ c>0, \quad (x, t)\in\Omega\times(0, T]. \end{array}\right\} \tag{1.6}$$

(iii) the coefficients $k_{ij}(x, t, u)$ are uniformly Lipschitz continuous as functions of t and u, i. e.

$$\left.\begin{array}{l} \sum_{i,j=1}^{N} |k_{ij}(x, t_1, u)-k_{ij}(x, t_2, u)| \leqq L|t_1-t_2|, \\ \quad t_1, t_2\in[0, T], \quad x\in\Omega, \quad -\infty<u<\infty, \\ \sum_{i,j=1}^{N} |k_{ij}(x, t, u_1)-k_{ij}(x, t, u_2)| \leqq L|u_1-u_2|, \\ \quad (x, t)\in\Omega\times[0, T], \quad -\infty<u_1, u_2<\infty. \end{array}\right\} \tag{1.7}$$

(iv) the functions f_i and g are uniformly Lipschitz continuous as functions of u, i. e.

$$\left.\begin{array}{l} \sum_{i=1}^{N} |f_i(x, t, u_1)-f_i(x, t, u_2)| \\ \quad +|g(x, t, u_1)-g(x, t, u_2)| \leqq L|u_1-u_2|, \\ \quad (x, t)\in\Omega\times[0, T], \quad -\infty<u_1, u_2<\infty. \end{array}\right\} \tag{1.8}$$

Before formulating the given problem in a variational form let us introduce some notation. By H^m we denote the Sobolev space of real functions which together with their generalized derivatives up to the m-th order inclusive are square integrable over Ω. The inner product and the norm are denoted by $(.,.)_m$ and $\| \cdot \|_m$, respectively. H_0^1 is the closure in the H^1-norm of infinitely differentiable functions having compact support contained in Ω.

Multiplying (1.1) by $\varphi \in H_0^1$ and using Green's theorem we come to the identity

$$\left.\begin{array}{c} (\alpha(x, t)\dot{u}, \varphi)_0+a(t, u; u, \varphi) = -(\mathbf{f}(x, t, u), \operatorname{grad}\varphi)_0+(g(x, t, u), \varphi)_0, \\ \forall\varphi\in H_0^1, \quad t\in(0, T]; \end{array}\right\} \tag{1.9}$$

here the dot means the derivative with respect to t,

$$(\mathbf{f}, \operatorname{grad}\varphi)_0 = \sum_{i=1}^{N}\left(f_i, \frac{\partial\varphi}{\partial x_i}\right)_0$$

and

$$a(t, w; u, \varphi) = \int_\Omega \sum_{i,j=1}^{N} k_{ij}(x, t, w)\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\,dx. \tag{1.10}$$

Hence the weak solution of the problem (1.1)-(1.3) (for the definition see, for instance, J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes*, Dunod, Paris 1968) satisfies (1.9).

To get the approximate solution we shall first discretize (1.9) in space. We shall use only finite element spaces which are subspaces of $H_0^1(\Omega)$. This restriction means that we can consider straight elements of different kind if Ω is a polyhedron and we have to consider curved elements which match exactly curved boundaries if Γ is curved. We denote the finite element spaces which will be used by V_h^p and we postulate the following properties:

A_2: (i) V_h^p is either a regular family of straight elements according to the definition by Ciarlet and Raviart (*see* [1] or [2], section 6, p. 9) or a family of curved triangular elements (*see* Zlámal appendix of [12] and [13]) satisfying the condition that the smallest angle of all triangles is bounded away from zero.

(ii) to any $u \in H^{p+1} \cap H_0^1$ there exists $\hat{u} \in V_h^p$ such that

$$\| u-\hat{u} \|_0 + h \| u-\hat{u} \|_1 \leqq C h^{p+1} \| u \|_{p+1}; \qquad (1.11)$$

h is the greatest diameter of all elements or the greatest side in case of triangles.

The discretization of (1.9) in space gives the continuous-time Galerkin solution $U(x, t)$. It is a function from V_h^p such that

$$\left.\begin{aligned} &(\alpha(x, t)\dot{U}, \varphi)_0 + a(t, U; U, \varphi) \\ &\quad = -(\mathbf{f}(x, t, U), \operatorname{grad}\varphi)_0 + (g(x, t, U), \varphi)_0, \\ &\qquad \forall \varphi \in V_h^p, \end{aligned}\right\} \qquad (1.12)$$

$$U(x, 0) = \hat{u}^0(x), \qquad \hat{u}^0(x) \in V_h^p. \qquad (1.13)$$

$\hat{u}^0(x)$ is an approximation of $u^0(x)$ and the simplest way is to choose the interpolate of $u^0(x)$ for it.

The continuous-time Galerkin solution has no practical significance. To get a computable approximate solution we must discretize also with respect to t. To this end we write (1.12), which represents a system of ordinary nonlinear differential equations, in a matrix form. Let $\{ v_i \}_{i=1}^d$ be a basis of V_h^p (of course, in finite element spaces we do not choose an arbitrary basis; however this circumstance does not play any role in our considerations) and put $U(x, t) = \mathbf{a}^T(t)\, \mathbf{v}(x)$ where $\mathbf{a} = (a_1, \ldots, a_d)^T$, $\mathbf{v} = (v_1, \ldots, v_d)^T$. Setting the basis functions v_i for φ in (1.12) we get

$$M(t)\dot{\mathbf{a}} + K(t, \mathbf{a})\mathbf{a} = \mathbf{F}(t, \mathbf{a}). \qquad (1.14)$$

Here

$$M(t) = (\alpha(x, t)\mathbf{v}, \mathbf{v})_0, \qquad K(t, \mathbf{a}) = \mathbf{a}(t, \mathbf{a}^T\mathbf{v}; \mathbf{v}, \mathbf{v}),$$

$$\mathbf{F}(t, \mathbf{a}) = -(\mathbf{f}(t, x, \mathbf{a}^T\mathbf{v}), \operatorname{grad}\mathbf{v})_0 + (g(x, t, \mathbf{a}^T\mathbf{v}), \mathbf{v})_0.$$

Both matrices $M(t)$ and $K(t, \mathbf{a})$ are positive definite, therefore

$$\dot{\mathbf{a}} = -A(t, \mathbf{a})\mathbf{a} + M^{-1}(t)\mathbf{F}(t, \mathbf{a}), \qquad A(t, \mathbf{a}) = M^{-1}(t)K(t, \mathbf{a}). \quad (1.15)$$

The system (1.15) is a stiff system and we shall use first linear two-step A-stable methods of the second order for its solution.

If

$$\rho(\zeta) = \sum_{s=0}^{2} \alpha_s \zeta^s \quad \text{and} \quad \sigma(\zeta) = \sum_{s=0}^{2} \beta_s \zeta^s$$

are characteristic polynomials of a linear two-step method (ρ, σ) normalized by

$$\sum_{s=0}^{2} \beta_s = 1, \quad (1.16)$$

then (ρ, σ) is of the second order iff

$$\left.\begin{array}{ll} \alpha_1 = 1 - 2\alpha_2, & \alpha_0 = -1 + \alpha_2, \\ \beta_1 = \frac{1}{2} + \alpha_2 - 2\beta_2, & \beta_0 = \frac{1}{2} - \alpha_2 + \beta_2. \end{array}\right\} \quad (1.17)$$

The result of Liniger [9] (*see* also Zlámal [15], section IV) can be stated as follows: Let (ρ, σ) satisfy (1.16), (1.17) and let ρ and σ have no common root. Then the necessary and sufficient condition that the method be Dahlquist and A-stable is

$$\alpha_2 \geqq \frac{1}{2}, \qquad \beta_2 > \frac{1}{2}\alpha_2. \quad (1.18)$$

Let us apply the scheme (ρ, σ) to the solution of (1.15). The result is

$$\sum_{s=0}^{2} \alpha_s \mathbf{a}^{n+s} + \Delta t \sum_{s=0}^{2} \beta_s A(t_{n+s}, \mathbf{a}^{n+s})\mathbf{a}^{n+s}$$
$$= \Delta t \sum_{s=0}^{2} \beta_s M^{-1}(t_{n+s})\mathbf{F}(t_{n+s}, \mathbf{a}^{n+s}). \quad (1.19)$$

This recurrence relation is algebraically nonlinear and has no practical significance. The idea of extrapolation was used often in recent years (we mention Douglas and Dupont [4] and Dupont, Fairweather and Johnson [5]) and here the extrapolation which linearizes (1.19) will be done in the following way: if $\gamma(t) \in C^2$ and $\gamma^n = \gamma(n\Delta t)$ choose c_0, c_1 such that $\gamma^{\bar{n}} = c_1 \gamma^{n+1} + c_0 \gamma^n$ satisfies

$$\sum_{s=0}^{2} \beta_s \gamma^{n+s} - \gamma^{\bar{n}} = O(\Delta t^2 \ddot{\gamma}). \quad (1.20)$$

Further determine $t_{\bar{n}}$ such that

$$\gamma^{\bar{n}}-\gamma(t_{\bar{n}}) = O(\Delta t^2 \ddot{\gamma}). \tag{1.21}$$

An easy calculation gives

$$c_1 = 2\beta_2+\beta_1, \qquad c_0 = \beta_0-\beta_2, \qquad t_{\bar{n}} = (n+c_1)\Delta t = t_n+(2\beta_2+\beta_1)\Delta t.$$

Now replace t_{n+s} and $\mathbf{a}^{n+s}$ in nonlinear terms of (1.19) by

$$t_{\bar{n}} = t_n+(2\beta_2+\beta_1)\Delta t, \qquad \mathbf{a}^{\bar{n}} = (2\beta_2+\beta_1)\mathbf{a}^{n+1}+(\beta_0-\beta_2)\mathbf{a}^n. \tag{1.22}$$

Multiplying the resulting recurrence relation by $M(t_{\bar{n}})$ we get the final algebraically linear relation

$$M^{\bar{n}} \sum_{s=0}^{2} \alpha_s \mathbf{a}^{n+s}+\Delta t K^{\bar{n}} \sum_{s=0}^{2} \beta_s \mathbf{a}^{n+s} = \Delta t\, \mathbf{F}^{\bar{n}}. \tag{1.23}$$

Here

$$M^{\bar{n}} = M(t_{\bar{n}}), \qquad K^{\bar{n}} = K(t_{\bar{n}}, \mathbf{a}^{\bar{n}}), \qquad \mathbf{F}^{\bar{n}} = \mathbf{F}(t_{\bar{n}}, \mathbf{a}^{\bar{n}}). \tag{1.24}$$

Evidently, at every step we have to compute the matrices $M^{\bar{n}}$, $K^{\bar{n}}$ and to solve a system of linear equations with the positive definite matrix $\alpha_2 M^{\bar{n}}+\beta_2\Delta t\, K^{\bar{n}}$. Of course, we need to know the starting values $\mathbf{a}^0$, $\mathbf{a}^1$. $\mathbf{a}^0$ is determined by the initial condition (1.13) whereas for the computation of $\mathbf{a}^1$ a suitable one-step method can be used (*see* section 3).

We can come back to a variational form and write (1.23) as

$$\left.\begin{aligned} &\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s U^{n+s}, \varphi\right)_0 +\Delta t a\left(t_{\bar{n}}, U^{\bar{n}}; \sum_{s=0}^{2} \beta_s U^{n+s}, \varphi\right) \\ &\qquad = -\Delta t(\mathbf{f}^{\bar{n}}, \operatorname{grad}\varphi)_0+\Delta t(g^{\bar{n}}, \varphi)_0 \qquad \forall \varphi \in V_h^p, \\ &\alpha^{\bar{n}} = \alpha(x, t_{\bar{n}}), \qquad \mathbf{f}^{\bar{n}} = \mathbf{f}(x, t_{\bar{n}}, U^{\bar{n}}), \qquad g^{\bar{n}} = g(x, t_{\bar{n}}, U^{\bar{n}}). \end{aligned}\right\} \tag{1.25}$$

Linear two-step schemes for nonlinear parabolic equations have been proposed recently by Comini, Del Guidice, Lewis and Zienkiewicz [3] and by Dupont, Fairweather and Johnson [5]. They are special cases of (1.23) and (1.25), respectively, with $\alpha_2 = 1/2$, $\beta_2 = 1/3$ in [3], $\alpha_2 = 1/2$, $\beta_2 = \Theta$ and $\alpha_2 = 1$, $\beta_2 = 1/2+\Theta$ in [5].

2. ERROR ESTIMATES

The technique for deriving error estimates used here is closely related to that of Wheeler [11] and Dupont, Fairweather, Johnson [5]. We shall decompose the exact solution in $u = \xi+\eta$, ξ being the Ritz approximation defined by

$$a(t, u; u, \varphi) = a(t, u; \xi, \varphi), \qquad \forall \varphi \in V_h^p. \tag{2.1}$$

We shall need estimates of $\| \dot{\eta} \|_0$ and $\| \eta \|_0$ of the form (4.15) in [5], i. e.

$$\| \eta \|_0 + \| \dot{\eta} \|_0 \leqq C h^{p+1} (\| u \|_{p+1} + \| \dot{u} \|_{p+1}), \qquad t \in (0, T]. \quad (2.2)$$

One can prove (2.2) exactly in the same way as Dupont, Fairweather and Johnson proved (4.15) in [5] under the following additional assumptions A_3: (i) if $z \in H_0^1$ is defined by

$$a(t, u; z, \varphi) = (f, \varphi)_0, \qquad \forall \varphi \in H_0^1$$

then $\| z \|_2 \leqq C \| f \|_0$ where C does not depend on t and on u.

(ii) The coefficients $k_{ij}(x, t, u)$ have partial derivatives

$$\frac{\partial k_{ij}}{\partial t}, \quad \frac{\partial k_{ij}}{\partial u}, \quad \frac{\partial^2 k_{ij}}{\partial x_i \partial t}, \quad \frac{\partial^2 k_{ij}}{\partial x_i \partial u}$$

and the matrices

$$\left\{ \dot{k}_{ij} + \frac{\partial k_{ij}}{\partial u} \dot{u} \right\}_{i,j=1}^N, \qquad \left\{ \frac{\partial}{\partial x_i} \left(\dot{k}_{ij} + \frac{\partial k_{ij}}{\partial u} \dot{u} \right) \right\}_{i,j=1}^N$$

are bounded on $\Omega \times (0, T]$.

REMARK: If Γ, u and k_{ij} are sufficiently smooth (i) follows from (1.6) and from Theorem 37, I in Miranda [10] p. 169. However, (i) may hold even when Ω has corners.

THEOREM: *Let the assumptions* A_1, A_2, A_3 *be satisfied. Let the scheme* (ρ, σ) *normalized by* (1.16) *satisfy* (1.17) *and* (1.18). *Finally, let the exact solution* u *be such that grad* u *is bounded in the maximum norm,* $\partial^3 u/\partial t^3$ *is continuous for* $(x, t) \in \bar{\Omega} \times [0, T]$ and $\| u \|_{p+1} + \| \dot{u} \|_{p+1} \leqq C, t \in [0, T]$. *Then for arbitrary* h, Δt

$$\max_{2 \leqq n \leqq T/\Delta t} \| u^n - U^n \|_0 \leqq C \left[\sum_{i=0}^{1} \| u^i - U^i \|_0 + h^{p+1} + \Delta t^2 \right]; \qquad (2.3)$$

here $u^n = u(x, n\Delta t)$, U^n *is defined by* (1.25) *and the constant* C *does not depend on* h *and* Δt.

Proof: a) Set

$$u^n - U^n = u^n - \xi^n + \xi^n - U^n = \eta^n + \varepsilon^n, \qquad \varepsilon^n = \xi^n - U^n \in V_h^p.$$

With respect to (2.2) it is sufficient to find a bound for $\| \varepsilon^n \|_0$.

For further purpose we prove now what we shall need later, namely

$$\max_{\bar{\Omega}} | \operatorname{grad} \xi | \leqq C, \qquad t \in (0, T] \qquad (2.4)$$

[ξ is defined by (2.1)]. We restrict ourselves to the case that V_h^p is formed by curved triangular elements. The proof for straight elements is analogous. If we prove that $\max_{\bar{\Omega}} |\operatorname{grad} \eta| \leqq Ch^{p-1} \|u\|_{p+1}$ then (2.4) follows because

$$\max_{\bar{\Omega}} |\operatorname{grad} \xi| \leqq \max_{\bar{\Omega}} |\operatorname{grad} u| + \max_{\bar{\Omega}} |\operatorname{grad} \eta| \leqq C$$

(notice that $p \geqq 1$). Set $\eta = u - u_I + u_I - \xi$ where u_I is the interpolate of u, i. e. that function from V_h^p which has the same nodal parameters as u. Standard arguments give $\max_{\bar{\Omega}} |\operatorname{grad}(u - u_I)| \leqq Ch^p \|u\|_{p+1}$ (*see* [12], Th. 2; here polynomials of the degree $p = 2n-1$, $n = 1, 2, \ldots$ are considered, however the generalization is immediate—*see* appendix of [13]). Therefore what we need to prove is

$$\max_{\bar{\Omega}} |\operatorname{grad}(u_I - \xi)| \leqq C h^{p-1}.$$

$u_I - \xi$ belongs to V_h^p. On every element it is of the form $r[s(x_1, x_2), t(x_1, x_2)]$ where $s = s(x_1, x_2)$, $t = t(x_1, x_2)$ maps the given element onto the unit triangle T_1 with vertices (0, 0), (1, 0), (0, 1) and r is a polynomial of the degree p. Let us consider the element e where $|\partial(u_I - \xi)/\partial x_i|$ assumes the maximum value M_i. We have

$$M_i = \left| \frac{\partial(u_I - \xi)}{\partial s} \frac{\partial s}{\partial x_i} + \frac{\partial(u_I - \xi)}{\partial t} \frac{\partial t}{\partial x_i} \right|.$$

As $|\partial s/\partial x_i|$, $|\partial t/\partial x_i| \leqq Ch^{-1}$ (*see* [12], equation (8); notice a different notation) it follows

$$M_i \leqq C h^{-1} \max_{T_1} \left(\left| \frac{\partial r}{\partial s} \right| + \left| \frac{\partial r}{\partial t} \right| \right).$$

$\partial r/\partial s$ and $\partial r/\partial t$ are polynomials. If $q(s, t)$ is a polynomial of the variables s, t then

$$\max_{T_1} q^2 \leqq C \int_{T_1} q^2 \, ds \, dt$$

(both sides of this inequality are positive definite quadratic forms of the coefficients of q bounded from below and above uniformly for $(s, t) \in T_1$). Therefore

$$\left(\frac{\partial r}{\partial s} \right)^2 + \left(\frac{\partial r}{\partial t} \right)^2 \leqq C \int_{T_1} (r_s^2 + r_t^2) \, ds \, dt.$$

As the Jacobian of the mapping $s = s(x_1, x_2)$, $t = t(x_1, x_2)$ is bounded by Ch^{-2} and for the inverse mapping it holds $|\partial x_i/\partial s|$, $|\partial x_i/\partial t| \leqq Ch$

(*see* [12], equations (8) and (7)) we get

$$\left(\frac{\partial r}{\partial s}\right)^2+\left(\frac{\partial r}{\partial t}\right)^2 \leqq C\int_e\left\{\left[\frac{\partial}{\partial x_1}(u_I-\xi)\right]^2+\left[\frac{\partial}{\partial x_2}(u_I-\xi)\right]^2\right\}dx_1\,dx_2$$
$$\leqq C(\|u-u_I\|_1^2+\|u-\xi\|_1^2).$$

The bound $\|u-\xi\|_1 \leqq Ch^p\|u\|_{p+1}$ follows by standard arguments and by (1.6), hence $M_i \leqq Ch^{p-1}$.

b) Here we want to prove that ε^n satisfies a recurrent relation of the form

$$\left.\begin{array}{c}\left(\alpha^{\bar n}\sum_{s=0}^{2}\alpha_s\varepsilon^{n+s},\,\varphi\right)_0+\Delta t a\left(t_{\bar n},\,U^{\bar n};\,\sum_{s=0}^{2}\beta_s\varepsilon^{n+s},\,\varphi\right)=\Delta t(\psi^{\bar n},\,\varphi)_1,\\ \forall\varphi\in V_h^p\end{array}\right\}\quad(2.5)$$

where ψ^n is a function such that

$$\|\psi^n\|_1 \leqq C(\vartheta+\|\varepsilon^{\bar n}\|_0),\qquad \vartheta=h^{p+1}+\Delta t^2. \tag{2.6}$$

The left-hand side of (2.5) differs from the left-hand side of (1.25) in that ε^{n+s} stands in place of U^{n+s}. As $\varepsilon^{n+s}=\xi^{n+s}-U^{n+s}$ we shall try to express

$$\left(\alpha^{\bar n}\sum_{s=0}^{2}\alpha_s\xi^{n+s},\,\varphi\right)_0+\Delta t a\left(t_{\bar n},\,U^{\bar n};\,\sum_{s=0}^{2}\beta_s\xi^{n+s},\,\varphi\right)$$

in a suitable way. We shall find that

$$\left.\begin{array}{l}\left(\alpha^{\bar n}\sum_{s=0}^{2}\alpha_s\xi^{n+s},\,\varphi\right)_0+\Delta t a\left(t_{\bar n},\,U^{\bar n};\,\sum_{s=0}^{2}\beta_s\xi^{n+s},\,\varphi\right)\\ \quad=-\Delta t(\mathbf{f}(x,\,t_{\bar n},\,\xi^{\bar n}),\,\mathrm{grad}\,\varphi)_0+\Delta t(g(x,\,t_{\bar n},\,\xi^{\bar n}),\,\varphi)_0\\ \quad+\Delta t(\psi^n,\,\varphi)_1,\qquad \psi^n \text{ satisfies (2.6).}\end{array}\right\}\quad(2.7)$$

Subtract (1.25) from (2.7). The left-hand side of this difference is that of (2.5). The right-hand side is equal to $\Delta t\,(\varkappa^n+\psi^n,\,\varphi)_1$ where $\varkappa^n$ is the function from V_h^p defined uniquely by

$$\left.\begin{array}{l}(\varkappa^n,\,\varphi)_1=-(\mathbf{f}(x,\,t_{\bar n},\,\xi^{\bar n})-\mathbf{f}(x,\,t_{\bar n},\,U^{\bar n}),\,\mathrm{grad}\,\varphi)_0\\ \quad+(g(x,\,t_{\bar n},\,\xi^{\bar n})-g(x,\,t_{\bar n},\,U^{\bar n}),\,\varphi)_0,\qquad\forall\varphi\in V_h^p.\end{array}\right\}\quad(2.8)$$

Setting $\varphi=\varkappa^n$ in (2.8) and using (1.8) you obtain $\|\varkappa^n\|_1 \leqq C\|\varepsilon^{\bar n}\|_0$. Writing ψ^n instead of $\varkappa^n+\psi^n$ you get (2.5) with ψ^n satisfying (2.6).

To prove (2.7) we first remark that for the operator

$$Lu^n=\sum_{s=0}^{2}\alpha_s u^{n+s}-\Delta t\sum_{s=0}^{2}\beta_s\dot u^{n+s}$$

it holds $|Lu^n| \leqq C\Delta t^3$ (*see* Henrici [6], Lemma 5.7, p. 247). It follows on basis of (1.20), (1.21) and (1.5) that

$$\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s u^{n+s} - \Delta t\, \alpha^{\bar{n}} \dot{u}(x, t_{\bar{n}}), \varphi\right)_0 = (\omega^n, \varphi)_0, \qquad \|\omega^n\|_0 \leqq C\Delta t^3.$$

We set for $\alpha^{\bar{n}} \dot{u}(x, t_{\bar{n}})$ from (1.1) and we easily derive

$$\begin{aligned}\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s u^{n+s}, \varphi\right)_0 &+ \Delta t a(t_{\bar{n}}, u(x, t_{\bar{n}}); u(x, t_{\bar{n}}), \varphi) \\ &= -\Delta t(\mathbf{f}(x, t_{\bar{n}}, u(x, t_{\bar{n}})), \operatorname{grad}\varphi)_0 \\ &\quad + \Delta t(g(x, t_{\bar{n}}, u(x, t_{\bar{n}})), \varphi)_0 + (\omega^n, \varphi)_0.\end{aligned}$$

The above equation can be written as

$$\begin{aligned}&\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \xi^{n+s}, \varphi\right)_0 + \Delta t a(t_{\bar{n}}, u(x, t_{\bar{n}}); u(x, t_{\bar{n}}), \varphi) \\ &\quad = -\Delta t(\mathbf{f}(x, t_{\bar{n}}, \xi^{\bar{n}}), \operatorname{grad}\varphi)_0 + \Delta t(g(x, t_{\bar{n}}, \xi^{\bar{n}}), \varphi)_0 + (\omega^n, \varphi)_0 \\ &\quad - \left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \eta^{n+s}, \varphi\right)_0 - \Delta t(\mathbf{f}(x, t_{\bar{n}}, u(x, t_{\bar{n}})) - \mathbf{f}(x, t_{\bar{n}}, u^{\bar{n}}), \operatorname{grad}\varphi)_0 \\ &\quad + \Delta t(g(x, t_{\bar{n}}, u(x, t_{\bar{n}})) - g(x, t_{\bar{n}}, u^{\bar{n}}), \varphi)_0.\end{aligned} \tag{2.9}$$

We have

$$\sum_{s=0}^{2} \alpha_s \eta^{n+s} = \alpha_2(\eta^{n+2} - \eta^n) + \alpha_1(\eta^{n+1} - \eta^n)$$

(from the consistency condition it follows

$$\left((1) = \sum_{s=0}^{2} \alpha_s = 0\right).$$

Using (2.2) we get

$$\left\|\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \eta^{n+s}\right\|_0 \leqq C\Delta t h^{p+1}.$$

Further, the last two terms in (2.9) are easy to estimate when we use (1.8) Therefore, (2.9) can be written as

$$\left.\begin{aligned}&\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \xi^{n+s}, \varphi\right)_0 + \Delta t a(t_{\bar{n}}, u(x, t_{\bar{n}}); u(x, t_{\bar{n}}), \varphi) \\ &\quad = -(\mathbf{f}(x, t_{\bar{n}}, \xi^{\bar{n}}), \operatorname{grad}\varphi)_0 + (g(x, t_{\bar{n}}, \xi^{\bar{n}}), \varphi)_0 + \Delta t(\psi^n, \varphi)_1, \\ &\qquad \|\psi^n\|_1 \leqq C\vartheta.\end{aligned}\right\} \tag{2.10}$$

If we prove that

$$\left.\begin{aligned} a\left(t_{\bar n},\, U^n;\, \sum_{s=0}^{2} \beta_s \xi^{n+s},\, \varphi\right) - a(t_{\bar u},\, u(x,\, t_{\bar n});\, u(x,\, t_{\bar n}),\, \varphi) = (\psi^n,\, \varphi)_1, \\ \forall\, \varphi \varepsilon\, V_h^p \end{aligned}\right\} \tag{2.11}$$

with ψ^n satifying (2.6) then multiplying (2.11) by Δt and adding to (2.10) we get (2.7).

(2.11) defines a unique $\psi^n \in V_h^p$. We can write

$$\begin{aligned} (\psi^n,\, \varphi)_1 = {} & a\left(t_{\bar n},\, U^{\bar n};\, \sum_{s=0}^{2} \beta_s \xi^{n+s},\, \varphi\right) - a\left(t_{\bar n},\, u(x,\, t_{\bar n});\, \sum_{s=0}^{2} \beta_s \xi^{n+s},\, \varphi\right) \\ & + a\left(t_{\bar n},\, u(x,\, t_{\bar n});\, \sum_{s=0}^{2} \beta_s u^{n+s} - u(x,\, t_{\bar n}),\, \varphi\right) \\ & - a\left(t_{\bar n},\, u(x,\, t_{\bar n});\, \sum_{s=0}^{2} \beta_s \eta^{n+s},\, \varphi\right). \end{aligned} \tag{2.12}$$

From (1.7) (taking into account the form of the functional (1.10)), further from (2, 4), (2.2) and (1, 21) there follow the estimates

$$\begin{aligned} & \left| a\left(t_{\bar n},\, U^{\bar n};\, \sum_{s=0}^{2} \beta_s \xi^{n+s},\, \varphi\right) - a\left(t_{\bar n},\, u(x,\, t_{\bar n});\, \sum_{s=0}^{2} \beta_s \xi^{n+s},\, \varphi\right) \right| \\ & \leqq C \| U^{\bar n} - u(x,\, t_{\bar n}) \|_0 \| \varphi \|_1 \\ & \leqq C \| U^{\bar n} - \xi^{\bar n} + \eta^{\bar n} + u^{\bar n} - u(x,\, t_{\bar n}) \|_0 \| \varphi \|_1 \\ & \leqq C (\| \varepsilon^{\bar n} \|_0 + \vartheta) \| \varphi \|_1 . \end{aligned}$$

The third term on the right-hand side of (2.12) is easy to estimate using (1.20) and (1.21). The result is

$$\left| a\left(t_{\bar n},\, u(x,\, t_{\bar n});\, \sum_{s=0}^{2} \beta_s u^{n+s} - u(x,\, t_n),\, \varphi\right) \right| \leqq C \Delta t^2 \| \varphi \|_1 .$$

Concerning the last term notice first that $a(t_n, u(x, t_n); \eta^n, \varphi) = 0$, $\forall\, \varphi \in V_h^p$. Therefore, we have

$$\begin{aligned} & a\left(t_{\bar n},\, u(x,\, t_{\bar n});\, \sum_{s=0}^{2} \beta_s \eta^{n+s},\, \varphi\right) \\ & = \sum_{s=0}^{2} \beta_s \left[a(t_{\bar n},\, u(x,\, t_{\bar n});\, \eta^{n+s},\, \varphi) - a(t_{n+s},\, u(x,\, t_{n+s});\, \eta^{n+s},\, \varphi) \right]. \end{aligned}$$

Every term of the sum on the right-hand side is bounded by

$$C \Delta t \| \eta^{n+s} \|_1 \| \varphi \|_1 \leqq C \Delta t h^p \| \varphi \|_1$$

[it follows by means of (1.7)]. As $2\,\Delta t h^p \leqq h^{2p}+\Delta t^2 \leqq \vartheta$ (if $h \leqq 1$) we see that $(\psi^n, \varphi)_1 \leqq C(\vartheta + \| \varepsilon^{\bar{n}} \|_0) \| \varphi \|_1$, $\forall \varphi \in V_h^p$, hence ψ^n satisfies (2.6). This completes the proof of (2.5).

c) Setting

$$\varphi = \sum_{s=0}^{2} \beta_s \varepsilon^{n+s}$$

in (2.5), using (1.6) and the inequality $| ab | \leqq (1/2)\gamma a^2 + (1/2)\gamma^{-1} b^2$ we get

$$\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \varepsilon^{n+s}, \sum_{s=0}^{2} \beta_s \varepsilon^{n+s}\right)_0 + c_1 \Delta t \left\| \sum_{s=0}^{2} \beta_s \varepsilon^{n+s} \right\|_1^2$$
$$\leqq \frac{1}{2}\Delta t\left[\gamma \| \psi^n \|_1^2 + \gamma^{-1} \left\| \sum_{s=0}^{2} \beta_s \varepsilon^{n+s} \right\|_1^2\right], \qquad c_1 = \text{const.} > 0.$$

Choosing $\gamma = 1/(2c_1)$ and taking into account that ψ^n satisfies (2.6) we see that

$$\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \varepsilon^{n+s}, \sum_{s=0}^{2} \beta_s \varepsilon^{n+s}\right)_0 \leqq C\,\Delta t(\vartheta^2 + \| \varepsilon^{\bar{n}} \|_0^2). \tag{2.13}$$

We write (2.13) for $n = 0, 1, \ldots, m-2$, $m \leqq (T/\Delta t)$, and we sum. As $\varepsilon^{\bar{n}}$ is a linear combination of ε^{n+1} and ε^n (*see* 1.22) we obtain

$$\sum_{n=0}^{m-2}\left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \varepsilon^{n+s}, \sum_{s=0}^{2} \beta_s \varepsilon^{n+s}\right)_0 \leqq C\vartheta^2 + C\,\Delta t \sum_{n=0}^{m-1} \| \varepsilon^n \|_0^2. \tag{2.14}$$

We need to estimate from below $\sum_{n=0}^{m-2} \alpha^{\bar{n}} S^n$ where

$$S^n = \sum_{s=0}^{2} \alpha_s \varepsilon^{n+s} \sum_{s=0}^{2} \beta_s \varepsilon^{n+s}.$$

Let us write for the moment ε_n instead of ε^n. The coefficients α_2, β_2 satisfy (1.18). Therefore $\beta_2 = (1/2)\alpha_2 + \delta$, $\delta > 0$. Using (1.17) we find by inspection that

$$S_n = \frac{1}{2}(\alpha_2^2+\delta)\,\varepsilon_{n+2}^2 - \left(\alpha_2 - \frac{1}{2}\right)\varepsilon_{n+1}^2 - \frac{1}{2}\left[(\alpha_2-1)^2+\delta\right]\varepsilon_n^2$$
$$-\left[\alpha_2(\alpha_2-1)+\delta\right](\varepsilon_{n+2}\,\varepsilon_{n+1} - \varepsilon_{n+1}\,\varepsilon_n)$$
$$+\delta\left(\alpha_2 - \frac{1}{2}\right)(\varepsilon_{n+2} - 2\,\varepsilon_{n+1} + \varepsilon_n)^2.$$

Therefore

$$S^n \geqq \frac{1}{2}(\alpha_2^2+\delta)\,\varepsilon_{n+2}^2 - \left(\alpha_2 - \frac{1}{2}\right)\varepsilon_{n+1}^2 - \frac{1}{2}\left[(\alpha_2-1)^2+\delta\right]\varepsilon_n^2$$
$$-\left[\alpha_2(\alpha_2-1)+\delta\right](\varepsilon_{n+2}\,\varepsilon_{n+1} - \varepsilon_{n+1}\,\varepsilon_n). \tag{2.15}$$

Hence

$$\sum_{n=0}^{m-2} \alpha^{\bar{n}} S^n \geqq \frac{1}{2}(\alpha_2^2+\delta) \sum_{n=2}^{m} \overline{\alpha^{n-2}}\, \varepsilon_n^2 - \left(\alpha_2 - \frac{1}{2}\right) \sum_{n=2}^{m-1} \overline{\alpha^{n-2}}\, \varepsilon_n^2$$

$$- \frac{1}{2}\left[(\alpha_2-1)^2+\delta\right] \sum_{n=2}^{m-2} \overline{\alpha^{n-2}}\, \varepsilon_n^2 - \left[\alpha_2(\alpha_2-1)+\delta\right] \sum_{n=2}^{m} \overline{\alpha^{n-2}}\, \varepsilon_n \varepsilon_{n-1}$$

$$+ \left[\alpha_2(\alpha_2-1)+\vartheta\right] \sum_{n=2}^{m-1} \overline{\alpha^{n-2}}\, \varepsilon_n \varepsilon_{n-1} + \left(\alpha_2 - \frac{1}{2}\right) \sum_{n=2}^{m-1} (\overline{\alpha^{n-2}} - \overline{\alpha^{n-1}})\, \varepsilon_n^2$$

$$+ \frac{1}{2}\left[(\alpha_2-1)^2+\delta\right] \sum_{n=2}^{m-2} (\overline{\alpha^{n-2}} - \alpha^{\bar{n}})\, \varepsilon_n^2$$

$$+ \left[\alpha_2(\alpha_2-1)+\delta\right] \sum_{n=2}^{m-1} (\overline{\alpha^{n-1}} - \overline{\alpha^{n-2}})\, \varepsilon_n \varepsilon_{n-1} - C(\varepsilon_0^2+\varepsilon_1^2).$$

The terms containing ε_m^2, ε_{m-1}^2, $\varepsilon_m \varepsilon_{m-1}$ give a form $1/2\, \overline{\alpha^{m-2}}\, Q$ where

$$Q = (\alpha_2^2+\delta)\varepsilon_m^2 + \left[(\alpha_2-1)^2+\delta\right]\varepsilon_{m-1}^2 - 2\left[\alpha_2(\alpha_2-1)+\delta\right]\varepsilon_m \varepsilon_{m-1}.$$

The remaining terms are easy to estimate by means of (1.5). The result is

$$\sum_{n=0}^{m-2} \alpha^{\bar{n}} S^n \geqq \frac{1}{2} \overline{\alpha^{m-2}}\, Q - C(\varepsilon_0^2+\varepsilon_1^2) - C \Delta t \sum_{n=2}^{m-1} \varepsilon_n^2. \qquad (2.16)$$

Assume first that $\alpha_2(\alpha_2-1)+\delta = 0$. Then $Q \geqq (\alpha_2^2+\delta)\varepsilon_m^2$. Now let $\alpha_2(\alpha_2-1)+\delta \neq 0$. Then using the inequality $|a b| \leqq (1/2)\gamma a^2 + 1/2\, \gamma^{-1} b^2$ with $\gamma^{-1} = [(\alpha_2-1)^2+\delta]/|\alpha_2(\alpha_2-1)+\delta|$ we have

$$Q \geqq \left\{\alpha_2^2+\delta - \frac{[\alpha_2(\alpha_2-1)+\delta]^2}{(\alpha_2-1)^2+\delta}\right\}\varepsilon_m^2$$

$$= (\alpha_2^2+\delta)\left\{1 - \frac{[\alpha_2(\alpha_2-1)+\delta]^2}{[\alpha_2(\alpha_2-1)+\delta]^2+\delta}\right\}\varepsilon_m^2.$$

In both cases it holds $Q \geqq c_2 \varepsilon_m^2$, $c_2 = \text{const.} > 0$.
As $\alpha \geqq m_1$ we see from (2.16) that

$$\sum_{n=0}^{m-2} \alpha^{\bar{n}} S^n \geqq c_3 \varepsilon_m^2 - C(\varepsilon_0^2+\varepsilon_1^2) - C \Delta t \sum_{n=2}^{m-1} \varepsilon_n^2, \qquad c_3 > 0, \qquad (2.17)$$

hence

$$\sum_{n=0}^{m-2} \left(\alpha^{\bar{n}} \sum_{s=0}^{2} \alpha_s \varepsilon^{n+s}, \sum_{s=0}^{2} \beta_s \varepsilon^{n+s}\right)_0$$

$$\geqq c_3 \|\varepsilon^m\|_0^2 - C(\|\varepsilon^0\|_0^2 + \|\varepsilon^1\|_0^2) - C \Delta t \sum_{n=2}^{m-1} \|\varepsilon^n\|_0^2$$

and from (2.14)

$$\| \varepsilon^m \|_0^2 \leqq C(\| \varepsilon^0 \|_0^2 + \| \varepsilon^1 \|_0^2 + \vartheta^2) + C\Delta t \sum_{n=2}^{m-1} \| \varepsilon^n \|_0^2, \qquad m \geqq 2. \quad (2.18)$$

The discrete analogue of Gronwal's inequality (*see* Lees [8] or [5], Lemma 2.1) gives $\| \varepsilon^m \|_0^2 \leqq C(\| \varepsilon^0 \|_0^2 + \| \varepsilon^1 \|_0^2 + \vartheta^2)$ for $2 \leqq m \leqq T/\Delta t$. It easily follows

$$\| \varepsilon^m \|_0 \leqq C(\| u^0 - U^0 \|_0 + \| u^1 - U^1 \|_0 + h^{p+1} + \Delta t^2)$$

which completes the proof of (2.3).

REMARK: In case that the vector $\mathbf{f}(x, t, u)$ is of the form $\mathbf{f} = \mathbf{b}(x, t, u)\, u$ where $\mathbf{b} = (b_1(x, t, u), \ldots, b_N(x, t, u))^T$ we can assume (instead of f_i being uniformly Lipschitz continuous as functions of u) that the functions b_i are uniformly Lipschitz continuous as functions of u and bounded as functions of all arguments x, t, u. We have namely used the assumption (1.8) in two places, in (2.8) and (2.9). In the first case, it means to estimate $b_i(x, t_n, \xi^{\bar{n}})\, \xi^{\bar{n}} - b_i(x, t_{\bar{n}}, U^{\bar{n}})\, U^{\bar{n}}$. Now $\xi^{\bar{n}}$ is bounded in the maximum norm because of (2.4) and $\xi\,|_\Gamma = 0$. Therefore

$$\begin{aligned} & | b_i(x, t_{\bar{n}}, \xi^{\bar{n}})\xi^{\bar{n}} - b_i(x, t_{\bar{n}}, U^{\bar{n}})\, U^{\bar{n}} | \\ & = | b_i(x, t_{\bar{n}}, U^{\bar{n}})(\xi^{\bar{n}} - U^{\bar{n}}) + [b_i(x, t_{\bar{n}}, \xi^{\bar{n}}) - b_i(x, t_{\bar{n}}, U^{\bar{n}})]\, \xi^{\bar{n}} | \\ & \leqq C | \xi^{\bar{n}} - U^{\bar{n}} | + CL | \xi^n - U^{\bar{n}} | \leqq C | \xi^{\bar{n}} - U^{\bar{n}} |. \end{aligned}$$

The same argument applies in the other case.

3. A-STABLE LINEAR ONE-STEP METHODS

We will briefly show that error estimates for linear one-step A-stable methods are easy to derive in the same way as for linear two-step A-stable methods (the first such estimates were given by Douglas and Dupont [4] and Wheeler [11]). All linear one-step A-stable methods correspond to

$$\rho(\zeta) = \zeta - 1, \qquad \sigma(\zeta) = (1-\Theta)\zeta + \Theta, \qquad \Theta \leqq \frac{1}{2}. \qquad (3.1)$$

(3.1) is often referred to as the "Θ-method" (*see* Lambert [7], p. 240). If $\Theta < 1/2$ the method is of the first order, if $\Theta = 1/2$ we have the trapezoidal rule which is of the second order. Instead of (1.22) we choose

$$\left.\begin{aligned} & t_{\bar{n}} = t_n + \frac{1}{2}\Delta t, \qquad U^{\bar{n}} = \frac{3}{2} U^n - \frac{1}{2} U^{n-1}, \qquad \Theta = \frac{1}{2}, \\ & t_{\bar{n}} = t_n, \qquad U^{\bar{n}} = U^n, \qquad \Theta < \frac{1}{2}. \end{aligned}\right\} \qquad (3.2)$$

The approximate solution U^n is defined by

$$\left.\begin{aligned}&(\alpha^{\bar{n}}[U^{n+1}-U^n], \varphi)_0+\Delta t a(t_{\bar{n}}, U^{\bar{n}}; (1-\Theta) U^{n+1}+\Theta U^n, \varphi)\\&=-\Delta t(\mathbf{f}^{\bar{n}}, \operatorname{grad}\varphi)_0+\Delta t(g^{\bar{n}}, \varphi)_0, \qquad \forall \varphi \in V_h^p.\end{aligned}\right\} \quad (3.3)$$

The matrix form of (3.3) is

$$[M^{\bar{n}}+(1-\Theta)\Delta t K^{\bar{n}}]\mathbf{a}^{n+1}=(M^{\bar{n}}-\Theta\Delta t K^{\bar{n}})\mathbf{a}^{\bar{n}}+\Delta t \mathbf{F}^{\bar{n}} \quad (3.4)$$

(for $\Theta = 1/2$ (3.3) and (3.4), respectively, represent the Crank-Nicolson -Galerkin scheme). We easily derive that

$$(\alpha^{\bar{n}}[\varepsilon^{n+1}-\varepsilon^n], \varphi)_0+\Delta t a(t_{\bar{n}}, U^{\bar{n}}; (1-\Theta)\varepsilon^{n+1}+\Theta\varepsilon^n, \varphi)=\Delta t(\psi^n, \varphi)_1,$$
$$\forall \varphi \in V_h^p,$$

where

$$\|\psi^n\|_1 \leqq C(h^{p+1}+\Delta t^2+\|\varepsilon^{\bar{n}}\|_0), \qquad \Theta=\frac{1}{2},$$

$$\|\psi^n\|_1 \leqq C(h^{p+1}+\Delta t+\|\varepsilon^{\bar{n}}\|_0), \qquad \Theta<\frac{1}{2}.$$

Instead of (2.15) we immediately find

$$S^n \equiv (\varepsilon_{n+1}-\varepsilon_n)[(1-\Theta)\varepsilon_{n+1}+\Theta\varepsilon_n] \geqq \frac{1}{2}(\varepsilon_{n+1}^2-\varepsilon_n^2)$$

from which we easily get

$$\begin{aligned}&\sum_{n=0}^{m-1}(\alpha^{\bar{n}}[\varepsilon^{n+1}-\varepsilon^n], (1-\Theta)\varepsilon^{n+1}+\Theta\varepsilon^n)_0\\&\geqq c_2\|\varepsilon^m\|_0^2-C\varepsilon_0^2-C\Delta t\sum_{n=0}^{m-1}\|\varepsilon^n\|_0^2.\end{aligned} \quad (3.5)$$

Assuming that we choose $\hat{u}^0$ such that

$$\|u^0-\hat{u}^0\|_0 \leqq C h^{p+1}$$

the final estimates are

$$\left.\begin{aligned}&\max_{2\leqq n\leqq T/\Delta t}\|u^n-U^n\|_0 \leqq C(\|u^1-U^1\|_0+h^{p+1}+\Delta t^2), \qquad \Theta=\frac{1}{2}\\&\max_{1\leqq n\leqq T/\Delta t}\|u^n-U^n\|_0 \leqq C(h^{p+1}+\Delta t), \qquad \Theta<\frac{1}{2}.\end{aligned}\right\} \quad (3.6)$$

We have to require the same assumptions as in Theorem with exception of (1.16)-(1.18) and in case of $\Theta < 1/2$ with exception that it is sufficient to assume $\partial^2 u/\partial t^2$ to be continuous for $(x, t) \in \bar{\Omega}\times[0, T]$.

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