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SOME ASYMPTOTIC ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATION OF MINIMAL SURFACES (1)

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Abstract. — *The solution of a minimal surface problem over a plane domain is approximated by piecewise linear finite elements. Using related results of Johnson/Thomée [5] and a weighted Sobolev norm technique introduced by Nitsche [11], we show the L^∞-convergence with rate* $0(h^2\ |\ln h|)$ *and the L^2-convergence with rate* $0(h^2)$.

1. INTRODUCTION

Let Ω be a bounded, strictly convex domain in the plane R^2 with smooth boundary $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$, and let $g \in C^0(R^2)$ be a real function. We consider the following minimal surface problem for functions $u \in C^{0,1}(\overline{\Omega})$:

$$\text{(V)} \qquad \int_\Omega (1 + |\nabla u|^2)^{1/2}\, dx = \text{Min.}, \qquad u = g \text{ on } \partial\Omega.$$

If g satisfies the bounded slope condition on $\partial\Omega$ and is in the Sobolev space $W^{2,q}(\Omega)$ for some $q \in]2, \infty[$ or in the Hölder space $C^{2,\alpha}(\overline{\Omega})$, then it is known that there is a unique minimizing function

$$u \in W^{2,q}(\Omega) \qquad \text{or} \quad u \in C^{2,\alpha}(\overline{\Omega}) \subset W^{2,\infty}(\Omega),$$

respectively, (see [7; 4.2.1]). This solution will be approximated by the simplest finite element method.

Let $\hat{\Omega}_h \supset \overline{\Omega}$, $o < h \leq h_0 < 1$, be polygonal domains, and let $T_h = \{ T_i \}$ be finite triangulations of $\hat{\Omega}_h$, such that the triangles have disjoint interiors and all their edges are the edge of another triangle or of the polygon $\partial\hat{\Omega}_h$. Further all vertices of the inscribed polygonal domain

$$\Omega_h := \cup \{ T \in T_h \mid T \subset \overline{\Omega} \} \subset \overline{\Omega} \subset \hat{\Omega}_h$$

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are on the boundary $\partial\Omega$. It is assumed that the triangulations T_h are quasi-regular :

(*T*) *Each triangle* $T \in T_h$ *contains a circle with radius* $c_1 h$ *and is contained in a circle with radius* $c_2 h$.

We define finite dimensional subspaces $S_h \subset W^{1,2}(\hat{\Omega}_h)$ by

$$S_h := \{ v_h \in C(\hat{\Omega}_h) \mid v_h \text{ linear on each } T \in T_h \}$$

and $S_h^0 \subset W_0^{1,2}(\Omega_h) \subset W_0^{1,2}(\Omega)$ by

$$S_h^0 := \{ v_h \in S_h \mid v_h = 0 \text{ on } \hat{\Omega}_h - \Omega_h \}.$$

The usual interpolant $I_h g \in S_h$ of g is determined by

$$I_h g(x) = g(x) \qquad \text{for each vertex} \quad x \in T \in T_h. \tag{1}$$

Now the approximating functions $u_h \in S_h$ are defined by the finite problems

$$(\mathrm{V}_h) \qquad \int_{\Omega_h} (1 + |\nabla u_h|^2)^{1/2}\, dx = \text{Min.}, \qquad u_h - I_h g \in S_h^0.$$

The function $F(\eta) := (1 + |\eta|^2)^{1/2}$ is strictly convex on R^2.

Thus the existence of unique solutions of (V_h) follows from the continuity and coerciveness of the functional.

Johnson/Thomee [5] have shown for $u \in W^{2,2}(\Omega) \cap W^{1,\infty}(\Omega)$ the rate of convergence

$$\|\nabla(u - u_h)\|_2 = O(h) \tag{2}$$

and the estimate

$$\|\nabla u_h\|_\infty = O(1). \tag{3}$$

Further for $u \in W^{2,q}(\Omega)$, $q > 2$, they derived, using a common duality argument,

$$\|u - u_h\|_p = O(h^2) \quad , \quad 1 \le p < 2. \tag{4}$$

For uniform triangulations and $u \in C^2(\overline{\Omega}) \cap W^{3,2}(\Omega)$ Mittelmann [6] has shown the L^∞-estimate

$$\|u - u_h\|_\infty = O(h^{3/2}\, |\ln h|^{1/2}). \tag{5}$$

In the linear case $F(u) = |\nabla u|^2 - 2fu$ with $u \in W^{2,\infty}(\Omega)$ the analogue of the method (V_h) has the order of convergence (see [8], [13], [11], [12], [4])

$$\|u - u_h\|_\infty = O(h^2\, |\ln h|). \tag{6}$$

Recently this was obtained by Frehse [3] for the general variational problem, too :

$$\int_\Omega F(., u, \nabla u)\, dx = \text{Min.}, \qquad u \in W_0^{1,2}(\Omega). \tag{7}$$

He used a Morrey norm estimate

$$\| |\nabla (u - u_h)| \|_\alpha = O(h) \qquad \text{with an} \quad \alpha > 0, \tag{8}$$

which is proven in [2] for zero boundary conditions and uniformly convex functions $F(x, \xi, .)$. *These assumptions are not valid in the case of minimal surfaces.* In this note we shall prove the L^∞-estimate (6) for the nonlinear problem (V) without making use of (8). Furthermore the L^p-estimate (4) will be extended to the case $p = 2$.

THEOREM : *Assume* $\partial\Omega \in C^{2,\alpha}$, $0 < \alpha < 1$, *and* $g \in W^{2,q}(R^2) \cap W^{2,q}(\partial\Omega)$ *and* $u \in W^{2,q}(\Omega)$ *for some* q *with* $2 < q \leq \infty$. *Further let the triangulations* T_h *be quasi-regular in the sense of* (T). *Then*

$$\|u - u_h\|_\infty = O(h^{2-2/q} |\ln h|), \tag{I}$$

$$\|u - u_h\|_2 = O(h^2). \tag{II}$$

REMARKS : The methods applied in the proof also work with slight modifications for the general variational problem (7) under the assumptions of [3]. This will be of its own interest since the estimate (8) is somewhat hard to prove. For this problem the above L^∞-estimate can be extended to higher dimensions $n \geq 3$, too. This will be carried out in a forthcoming paper.

Moreover, there is no difficulty to obtain analogously to [11] the order of pointwise convergence $O(h^m)$ for the finite element approximation of a solution $u \in W^{m,\infty}(\Omega)$ of (7) with elements of order $m \geq 3$.

Here and below $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $W_0^{m,p}(\Omega)$ denote the usual real Lebesgue and Sobolev spaces with the corresponding norms

$$\|\cdot\|_p = \|\cdot\|_{p,\Omega} \quad , \quad \|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega} \quad , \quad 1 \leq p \leq \infty \quad , \quad m \in N_0 .$$

Further we use the abbreviations

$$\partial_i v := \partial v / \partial x^i \quad , \quad i = 1, 2, \nabla^1 v = \nabla v := \text{grad } v \quad , \quad \nabla^2 v := (\partial_i \partial_k v)_{i,k=1,2},$$

for the partial (generalized) derivatives, and c for a positive (generic) constant which is independent of the parameters h and ρ, defined below. Finally, we shall use the usual summation convention.

2. PROOF OF THE THEOREM

First we introduce some notations and technical facts.

The minimizing functions $u \in W^{2,q}(\Omega) \subset C^1(\bar{\Omega})$, $2 < q \leq \infty$, of (V) and $u_h \in S_h$ of (V_h) necessarily satisfy the Euler equations

$$\int_\Omega (1 + |\nabla u|^2)^{-1/2} \nabla u \,.\, \nabla v \, dx = 0, \qquad \forall v \in W_0^{1,2}(\Omega),$$

$$\int_{\Omega_h} (1 + |\nabla u_h|^2)^{-1/2} \nabla u_h \,.\, \nabla v_h \, dx = 0, \qquad \forall v_h \in S_h^0 .$$

Denoting by $F_{ik} := \partial_i \partial_k F$, $i, k = 1, 2$, the second derivatives of the function $F(\eta) := (1 + |\eta|^2)^{1/2}$, $\eta \in R^2$, we obtain

$$F_{ik}(\eta)\xi_i\xi_k \geq (1 + |\eta|^2)^{-3/2} |\xi|^2 \quad , \quad \eta, \xi \in R^2.$$

Combination of the Euler equations yields

$$a^h(v_h, u - u_h) := \int_{\Omega_h} a^h_{ik}(.)\, \partial_i v_h\, \partial_k(u - u_h)\, dx = 0, \qquad \forall v_h \in S^0_h, \tag{9}$$

with the $L^\infty(\Omega)$-functions

$$a^h_{ik}(.) := \int_0^1 F_{ik}(\nabla u_h(.) + t\,\nabla(u - u_h)(.))\, dt, \qquad i, k = 1, 2.$$

Further by the result (3),

$$a^h_{ik}\xi_i\xi_k \geq c\,|\xi|^2 \quad , \quad \xi \in R^2 \quad , \quad \text{on} \quad \bar{\Omega}.$$

Since the a^h_{ik} are discontinuous we introduce the bilinear form

$$a(v, w) := \int_\Omega a_{ik}(.)\, \partial_i v\, \partial_k w\, dx, \qquad v, w \in W^{1,2}(\Omega),$$

with coefficients

$$a_{ik}(.) := F_{ik}(\nabla u(.)) \in W^{1,q}(\Omega) \quad , \quad i, k = 1, 2.$$

Then the differential operator

$$A := -\,\partial_k \{\, a_{ik}(.)\partial_i \,\} \tag{11}$$

is uniformly elliptic in Ω and hence satisfies the well known a priori estimate (see [7; 5.2 ff]) :

$$\|v\|_{2,2} \leq c\,\|Av\|_2 \quad , \quad \forall v \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega).$$

From the boundedness of the derivatives of F_{ik} it follows that

$$|a_{ik} - a^h_{ik}| \leq c\,|\nabla(u - u_h)| \qquad \text{on} \quad \Omega. \tag{12}$$

For functions $v \in W^{2,p}(\Omega)$, $2 \leq p \leq \infty$, and $w \in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ and the corresponding interpolants $I_h v \in S_h$ and $I_h w \in S^0_h$ the following estimates are known (see [1] and [10]) :

$$\|v - I_h v\|_{j,p,\Omega_h} \leq c h^{2-j}\, \|\nabla^2 v\|_{p\,\Omega_h} \quad , \quad 0 \leq j < 2 \quad , \quad 2 \leq p \leq \infty. \tag{13}$$

$$\|v - I_h v\|_{j,\infty,\Omega_h} \leq c h^{2-j-2/p}\, \|\nabla^2 v\|_{p;\Omega_h}$$

Using in addition Lemma A4 of the Appendix we find with (15)

$$\|w - I_h w\|_{j,2} \leq c h^{2-j} \|w\|_{2,2} \quad , \quad 0 \leq j < 2. \tag{14}$$

Proof of Proposition (I)

In order to make the outline of the proof clear we write its main steps as lemmas.

Set $e_h := u - u_h$ and let $E_h := I_h e_h = I_h u - u_h$ be its interpolant on Ω_h. Observe that $E_h \in S_h^0$.

First we estimate e_h on $\Omega - \Omega_h$:

The assumptions (T) and $\partial\Omega \in C^2$ imply

$$d(\partial\Omega_h, \partial\Omega) := \sup_{x \in \partial\Omega_h} \operatorname{dist}(x, \partial\Omega) \leq c h^2. \tag{15}$$

Thus, we conclude using $u = g$ on $\partial\Omega$, $u_h = I_h g$ on $\Omega - \Omega_h$ and the estimate (13)

$$\begin{aligned} \|e_h\|_{\infty;\Omega-\Omega_h} &\leq \|u - g\|_{\infty;\Omega-\Omega_h} + \|g - I_h g\|_{\infty;\Omega-\Omega_h} \\ &\leq c h^{2-2/q}(\|u\|_{2,q} + \|g\|_{2,q,R_?^2}). \end{aligned}$$

Next, let $z_h \in \Omega_h$ be points with the properties

$$|E_h(z_h)| = \|E_h\|_\infty .$$

Then, with any disk $B := B_\tau(z_h)$, $\tau > 0$,

$$\|E_h\|_\infty \leq c\tau^{-2} \int_B |E_h| \, dx + c\tau \|\nabla E_h\|_\infty .$$

Using the well known inverse relation $\|\nabla E_h\|_\infty \leq \mathrm{c}h^{-1} \|E_h\|_\infty$, we find that for $\tau := \delta h$, $\delta > 0$ a constant sufficiently small,

$$\|E_h\|_\infty \leq c h^{-2} \int_B |E_h| \, dx.$$

Thus by the estimate (13)

$$\begin{aligned} \|e_h\|_{\infty;\Omega_h} &\leq \|u - I_h u\|_{\infty;\Omega_h} + \|E_h\|_\infty \\ &\leq c h^{2-2/q} \|u\|_{2,q} + c h^{-2} \int_B |e_h| \, dx. \end{aligned} \tag{17}$$

In the following our main tool will be a modification of the weighted norm technique by Nitsche [11]. With a real parameter $0 < \rho \leq \rho_0$, which will be appropriately coupled with h below, we define the weight function

$$\sigma_h(.) := (|. - z_h|^2 + \rho^2)^{1/2}$$

and the weighted norms (T denoting triangles of T_h)

$$\|\cdot\|_{(\nu)} := \left(\sum_{T \subset \Omega_h} \int_T \sigma_h^\nu |\cdot|^2 \, dx \right)^{1/2}, \nu \in R. \tag{18}$$

Since the points z_h and the corresponding disks $B := B_{\delta h}(z_h)$ will be fixed during the proof, we shall omit the index of σ.

Obviously with constants independent of ρ

$$|\nabla\sigma| \leq c \quad , \quad |\nabla^2 \sigma| \leq c\sigma^{-1} \leq c\rho^{-1} \qquad \text{on} \quad R^2,$$

and for $\rho \geq c_3 h$, c_3 sufficiently large,

$$\max_{T \in T_h} \{ \max_{x \in T} \sigma^\nu(x) / \min_{x \in T} \sigma^\nu(x) \} \leq c, \qquad -4 \leq \nu \leq 4.$$

From this and (13) we conclude the following interpolation estimate for functions $v \in C(\Omega_h) \cap \bigoplus_{T \subset \Omega_h} W^{2,2}(T)$ (see [11]) :

$$\|v - I_h v\|_{(\nu)} + h \|\nabla(v - I_h v)\|_{(\nu)} \leq ch^2 \|\nabla^2 v\|_{(\nu)} \quad , \quad -4 \leq \nu \leq 4. \tag{19}$$

LEMMA 1 : *Let* $1 < \beta < 2$. *Then there are constants* c_4, c_β, *independent of* h *and* $\rho \geq c_3 h$, *so that*

$$\|e_h\|_{\infty;\Omega_h} \leq c_4 h^{2-2/q} |\ln h| + c_\beta \rho h^{-1} |\ln h| \, \|\nabla e_h\|^2_{(-\beta)}.$$

Proof. We shall estimate the integral in (17) making use of a common duality argument. Let $G^h \in \overset{\circ}{W}{}^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)$ be the solution of the boundary value problem (smoothed Green function)

$$AG^h = h^{-2} sgn(e_h)\chi_B \qquad \text{in} \quad \Omega \quad , \quad G^h = 0 \qquad \text{on} \quad \partial\Omega, \tag{20}$$

with χ_B the characteristic function of the disk B, and let $G_h^h \in S_h^0$ be its Ritz projections defined by

$$a(v_h, G_h^h) = a(v_h, G^h) \quad , \quad \forall v_h \in S_h^0. \tag{21}$$

With this setting we obtain by integration by parts denoting

$$\partial_n := -n_k a_{ik} \partial_i \quad , \quad (n_1, n_2) := \text{outward normal to} \quad \partial\Omega,$$

$$h^{-2} \int_B |e_h| \, dx = \int_{\partial\Omega} e_h \partial_n G^h \, ds + a(e_h, G^h) \tag{22}$$

and by (9)

$$h^{-2} \int_B |e_h| \, dx = \int_{\partial\Omega} e_h \partial_n G^h \, ds + a(e_h, G^h - G_h^h) + (a - a^h)(e_h, G_h^h). \tag{23}$$

Now we shall estimate the three terms on the right.

Since $e_h = g - I_h g$ on $\partial\Omega$, we find using Hölder's inequality and a well known trace theorem (see [9])

$$\left| \int_{\partial\Omega} e_h \partial_n G^h \, ds \right| \leq c \, \|g - I_h g\|_{q;\partial\Omega} \, \|G^h\|_{2,r}, \qquad 1/r = 1 - 1/(2q) \quad (24)$$

For $g \in W^{2,q}(R^2) \cap W^{2,q}(\partial\Omega)$ it is proven in [5] that

$$\|g - I_h g\|_{q;\partial\Omega} \leq c h^2.$$

Using Hölder's inequality (notice $r = 2q/(2q - 1) < 2$), Poincaré's inequality for $G^h \in W_0^{1,2}(\Omega)$ and Lemma A2(a) with $\rho \geq c_3 h$, we find (We note that the logarithm only appears for $r = 1$.)

$$\|G^h\|_{2,r} \leq c h^{-1/q} \, |\ln h|^{1/2} \, \{ \, \|\nabla G^h\|_2 + \|\nabla^2 G^h\|_{(2)} \} \leq c h^{-1/q} \, |\ln h|. \quad (25)$$

Thus

$$\left| \int_{\partial\Omega} e_h \partial_n G^h \, ds \right| \leq c h^{2 - 1/q} \, |\ln h|. \quad (26)$$

The second term in (23) is the same that occurs in the case of linear problems (see [4]). The modified interpolate of u, defined by

$$\hat{I}_h u := \begin{cases} I_h u & \text{on } \Omega_h, \\ I_h g & \text{on } \hat{\Omega}_h - \Omega_h, \end{cases}$$

obviously satisfies $\hat{I}_h u - u_h \in S_h^0$. Hence, we find using (21)

$$\begin{aligned} |a(e_h, G^h - G_h^h)| &= |a(u - \hat{I}_h u, G^h - G_h^h)| \\ &\leq c \, \|\nabla(u - I_h u)\|_{\infty;\Omega_h} \, \|\nabla(G^h - G_h^h)\|_1 \\ &\quad + c(\|u\|_{2,q} + \|g\|_{2,q;R^2}) \times \int_{\Omega - \Omega_h} |\nabla G^h| \, dx. \end{aligned}$$

By Lemma A4 (notice (15)) and Lemma A2(a) (with $\varepsilon = 0$, $\rho \geq c_3 h$) it follows in the same way as in (25) with $r = 1$

$$\int_{\Omega - \Omega_h} |\nabla G^h| \, dx \leq c h^2 \, \|G^h\|_{2,1} \leq c h^2 \, |\ln h|.$$

Thus, we conclude using the estimate (13) and the result, stated in Lemma A2(b), concerning the convergence of Green functions :

$$|a(e_h, G^h - G_h^h)| \leq c h^{2 - 2/q} \, |\ln h|. \quad (27)$$

The third term in (23) comes from the nonlinearity of the problem (V), i.e. from the replacement of the discontinuous coefficients a^h_{ik} by $a_{ik} \in W^{1,q}(\Omega)$. By making use of (12) and Lemma A2(c) (with $\rho \geq c_3 h$) we see (notice $G^h_h = 0$ on $\Omega - \Omega_h$)

$$|(a - a^h)(e_h, G^h_h)| \leq c \int_{\Omega_h} |\nabla e_h|^2 \sigma^{-\beta} |\sigma^\beta \nabla G^h_h| \, dx$$
$$\leq c_\beta \rho h^{-1} |\ln h| \, \|\nabla e_h\|^2_{(-\beta)}.$$

Together with (27) and (26) this establishes the desired estimate.

Since $\beta < 2$ our proposition (I) suggests the estimate $\|\nabla e_h\|_{(-\beta)} = O(h^{1-1/q})$ without any logarithmic term. Obviously this would complete the proof of (I). A first step in this direction will be the following :

LEMMA 2 : *Let $1 < \beta < 2$. Then there are constants c_6, c_β, independent of h, so that for $\rho = c_6 h |\ln h|^{3/2}$*

$$\|\nabla e_h\|^2_{(-\beta)} \leq c_\beta h^{3-\beta-2/q} |\ln h|.$$

Proof. The result (2) gives for $\rho \geq h$

$$\|\nabla e_h\|^2_{(-\beta)} \leqq c \, \|\nabla e_h\|_{(-2)} \, \|\nabla e_h\|_2 h^{1-\beta} \leq c h^{2-\beta} \, \|\nabla e_h\|_{(-2)}. \tag{28}$$

The weighted norm on the right will be estimated in the same way as in [11] and [12]. By (10) we have

$$\|\nabla e_h\|^2_{(-2)} \leq c \, |a^h(e_h, \sigma^{-2} e_h)| + c \int_{\Omega_h} |\nabla e_h| \, |e_h| \, |\nabla \sigma^{-2}| \, dx$$
$$\leq c \, |a^h(e_h, \sigma^{-2} e_h)| + c \, \|\nabla e_h\|_{(-2)} \, \|e_h\|_{(-4)}.$$

Since $v_h := I_h(\sigma^{-2} E_h) \in S^0_h$, $E_h := I_h u - u_h$, it follows by (9) that

$$|a^h(e_h, \sigma^{-2} e_h)| = |a^h(e_h, \sigma^{-2}(u - I_h u) + \sigma^{-2} E_h - v_h)|$$
$$\leq c \, \|\nabla e_h\|_{(-2)} \{ \|\nabla(\sigma^{-2}(u - I_h u))\|_{(2)} + \|\nabla(\sigma^{-2} E_h - v_h)\|_{(2)} \}.$$

Using the estimate (19), we get for $\rho \geq c_3 h$ (notice $\nabla^2 E_h = 0$ on each $T \in T_h$)

$$\|\nabla(\sigma^{-2}(u - I_h u))\|_{(2)} \leq c \{ \|u - I_h u\|_{(-4)} + \|\nabla(u - I_h u\|_{(-2)} \}$$
$$\leq c h \, \|\nabla^2 u\|_{(-2)}$$

and

$$\|\nabla(\sigma^{-2} E_h - v_h)\|_{(2)}$$
$$\leq c h \, \|\nabla^2(\sigma^{-2} E_h)\|_{(2)}$$
$$\leq c h \{ \|e_h\|_{(-6)} + \|\nabla e_h\|_{(-4)} + \|u - I_h u\|_{(-6)} + \|\nabla(u - I_h u)\|_{(-4)} \}.$$
$$\leq c \{ \|e_h\|_{(-4)} + h \rho^{-1} \|\nabla e_h\|_{(-2)} + h \, \|\nabla^2 u\|_{(-2)} \}.$$

Thus

$$\|\nabla e_h\|_{(-2)} \leq c\,\|e_h\|_{(-4)} + ch\rho^{-1}\,\|\nabla e_h\|_{(-2)} + ch\,\|\nabla^2 u\|_{(-2)}, \tag{29}$$

and with $\rho \geq c_5 h$, c_5 sufficiently large,

$$\|\nabla e_h\|_{(-2)} \leq c\,\|e_h\|_{(-4)} + ch^{1-2/q}\,|\ln h|^{1/2}\,\|u\|_{2,q}. \tag{30}$$

In order to estimate the first term on the right we use a duality argument. With the solution $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ of the boundary value problem

$$Av = \sigma^{-4} e_h \quad \text{in } \Omega \ , \quad v = 0 \quad \text{on } \partial\Omega,$$

we obtain by integration by parts

$$\|e_h\|_{(-4)}^2 = \int_{\partial\Omega} e_h \partial_n v\, ds + a(e_h, v)$$

and in addition with $I_h v \in S_h^0$ by (9) (see (23))

$$\|e_h\|_{(-4)}^2 = \int_{\partial\Omega} e_h \partial_n v\, ds + a(e_h, v - I_h v) + (a - a^h)(e_h, I_h v). \tag{31}$$

The three terms on the right will be estimated analogously to those in (23). With $1/r = 1 - 1/(2q)$ we get

$$\left| \int_{\partial\Omega} e_h \partial_n v\, ds \right| \leq c\,\|g - I_h g\|_{q;\partial\Omega}\,\|v\|_{2,r}$$
$$\leq ch^{2-1/q}\,|\ln h|^{1/2}\,\{\,\|\nabla v\|_2 + \|\nabla^2 v\|_{(2)}\,\}.$$

From the estimate (19) and Lemma A4 it follows that

$$|a(e_h, v - I_h v)| \leq c\,\|\nabla e_h\|_{(-2)}\,\|\nabla(v - I_h v)\|_{(2)} + c\,\|\nabla(u - I_h g)\|_\infty \int_{\Omega - \Omega_h} |\nabla u|\, dx$$
$$\leq ch\,\|\nabla e_h\|_{(-2)}\,\|\nabla^2 v\|_{(2)} + ch^2\,\|v\|_{2,1}$$
$$\leq c\,\{\,h\,\|\nabla e_h\|_{(-2)} + h^2\,|\ln h|^{1/2}\,\}\,\{\,\|\nabla v\|_2 + \|\nabla^2 v\|_{(2)}\,\}.$$

Using the estimate (12), Lemma A1 (with $\nu = 2$) and the result (2), we find

$$|(a - a^h)(e_h, I_h v)| \leq c\,\|\nabla e_h\|_{(-1)}^2\,|\ln h|\,\{\,\|\nabla v\|_2 + \|\nabla^2 v\|_{(2)}\,\}$$
$$\leq ch\,|\ln h|\,\|\nabla e_h\|_{(-2)}\,\{\,\|\nabla v\|_2 + \|\nabla^2 v\|_{(2)}\,\}.$$

Thus, by Lemma A3, observing that $\|Av\|_{(4)} = \|e_h\|_{(-4)}$,

$$\|e_h\|_{(-4)} \leq ch^{1-1/q}\,|\ln h| + ch\rho^{-1}\,|\ln h|^{3/2}\,\|\nabla e_h\|_{(-2)}.$$

We substitute this in (30), choose $\rho = c_6 h\,|\ln h|^{3/2}$, c_6 appropriately large, and obtain

$$\|\nabla e_h\|_{(-2)} \leq ch^{1-2/q}\,|\ln h|. \tag{32}$$

From this the desired estimate follows.

By combination of Lemma 1 and Lemma 2, we find as a first result for $1 < \beta < 2$

$$\|e_h\|_{\infty;\Omega_h} \leq c_\beta h^{3-\beta-2/q} |\ln h|^4. \tag{33}$$

Now, this will be used to improve the estimate of Lemma 2.

LEMMA 3 : *Let* $1 < \beta_q < 1 + (q-2)/(3q)$. *Then there are constants* c_7, c_8 *independent of h, so that for* $\rho = c_7 h$

$$\|\nabla e_h\|_{(-\beta_q)} \leq c_8 h^{1-1/q}.$$

Proof. Set $\rho \geq c_3 h$ to guarantee the interpolation estimate (19). We start in the same way as in the proof of Lemma 2 concerning the term $\|\nabla e_h\|_{(-2)}$. With $v_h := I_h(\sigma^{-\beta} E_h) \in S_h^0$, $E_h := I_h u - u_h$, we find analogously to (29)

$$\|\nabla e_h\|_{(-\beta)} \leq c\,\|e_h\|_{(-\beta-2)} + ch\rho^{-1}\,\|\nabla e_h\|_{(-\beta)} + ch\,\|\nabla^2 u\|_{(-\beta)}$$

and for $\rho = c_7 h$, c_7 appropriately large,

$$\|\nabla e_h\|_{(-\beta)} \leq c\,\|e_h\|_{(-\beta-2)} + ch\,\|\nabla^2 u\|_{(-\beta)}.$$

Thus, by the result (33)

$$\begin{aligned}\|\nabla e_h\|_{(-\beta)} &\leq ch^{3-\beta-2/q}\,|\ln h|^4\,\|1\|_{(-\beta-2)} + ch^{2-\beta/2-2/q}\,\|u\|_{2,q}\\ &\leq ch^{1-1/q}\,\{\,h^{2-3\beta/2-1/q}\,|\ln h|^4 + h^{1-\beta/2-1/q}\,\}.\end{aligned}$$

Since $q > 2$, we can find some β_q with $1 < \beta_q < 1 + (q-2)/(3q) < 2$, so that

$$h^{(2-3\beta q/2-1/q)}\,|\ln h|^4 + h^{(1-\beta q/2-1/q)} \leq c.$$

This proves Lemma 3.

Finally, combination of Lemma 3 and Lemma 1 completes the proof of proposition (I).

Proof of Proposition (II)

The proof of Proposition (II) makes use of the estimate (I) for the given $q > 2$ $(e_h := u - u_h)$:

$$\|e_h\|_\infty \leq ch^{2-2/q}\,|\ln h| \leq c_q h. \tag{36}$$

Let $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ be the solution of the problem

$$Av = e_h \quad \text{in} \quad \Omega \quad , \quad v = 0 \quad \text{on} \quad \partial\Omega. \tag{37}$$

Using (9), we find analogously to (31) with $I_h v \in S_h^0$

$$\|e_h\|_2^2 = \int_{\partial\Omega} e_h\,\partial_n v\,ds + a(e_h, v - I_h v) + (a - a^h)(e_h, I_h v). \tag{38}$$

The three terms on the right can be estimated in a similar way as those in (31). We get

$$\left| \int_{\partial\Omega} e_h \partial_n v \, ds \right| \leq c \, \|g - I_h g\|_{2;\partial\Omega} \|v\|_{2,2} \leq c h^2 \, \|v\|_{2,2}$$

and, using (14) and the result (2),

$$|a(e_h, v - I_h v)| \leq c \, \|\nabla e_h\|_2 \, \|\nabla(v - I_h v)\|_2 \leq c h^2 \, \|v\|_{2,2}.$$

Further

$$|(a - a^h)(e_h, I_h v)| \leq c \sum_{i,k} \int_{\Omega_h} |a_{ik} - a^h_{ik}| \, |\nabla e_h| \, |\nabla I_h v| \, dx.$$

It will be convenient to replace the discontinuous function $|\nabla I_h v|$ by a continuous approximant $\;_h := I_h \, |\nabla v|$ 0 of $|\nabla v| \in W^{1,}(\Omega)$ on Ω_h. By the boundeness of a_{ik}, a^h_{ik} and the estimate (12), we have

$$|(a - a^h)(e_h, I_h v)| \leq c \int_{\Omega_h} \{ |\nabla e_h| \, |\nabla(I_h v - v)| + |\nabla e_h| \, ||\nabla v| - \psi_h| + |\nabla e_h|^2 \, \psi_h \} \, dx$$

and, using (13) and the result (2),

$$\begin{aligned} |(a - a^h)(e_h, I_h v)| &\leq c \, \|\nabla e_h\|_2 h \, \|v\|_{2,2} + c \int_{\Omega_h} |\nabla e_h|^2 \psi_h \, dx \\ &\leq c h^2 \, \|v\|_{2,2} + c \int_{\Omega_h} |\nabla e_h|^2 \psi_h \, dx. \end{aligned}$$

The integral on the right can be estimated in the same way as the term $\|\nabla e_h\|_{(-2)}$ in the proof of Lemma 2. Defining $v_h := I_h(\psi_h E_h) \in S^0_h$, $E_h := I_h u - u_h$, we have by (9)

$$\left| \int_{\Omega_h} |\nabla e_h|^2 \psi_n \, dx \right| \leq c \, |a^h(e_h, \psi_h(u - I_h u) + \psi_h E_h - v_h)| + c \int_{\Omega_h} |\nabla e_h| \, |e_h| \, |\nabla \psi_h| \, dx$$

and hence

$$\begin{aligned} \int_{\Omega_h} |\nabla e_h|^2 \psi_h \, dx &\leq c \, \|\nabla e_h\|_2 \, \{ \|\nabla \psi_h (u - I_h u)\|_{2;\Omega_h} + \|\psi_h \nabla(u - I_h u)\|_{2;\Omega_h} \\ &\qquad + \|\nabla(\psi_h E_h - v_h)\|_{2;\Omega_h} + \|\nabla \psi_h e_h\|_{2;\Omega_h} \} \\ &\leq c \, \|\nabla e_h\|_2 \, \{ \text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)} \}. \end{aligned}$$

Using Hölder's inequality, the Sobolev embedding theorem and (13), it follows that

(i) $\leq c \|u - I_h u\|_{\infty,\Omega_h} \|\nabla \psi_h\|_{2;\Omega_h} \leq c h^{2-2/q} \|u\|_{2,q} \|v\|_{2,2}$,

(ii) $\leq c \|\nabla (u - I_h u)\|_{q;\Omega_h} \|\psi_h\|_{1;2\,\Omega_h} \leq c h \|u\|_{2,q} \|v\|_{2,2}$.

Observing $\psi_h \geqq 0$, $\nabla^2 \psi_h = \nabla^2 E_h = 0$ on each $T \in T_h$ and the result (3), we find

$$\text{(iii)} \qquad \leq c h \left(\sum_{T \subset \Omega_h} \int_T |\nabla^2 (\psi_h E_h)|^2 \, dx \right)^{1/2} \leq \text{ch} \|\nabla \psi_h\|_{2,\Omega_h} \|\nabla E_h\|_\infty$$
$$\leq c h \|u\|_{2,q} \|v\|_{2,2}.$$

Finally, our result (36) applies to the crucial term (iv)

(iv) $\leq \|\nabla \psi_h\|_{2;\Omega_h} \|e_h\|_{\infty;\Omega_h} \leq c h \|v\|_{2,2}$.

This gives

$$|(a - a^h)(e_h, I_h v)| \leq c h \|v\|_{2,2} \|\nabla e_h\|_2,$$

Thus, by $\|v\|_{2,2} \leq c \|e_h\|_2$ and the result (2),

$$\|e_h\|_2 = O(h^2).$$

This completes the proof of the theorem.

3. APPENDIX

Here we state some lemmas used in the proof of the theorem. Assume the condition (T) to be satisfied.

LEMMA A1 : *Let $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ and let $I_h v \in S_h^0$ be its interpolant. Then*

$$\|\nabla I_h v\|_\infty \leq c |\ln h| \|v\|_{2,2}, \tag{a}$$

and with the weighted norms (18) *for* $0 \leq \nu \leq 4$ *and* $\rho \geq c_3 h$

$$\|\sigma^{\nu/2} \nabla I_h v\|_\infty \leq c |\ln h| (\|\nabla v\|_{(\nu-2)} + \|\nabla^2 v\|_{(\nu)}). \tag{b}$$

Proof. We shall prove (b). The proof of (a) is similar.

Let $T \in T_h$ be any triangle with $T \subset \Omega_h$ and let $\xi \in T$ be the center of the inscribed circle with radius $c_1 h$ (assumption (T)). The boundary $\partial\Omega$ is of class C^2 and hence satisfies a strong cone condition. The corresponding spherical cone $K := K(\xi, \tau) \subset B(\xi, \tau)$ with vertex ξ, opening $|\Sigma_K|$ and height

$\tau > 0$ (independent of $\xi \in \Omega$ and h) can be cut off to a cone $K_T \subset T$ with volume $|K_T| = ch^2$ and $K_T \subset\subset B(\xi, \tau)$. Then by (19)

$$|\sigma^{\nu/2} \nabla I_h v(\xi)| \leq c h^{-2} \int_{K_T} \sigma^{\nu/2} |\nabla I_h v| \, dx$$
$$\leq c h^{-1} \|\nabla(v - I_h v)\|_{(\nu)} + c h^{-2} \int_{K_T} \sigma^{\nu/2} |\nabla v| \, dx$$
$$\leq c \|\nabla^2 v\|_{(v)} + c h^{-1} \left(\int_{K_T} \sigma^{\nu} |\nabla v|^2 \, dx \right)^{1/2} .$$

Now choose a function $\varphi \in C^\infty$ with support in the ball $B(\xi, \tau)$ and the properties (independent of h)

$$0 \leq \varphi \leq 1 \quad , \quad |\nabla \varphi| \leq c \quad , \quad \varphi = 1 \qquad \text{on} \quad K_T .$$

Using polar coordinates $(r, 0)$ centered in ξ we find with the function

$$p(|x - \xi|) := (|x - \xi|^2 + h^2)^{-1}$$

$$h^{-2} \int_{K_T} \sigma^{\nu} |\nabla v|^2 \, dx \leq c \int_{\Sigma_K} \int_0^\tau r p(r) |\varphi \sigma^{\nu/2} \nabla v|^2 \, dr \, do.$$

According to the special choice of φ it follows by integration by parts

$$\int_0^\tau r p(r) |\varphi \sigma^{\nu/2} \nabla v|^2 \, dr = -2 \int_0^\tau \left\{ \int_0^r s p(s) \, ds \right\} \varphi \sigma^{\nu/2} |\nabla v| \, \partial_r (\varphi \sigma^{\nu/2} |\nabla v|) \, dr$$

and, using the inequality $ab \leq \varepsilon a^2 + (4\varepsilon)^{-1} b^2$, $\varepsilon > 0$,

$$\int_0^\tau r p(r) |\varphi \sigma^{\nu/2} \nabla v|^2 \, dr \leq 4 \int_0^\tau \left\{ \int_0^r s p(s) \, ds \right\}^2 r^{-2} p(r)^{-1} r \, |\nabla (\varphi \sigma^{\nu/2} |\nabla v|)|^2 \, dr.$$

The function $\left\{ \int_0^r sp(s) \, ds \right\}^2 r^{-2} p(r)^{-1}$ is continuous and nondecreasing for $0 \leqq r < h$ and hence uniformly bounded by $c\,|\ln h|^2$ for $r \leqq \operatorname{diam}(\Omega)$. This gives

$$h^{-2} \int_{K_T} \sigma^{\nu} |\nabla v|^2 \, dx \leq c \, |\ln h|^2 (\|\nabla v\|_{(\nu-2)} + \|\nabla^2 v\|_{(\nu)})^2$$

and completes the proof.

LEMMA A2 : *Let* $G^h \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ *be the smoothed Green functions defined by* (20) *and let* G_h^h *be its Ritz projections defined by* (21). *Then with the weighted norms* (18) *for* $\rho \geq c_3 h$

$$\|\nabla G^h\|_{(\varepsilon)} + \|\nabla^2 G^h\|_{(2+\varepsilon)} \leq c_\varepsilon \rho h^{-1} \begin{cases} 1 & , \ \varepsilon > 0 \\ |\ln h|^{1/2} & , \ \varepsilon = 0, \end{cases} \tag{a}$$

$$\|\nabla(G^h - G_h^h)\|_1 \leq c\,|\ln h|^{1/2}\,\|\nabla(G^h - G_h^h)\|_{(2)} \leqslant c\,\rho\,|\ln h|, \tag{b}$$

$$\|\sigma^{1+\varepsilon}\,\nabla G_h^h\|_\infty \leq c_\varepsilon \rho h^{-1}\,|\ln h| \quad , \quad \varepsilon > 0. \tag{c}$$

Proof. The estimate (a) and (b) are proven in [4].

Analogously to Lemma A1 we find

$$|\sigma^{1+\epsilon}\nabla G_h^h(\xi)| \leq ch^{-1}\,\|\nabla(G_h^h - G^h)\|_{(2+2\epsilon)} + c\,|\ln h|\,\{\,\|\nabla G^h\|_{(2\epsilon)} + \|\nabla^2 G^h\|_{(2+2\varepsilon)}\,\}$$

and, using (a) and (b),

$$|\sigma^{1+\varepsilon}\nabla G_h^h(\xi)| \leq c\rho h^{-1}\,|\ln h|^{1/2} + c_\epsilon \rho h^{-1}\,|\ln h|.$$

This proves (c).

LEMMA A3 : *Let* $v \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ *and let* A *be the uniformly elliptic differential operator defined by* (11). *Then with the weighted norms* (18) *for* $0 < \rho < \rho_0$

$$\|\nabla v\|_2 + \|\nabla^2 v\|_{(2)} \leq c\rho^{-1}(1 + |\ln \rho|^{1/2})\,\|Av\|_{(4)}.$$

For the sake of completeness we shall sketch a proof of this important a priori estimate. It rests on ideas contained in [12] and [4]. A similar assertion is stated by Nitsche [11] for the Laplace operator.

Proof. Set $\sigma(.) := (|. - z|^2 + \rho^2)^{1/2}$, $0 < \rho \leq \rho_0$, $z \in \Omega$.

Denoting by $y^j := x^j - z^j$, $j = 1, 2$, the components of the vector $x - z$, $x \in \Omega$, we get

$$\|\nabla^2 v\|_{(2)}^2 = \sum_{j=1}^{2} \|y^j\,\nabla^2 v\|_2^2 + \rho^2\,\|\nabla^2 v\|_2^2,$$

and by the well known L^2-estimate,

$$\rho\,\|\nabla^2 v\|_2 \leq c\rho\,\|Av\|_2 \leq c\rho^{-1}\,\|Av\|_{(4)},$$

$$\|y^j\nabla^2 v\|_2 \leq c\,\{\,\|\nabla^2(y^j v)\|_2 + \|\nabla v\|_2\,\} \leq c\,\{\,\|A(y^j v)\|_2 + \|\nabla v\|_2\,\}.$$

Further, using $M := \max\limits_{i,k} \|a_{ik}\|_{1,q} < \infty$ and $\max\limits_{i,k} \|a_{ik}\|_\infty \leq cM$, we find with $p := 2q/(q-2)$

$$\|A(y^j v\|_2 \leqq c\,\{\,\|y^j A v\|_2 + M\,\|v\|_p + M\,\|\nabla v\|_2\,\} \leqq c\,\{\,\|y^j A v\|_2 + \|v\|_{1,2}\,\},$$

and thus by Poincaré's inequality and $\sigma^{-1} \leq c\rho^{-1}$

$$\|\nabla^2 v\|_{(2)} \leq c\,\{\,\|Av\|_{(2)} + \|v\|_{1,2}\,\} \leq c\,\{\,\rho^{-1}\,\|Av\|_{(4)} + \|\nabla v\|_2\,\}. \tag{39}$$

By the inequality $ab \leq (a^2 + b^2)/2$, it follows that

$$\|\nabla v\|_2^2 \leq a(v, v) \leq c\rho^{-2} |\ln \rho| \, \|Av\|_{(4)}^2 + c\,\rho^2 |\ln \rho|^{-1} \|v\|_{(-4)}^2.$$

Denoting by $g(.,.)$ the Green function of A over Ω we obtain

$$\|v\|_{(-4)}^2 = \int \sigma^{-4}(x) \left| \int Av(\xi) g(x, \xi)\, d\xi \right|^2 dx,$$

and by Hölder's inequality and an interchange of the order of integration

$$\|v\|_{(-4)}^2 \leq \int \sigma^4(\xi) |Av(\xi)|^2 \left(\int \sigma^{-4}(x) g(x, \xi) \left\{ \int \sigma^{-4}(\eta) g(x, \eta)\, d\eta \right\} dx \right) d\xi.$$

It is well known that the Green function g can be estimated on Ω by (see [4])

$$0 \leq g(x, y) \leq c(1 + |\ln |x - y||).$$

From this we conclude

$$\int \sigma^{-4}(\eta) g(x, \eta)\, d\eta \leq c\rho^{-2}(1 + |\ln \rho|).$$

It follows that

$$\|v\|_{(-4)}^2 \leq c\rho^{-4}(1 + |\ln \rho|)^2 \|Av\|_{(4)}^2,$$

and thus

$$\|\nabla v\|_2 \leq c\rho^{-1}(1 + |\ln \rho^{1/2}) \|Av\|_{(4)}.$$

Together with (39) this completes the proof.

Finally, we state a simple boundary estimate, which can be proven by locally reduction to one dimensional integrations.

LEMMA A4. *Let* $v \in W^{2,p}(\Omega)$, $1 \leq p < \infty$. *Then* (*see* (15))

$$\int_{\Omega - \Omega_h} |v|^p \, dx \leq c_p d(\partial\Omega, \partial\Omega_h) \int_\Omega \{ |v|^p + |\nabla v|^p \} \, dx.$$

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