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Error estimates for elasto-plastic problems

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ERROR ESTIMATES
FOR ELASTO-PLASTIC PROBLEMS (1)

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Communiqué par P. G. Ciarlet

Abstract. — Under some reasonable smoothness assumptions on the displacements, we are able to derive an error estimate of the form \( \| \sigma - \sigma_h \|_{L^2(\Omega)} \leq C h \), for the approximation of the stress field \( \sigma \) in some problems in elasto-plasticity. Using the same ideas, we also find a piecewise linear approximation of Mosolov’s problem, for which we still get an \( O(h) \) error estimate.

I. INTRODUCTION

In this paper, we consider the approximation of some stationary elastic-perfectly problems formalized by Duvaut-Lions [7]. Our main purpose is to derive error estimates for the approximation of the stress field \( \sigma \) given by a finite element method, appearing in Mercier [16]. The approximate problems we solve, however, will be in terms of the displacements, which are the natural variables for computation.

This work appears to parallel that of Johnson [12], who considered the derivation of error estimates for evolution problems in plasticity. In this stationary case, we are able to obtain improved error estimates over those derived in [12].

Using some ideas from Johnson [11], we are able to establish the existence of a displacement in \( L^7(\Omega) \) for a class of problems in stationary elasto-plasticity.

Finally, we apply the method to obtain error estimates for the elasto-plastic torsion, and Mosolov’s problem.

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II. PHYSICAL PROBLEM

Let us consider (as in [7]) a continuous medium $\Omega \subset \mathbb{R}^N$, submitted to body forces inside $\Omega$, and to pressure loads on a part $\Gamma_F$ of its boundary.

On the other part $\Gamma_U$, it is assumed to be fixed.

The stress field $\sigma \in K$, and the displacement field $u \in V$, are shown ([7]) to be solutions, if they exist, of the following relations:

\[
\begin{align*}
(g(\sigma), \tau - \sigma) - (\varepsilon(u), \tau - \sigma) & \geq 0 \quad \forall \tau \in K; \\
(\sigma, \varepsilon(v)) &= L(v) \quad \forall v \in V;
\end{align*}
\]

with the following notation:

\[
V = \{ v \in (H^1(\Omega))^N \mid v = 0 \text{ on } \Gamma_U \}
\]

is the set of the admissible displacements.

\[
K = \{ \tau \in Y \mid \tau(x) \in P \text{ a.e.} \}
\]

is the convex set of plastically admissible stress fields, where

\[
Y = \{ \tau \mid \tau_{ij} \in L^2(\Omega); \tau_{ij} = \tau_{ji}; i, j = 1, \ldots, N \}
\]

and $P$ is a fixed closed convex subset of $\mathbb{R}^{N^2}$.

We denote by $|.|$ the euclidean norm of $\mathbb{R}^{N^2}$, and observe that $Y$ is a Hilbert space with the scalar product

\[
(\tau, \sigma) = \int_{\Omega} \sum_{i,j=1}^{N} \sigma_{ij} \tau_{ij} \, dx,
\]

and associated norm

\[
\|\tau\| = \left( \int_{\Omega} |\tau|^2 \, dx \right)^{1/2}.
\]

$\varepsilon : V \to Y$ is the strain operator given by

\[
\varepsilon_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).
\]

$L(v)$ is the work of the external loads in a “virtual” displacement $v \in V(L \in V')$.

$g : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$ is an isomorphism representing the elasticity coefficients (the analogue of (1) in the elastic case would be $\varepsilon(u) = g(\sigma)$).

We make the following monotonicity hypothesis on $g$, i.e. there exists $\alpha > 0$ such that

\[
J(\tau) \equiv \frac{1}{2} (g(\tau), \tau) \geq \alpha \|\tau\|^2 \quad \forall \tau \in Y.
\]

We note this implies a coercivity condition on the “complementary energy” $J(\tau)$. 

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Finally, we introduce the set of statically admissible stress fields
\[ M = \{ \tau \in Y : (\tau, \varepsilon(v)) = L(v), \forall v \in V \}. \]
We choose \( \tau \in K \cap M \) in (1). (We suppose the set \( K \cap M \) is non empty.)

We then eliminate \( u \), and we see that \( \sigma \) is the solution of the problem \((P)\):
Find \( \sigma \in K \cap M \) such that
\[ J(\sigma) = \inf_{\tau \in K \cap M} J(\tau). \]

Using hypothesis (3), we have the existence and uniqueness of \( \sigma \). We are not able to prove, in the general case, that there exists a \( u \in V \) such that \((\sigma, u)\) is a solution of (1), (2). However, in a slightly more restrictive case, we are able to prove the existence of a weak solution \( u \in [L^p(\Omega)]^N \) (see section IV).

For the derivation of error estimates, we will assume that \( u \) satisfies the regularity condition
\[ u \in V \cap [H^2(\Omega)]^N \]  
(4)
From the exact solutions, given by Mandel [14], we see that this hypothesis is not an unreasonable one, provided we are not near plastic collapse.

III. APPROXIMATION

Let us assume for simplicity that \( \Omega \) is a bounded polytope. Corresponding to each value of a parameter \( h, 0 < h < 1 \), let \( \mathcal{T}_h \) be a regular triangularization of \( \Omega \) by \( N \)-simplices \( T \) of sides less than or equal to \( h \). Define \( V_h \subset V \) as the subspace of functions in \( V \) which are continuous on \( \Omega \) and linear on each \( T \) of \( \mathcal{T}_h \), and \( Y_h \subset Y \) as the subspace of tensors in \( Y \) which are constant on each \( T \in \mathcal{T}_h \).

For properties of such finite element spaces, we refer the reader to [5], [6]. We note that
\[ \varepsilon : V_h \to Y_h. \]  
(5)

Using the above notation, we define our approximate problem
\( (P_h) \): Find \( \sigma_h \in K \cap M_h \) such that
\[ J(\sigma_h) = \inf_{\tau_h \in K \cap M_h} J(\tau_h), \]
where
\[ M_h = \{ \tau_h \in Y_h : (\tau_h, \varepsilon(v_h)) = L(v_h), \forall v_h \in V_h \}. \]

Applying the results of [16], we know that there exists a unique solution \( \sigma_h \) to problem \((P_h)\) and that it converges to \( \sigma \) as \( h \to 0 \). Our purpose, in this paper, is to derive an error estimate for \( \| \sigma - \sigma_h \| \).

THEOREM 1: If \( u \in [H^2(\Omega)]^N \), we have the error estimate
\[ \| \sigma - \sigma_h \| \leq C h \| u \|_{[H^2(\Omega)]^N} \]
where \( C \) is a constant independent of \( h, u, \) and \( \sigma \).

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Proof: From (1), we get with $x = G^*fe(a)$,
\[(g(\sigma), \sigma_h - \sigma) - \varepsilon(u), \sigma_h - \sigma) \geq 0 \tag{6}\]
and from the definition of $\sigma_h$, we have
\[(g(\sigma_h), \tau_h - \sigma_h) \geq 0 \quad \forall \tau_h \in K \cap M_h. \tag{7}\]

Writing $\tau_h - \sigma_h$ as $\tau_h - \sigma + \sigma - \sigma_h$, and adding (7) to (6), we get
\[(g(\sigma - \sigma_h), \sigma_h - \sigma) + (g(\sigma_h), \tau_h - \sigma) - \varepsilon(u), \sigma_h - \sigma) \geq 0 \quad \forall \tau_h \in K \cap M_h.
\]
Hence, applying (3)
\[\alpha \|\sigma - \sigma_h\|^2 \leq (g(\sigma_h), \tau_h - \sigma) - \varepsilon(u), \sigma_h - \sigma). \tag{8}\]

Since $\sigma_h \in M_h$, and $\sigma \in M$, we have
\[(\sigma - \sigma_h, \varepsilon(v_h)) = 0 \quad \forall v_h \in V_h,
\]
so that
\[(\varepsilon(u), \sigma_h - \sigma) = (\varepsilon(u - v_h), \sigma_h - \sigma) \quad \forall v_h \in V_h.
\]
Since
\[\varepsilon(u - v_h), \sigma_h - \sigma) \leq \|\varepsilon(u - v_h)\| \|\sigma_h - \sigma\|
\]
\[\leq \frac{1}{2\alpha} \|\varepsilon(u - v_h)\|^2 + \frac{\alpha}{2} \|\sigma_h - \sigma\|^2,
\]
we obtain, after collecting terms, that
\[\frac{\alpha}{2} \|\sigma - \sigma_h\|^2 \leq (g(\sigma_h), \tau_h - \sigma) + \frac{1}{2\alpha} \|\varepsilon(u - v_h)\|^2,
\]
\[\forall v_h \in V_h, \quad \tau_h \in K \cap M_h. \tag{9}\]

We now choose $\tau_h = \Pi_h \sigma$ where $\Pi_h$ denotes the projection of $Y \to Y_h$ associated with the norm $\| \cdot \|$. Then
\[(\sigma - \tau_h, \gamma_h) = 0, \quad \forall \gamma_h \in Y_h. \tag{10}\]
Applying (5) and using the fact that $\sigma \in M$, we see that
\[(\tau_h, \varepsilon(v_h)) = (\sigma, \varepsilon(v_h)) = L(v_h) \quad \forall v_h \in V_h,
\]
and hence $\tau_h \in M_h$. Since $Y_h$ is a space of piecewise constants,
\[\tau_h \big|_T = \frac{1}{\text{meas} (T)} \int_T \sigma \, dx.
\]
Then, since $\sigma \in P$ a.e., and $P$ is closed and convex, we get $\tau_h \in P$ for all $T \in \mathcal{T}_h$. Thus $\tau_h \in K \cap M_h$, and from (10),
\[(g(\sigma_h), \tau_h - \sigma) = 0.
\]
Thus (9) becomes
\[ \| \sigma - \sigma_h \| \leq \frac{1}{\alpha} \| \varepsilon (u - v_h) \| \quad \forall v_h \in V_h. \] (11)

Using the continuity of \( \varepsilon \) and the well known approximation properties of the space \( V_h \) [5], we obtain
\[ \| \sigma - \sigma_h \| \leq C_h \| u \|_{H^2(\Omega)}^\infty. \]

IV. REMARKS ON THE EXISTENCE OF A DISPLACEMENT

As in [7], we make the following additional hypotheses. Let \( \| \cdot \| \) be the \( L^\infty \) norm defined by
\[ \| e \|_\infty \equiv \text{ess sup}_{x \in \Omega} |e(x)|. \]

We assume
\[ \exists \delta > 0 \text{ and } \chi \in M \text{ such that } \chi + e \in K, \quad \forall e \in Y \text{ with } \| e \|_\infty \leq \delta. \] (12)

Furthermore, we shall restrict ourselves to the case where
\[ \Gamma_F = \emptyset \text{ and where } L(v) = \int_{\Omega} f(v) \, dx, \ f \in [L^q(\Omega)]^N \text{ with } q = N. \]

Choosing \( \chi_h = \Pi_h \chi \), we see that \( \chi_h \in M_h \), and using the convexity of \( P \), that \( \chi_h \) belongs to the relative interior of \( K \) in \( Y_h \). We may then apply the Kuhn-Tucker theorem [18] to show the existence of \( u_h \in V_h \) such that
\[ (g(\sigma_h), \tau_h - \sigma_h) - (\varepsilon(u_h), \tau_h - \sigma_h) \geq 0 \quad \forall \tau_h \in K. \] (13)

We now define \( (D\varepsilon)_i = - \sum_{j=1}^{N} \frac{\partial \tau_{ij}}{\partial x_j} \) and notice that \( D : Y \to V' \) is the adjoint of \( \varepsilon \).

Using the regularity we assumed on \( L \), we see that the solution \( \sigma \) of \( (P) \) satisfies
\[ -D\sigma + f = 0 \]
in the distribution sense on \( \Omega \). Then
\[ \sigma \in K_1 = \{ \tau \in Y : D\tau \in [L^q(\Omega)]^N \}. \]

We shall now prove the existence of a displacement \( u \) which satisfies the following relation
\[ (g(\sigma), \tau - \sigma) - (u, D(\tau - \sigma)) \geq 0 \quad \forall \tau \in K_1, \] (14)
which can be considered as a weak formulation of (1).

**Theorem 2**: Under hypothesis (12), the sequence \( \varepsilon(u_h) \) is bounded in \( [L^1(\Omega)]^N \).

Hence a subsequence of \( u_h \) is converging weakly to \( u \in [L^q'(\Omega)]^N \) when \( q' = \frac{N}{N - 1} \) and \( (\sigma, u) \) is a solution of (14).
Proof: Let $e \in Y$ satisfy $\|e\|_\infty \leq \delta$ and let $\chi$ be as defined in (12). Since $\tau_h = \Pi_h e + \chi_h \in K$, we may use this choice of $\tau_h$ in (13) to obtain
\[
(g(\sigma_h), \Pi_h e) + (g(\sigma_h), \chi_h - \sigma_h) - (e(u_h), \Pi_h e) - (e(u_h), \chi_h - \sigma_h) \geq 0. \tag{15}
\]
Using the definition (10) of $\Pi_h$, we can replace $\Pi_h e$ by $e$ everywhere in (15). Since $\chi_h$ and $\sigma_h \in M_h$, the last term of (15) is zero. Applying the continuity of $g$, we get
\[
(e, e(u_h)) \leq (g(\sigma_h), \chi_h - \sigma_h) + C \delta \|\sigma_h\| \tag{16}
\]
Since $\Omega$ is bounded, $\sigma_h$ being bounded in $Y$ implies $\sigma_h$ is also bounded in $[L^1(\Omega)]^N$. As (16) is true for all $e \in Y$ with $\|e\|_\infty \leq \delta$, we get
\[
\|e(u_h)\|_{[L^1(\Omega)]^N} \leq C.
\]
We then apply a result of Strauss [19] to obtain
\[
\|u_h\|_{[L^q(\Omega)]^N} \leq C \|e(u_h)\|_{[L^1(\Omega)]^N} \leq C.
\]
From this, we deduce that a subsequence of $u_h$ (which we still denote by $u_h$) is converging weakly to $u$ in $[L^q(\Omega)]^N$.

For any $\tau \in K_1$, we choose $\tau_h = \Pi_h \tau$ in (13) and obtain
\[
(g(\sigma_h), \tau_h) = (g(\sigma_h), \tau) - (e(u_h), \tau_h - \sigma_h). \tag{17}
\]
Now
\[
(e(u_h), \tau_h) = (e(u_h), \tau) = (u_h, D\tau) \to (u, D\tau),
\]
and since $\sigma_h \in M_h$,
\[
(e(u_h), \sigma_h) = (f, u_h) \to (f, u) = (D\sigma, u).
\]
Also
\[
(g(\sigma_h), \tau_h) = (g(\sigma_h), \tau) \to (g(\sigma), \tau),
\]
because $\sigma_h$ converges to $\sigma$ and $g$ is continuous. In the same way $(g(\sigma_h), \chi_h)$ converges to $(g(\sigma), \sigma)$. Hence letting $h \to 0$ in (17), we obtain (14), which is the desired result.

V. OTHER APPLICATIONS

5.1. Elastic-plastic torsion

This problem is usually formulated as the following minimization problem, where $N = 2$, [7]:

Find $u \in K$ minimizing
\[
\frac{1}{2} \|\nabla v\|^2 - (f, v) \quad \text{over } K, \tag{18}
\]
where
\[
K = \{ v \in H_0^1(\Omega) : |\nabla v| \leq 1 \text{ a.e. in } \Omega \},
\]
and
\[
\|v\| = \|v\|_{[L^2(\Omega)]^N}.
\]
Lemma 1: Problem (18) is equivalent to the problem:

Find \( p \in K_1 \cap M \) minimizing \( \frac{1}{2} \| p \|^2 - (\varphi, p) \) over \( K_1 \cap M \), where \( \varphi \) is any solution of

\[
\text{rot} \ \varphi = \frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} = -f
\]

\( K_1 = \{ p \in [L^2(\Omega)]^2 : |p| \leq 1 \ \text{a.e. in} \ \Omega \}, \) and

\( M = \{ p \in [L^2(\Omega)]^2 : (p, \nabla \Psi) = 0, \ \forall \ \Psi \in H^1(\Omega) \}. \)

\( \text{Proof:} \) The result follows easily by using the fact that \( p \in M \) is equivalent to \( p = \text{rot} \ v \) for some \( v \in H^1_0(\Omega) \) (see [13]),

\[
(\varphi, \text{rot} \ v) = (f, v), \quad \forall v \in H^1_0(\Omega)
\]

\[
|\nabla v| = |\text{rot} \ v| \quad \text{for} \quad v \in H^1_0(\Omega)
\]

(Recall that when \( v \) is a scalar, \( \text{rot} \ v \) is the vector deduced from the gradient by a rotation of \( + \pi \)).

Remark: We note that problem (19) is in fact the original problem (see [7]).

We further note that problem (19) can be derived from the more general problem:

Find \( (\cdot, \chi) \in (K_1 \cap M) \times H^1(\Omega) \) satisfying

\[
(p, q - p) - (\varphi + \nabla \chi, q - p) \geq 0 \quad \forall q \in K_1.
\]

(20)

Using a result of Brezis [2], it was proved in [15] that there exists a solution to problem (20), when \( f \) is constant.

We will assume, as in section II, that \( \chi \), which may be interpreted as a displacement, belongs to \( H^2(\Omega) \). We know from [3] that \( p \in [H^1(\Omega)]^2 \) for \( f \in L^2(\Omega) \).

Following the ideas of section III, we approximate problem (19) by the problem

Find \( p_h \in K_1 \cap M_h \) minimizing

\[
\frac{1}{2} \| p_h \|^2 - (\varphi, p_h) \text{ over } K_1 \cap M_h, \text{ where}
\]

\( M_h = \{ p_h \in Y_h : (p_h, \nabla \Psi_h) = 0 \ \forall \ \Psi_h \in V_h \}, \)

\( Y_h \) is the subspace of \( [L^2(\Omega)]^2 \) of piecewise constants, and

\( V_h \) is the subspace of \( H^1(\Omega) \) of continuous piecewise linear functions.

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THEOREM 3: If \( \varphi \in [H^1(\Omega)]^2 \) and \( \chi \in H^2(\Omega) \), then we have the error estimate

\[
\| p - p_h \| \leq Ch \left( \| \varphi \|_1 + \| \chi \|_2 \right),
\]

where \( c \) is a constant independent of \( \varphi, \chi \) and \( h \). (\( \| \varphi \|_1 \) is the norm of \( \varphi \) in \( [H^1(\Omega)]^2 \) and \( \| \chi \|_2 \) is the norm of \( \chi \) in \( H^2(\Omega) \)).

Proof: Proceeding in identical fashion to the proof of theorem 1, we easily obtain the estimate

\[
\frac{1}{2} \| p - p_h \|^2 \leq \frac{1}{2} \| \nabla (\chi - \chi_h) \|^2 + (\varphi, p - q_h) \quad \forall \chi_h \in V_h,
\]

where \( q_h \) has been chosen as the \([L^2(\Omega)]^2\) projection of \( p \) onto \( Y_h \). Since \( (\varphi_h, p - q_h) = 0 \quad \forall \varphi_h \in Y_h \), we get

\[
(\varphi, p - q_h) = (\varphi - \varphi_h, p - q_h) \leq \| \varphi - \varphi_h \| \| p - q_h \|
\]

\[
\leq Ch_2 \| \varphi \|_1 \| p \|_1
\]

(using the standard approximation properties of \( Y_h \) and the assumed regularity of \( p \) and \( \varphi \)). Estimating

\[
\| \nabla (\chi - \chi_h) \|^2 \leq Ch^2 \| \chi \|_2^2
\]

as before, we obtain the desired result.

We remark that the approximation given by (21) is not equivalent to the usual direct approximation of problem (18) by piecewise linear finite elements [10], since this would lead to an internal approximation of \( M \), which is not the case here (\( M_h \neq M \)). For the direct approximation, non-optimal error estimates have previously been derived in [8].

5.2. Mosolov’s problem [7]

This problem is usually formulated as the following:

Find \( u \in H^1_0(\Omega) \) minimizing

\[
\frac{1}{2} \| \nabla v \|^2 + j(\nabla v) - (f, v) \quad \text{over} \quad H^1_0(\Omega), \quad \text{where}
\]

\[
j(p) \equiv g \int_{\Omega} |p| \, dx.
\]

Since \( \Omega \subset \mathbb{R}^2 \), we form an equivalent problem in a similar fashion to lemma 1. We get problem

Find \( p \in M \) minimizing

\[
\frac{1}{2} \| q \|^2 + j(q) - (\varphi, q) \quad \text{over} \quad M, \quad \text{where} \quad \varphi \quad \text{and} \quad M \quad \text{are chosen as in section 5.1}. \quad (23)
\]

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Using duality theory, we have that problem (23) is the dual of the problem
\[
\sup_{\varphi \in H^1(\Omega)} - \frac{1}{2} \left\| \{ \varphi + \nabla \Psi \}^+ - g \right\|^2 \quad (\text{see [17]}). \tag{24}
\]
Since the problem is coercive in $H^1(\Omega)/\mathbb{R}$, we know that it has a solution $\chi \in H^1(\Omega)$. Hence $(p, \chi)$ satisfies the following extremality relation
\[
(p, q - p) + j(q) - j(p) - (\varphi + \nabla \chi, q - p) \geq 0 \quad \forall q \in [L^2(\Omega)]^2.
\]
We will again assume that $\chi \in H^2(\Omega)$, which is a valid assumption at least for the exact solution computed by Glowinski [9]. Using our general technique once more we approximate (23) by the following problem.

Find $p_h \in M_h$ minimizing
\[
\frac{1}{2} \| q_h \|^2 + j(q_h) - (\varphi, q_h) \quad \text{over} \quad q_h \in M_h, \tag{25}
\]
where $M_h$ is defined as in section 5.1.

**Theorem 4:** If $\varphi \in [H^1(\Omega)]^2$ and $\chi \in H^2(\Omega)$, then we have the error estimate
\[
\| p - p_h \| \leq C h [\| \varphi \|_1 + \| \chi \|_2]
\]

**Proof:** Proceeding in an identical fashion to the proof of theorem 3, we easily obtain the estimate
\[
\frac{1}{2} \| p - p_h \|^2 \leq C h^2 [\| \varphi \|_1 + \| \chi \|_2]^2 + j(q_h) - j(p),
\]
where $q_h$ is again the $[L^2(\Omega)]^2$ projection of $p$ onto $Y_h$. Hence
\[
\forall T \in \mathcal{C}_h \quad q_h |_T = \frac{1}{\text{meas}(T)} \int_T p \, dx,
\]
and the convexity of $j$ implies that $j(q_h) \leq j(p)$. Thus, we get the desired result.

We remark that this approximation is again different from the direct approximation of (22) for which quasi-optimal error estimates have already been derived [9].

As far as we know, the approximate problem (25) has not been solved numerically. What we should suggest for such a numerical computation is to try to solve directly the approximation of the dual problem (24), when $H^1(\Omega)$ is approximated by $V_h$, because this problem would be the dual of (25). Furthermore, it is a problem of unconstrained minimization of a differentiable (but not strictly convex) function.

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