RAIRO. ANALYSE NUMÉRIQUE

FRANCESCO SCARPINI
MARIA AGOSTINA VIVALDI

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RAIRO. Analyse numérique, tome 11, nº 2 (1977), p. 197-208
<http://www.numdam.org/item?id=M2AN_1977__11_2_197_0>

R.A.I.R.O. Analyse Numérique/Numerical Analysis
(vol. 11, n° 2, 1977, p. 197 à 208)

ERROR ESTIMATES FOR THE APPROXIMATION OF SOME UNILATERAL PROBLEMS (*) (1)

by Francesco SCARPINI and Maria Agostina VIVALDI (2)

Communiqué par P. A. RAVIART

Abstract. — *An error estimate for the affine finite element approximation of some unilateral problems is given.*

1. INTRODUCTION

In the mechanics of Fluids through semipermeable boundary the following problem is studied:

$$- \Delta u = f \quad \text{in} \quad \Omega; \; u \geq \psi, \; \frac{\partial u}{\partial n} \geq 0, \; (u - \psi)\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma \quad (1)$$

where Γ is a thin membrane around the space Ω filled by the fluid: semi-permeable means that the fluid is allowed to enter but not to escape.

Δ denotes the Laplace operator, u the pressure of the fluid in its stable condition, f the amount of the fluid that has been put in, ψ denotes the external fluid pressure on Γ, $\frac{\partial}{\partial n}$ the outer normal derivative on Γ.

(1) can be used also to sketch some problems in thermo-dynamics or electric dynamics.

A system such as (1) is known in the literature as a complementarity system and can be solved when some compatibility conditions are imposed on f (see [8]).

If the internal pressure u is greater than the external one ψ then the semi-permeable membrane holds the fluid and the fluid cannot escape $\left(\text{so} \frac{\partial u}{\partial n} = 0\right)$.

If ψ is greater or equal than u, the external fluid enters through Γ $\left(\frac{\partial u}{\partial n} \geq 0\right)$ until there is $u = \psi$; in theory there is no $u < \psi$.

(*) This paper has been partially supported by GNAFA-C.N.R.
(1) Manuscrit reçu le 10 décembre 1975.
(2) Istituto Matematico, Università di Roma.

The membrane could be devided into two parts Γ_1 and Γ_2

on Γ_1 we have $u = \psi$ (Dirichlet condition),

on Γ_2 we have $\frac{\partial u}{\partial n} = 0$ (Neumann condition).

but we look at (1) as a free boundary problem because we do not know Γ_1 or Γ_2.

We shall study some variational inequalities (with coerciveness assumption) (*cfr* [23]), we shall consider an approximation using the triangular affine elements: the solutions of the corresponding discrete complementarity systems are supposed known.

Our purpose is to estimate the distance between the exact solution u and the discrete one u_h.

2. THE BASIC NOTATION AND TERMINOLOGY

Ω denotes a bounded, open set of $\mathbf{R}^2$, Γ denotes the boundary of Ω and $\bar{\Omega}$ the closure of Ω so that $\bar{\Omega} = \Omega \cup \Gamma$: Ω is supposed with "not too bad" a boundary.

$C^k(\bar{\Omega})$, $(k = 0, 1, 2, \ldots)$ is a Banach space, the elements of which are functions that are continuous in $\bar{\Omega}$ and have continuous derivatives in $\bar{\Omega}$ of the first k order: the norm is defined by:

$$\|u\|_{C^k(\bar{\Omega})} = \sum_{j=0}^{k} \sum_{|\alpha|=j} \max_{\bar{\Omega}} |D^\alpha u|$$

(if $k = +\infty$ the functions are infinitely differentiable)

$\mathscr{D}(\Omega)$ is the space of the functions of $C^\infty(\bar{\Omega})$ which are zero in a neighbourhood of $\partial\Omega$, and we put on it Schwartz's topology, $L^p(\Omega)$, $(1 < p < +\infty)$ denotes the Banach space of all functions on Ω that are measurable and p-summable in Ω. the norm in this space is defined by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx\right)^{1/p}$$

or if $p = +\infty$ a e. bounded in Ω, (the elements $L^p(\Omega)$ are the class of equivalent functions on Ω).

$W^{k,p}(\Omega)$, $(k \in \mathbf{N}, 1 < p < +\infty)$ denotes the Banach space of all elements of $L^p(\Omega)$ that have generalized derivatives of all kinds of the first k orders that are p-summable in Ω: the norm is defined by

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}, \quad \alpha = (\alpha_1, \alpha_2) : |\alpha| = \sum_{i=1}^{2} \alpha_i$$

$W_0^{k,p}(\Omega)$ denotes the closure in the space $W^{k,p}(\Omega)$ of $\mathscr{D}(\Omega)$; for further details and for the spaces $W^{s,p}(\Omega)$, with s real, see e.g. [15] and [20].

We shall use the following notations

$$H^k(\Omega) = W^{k,2}(\Omega) \quad ; \quad H_0^k(\Omega) = W_0^{k,2}(\Omega)$$

$$\|v\|_{k,\Omega} = \|v\|_{W^{k,2}(\Omega)} \qquad |v|_{k,\Omega} = \left(\sum_{|\alpha|=k} \|D^\alpha v\|_{0,\Omega}^2\right)^{1/2}.$$

3. VARIATIONAL FORMULATION OF THE PROBLEM

Let us look at the following form

$$a(u, v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_j} + a_0(x)uv\right)dx$$

where $a_{ij}(x) \in C^1(\bar{\Omega})$, and $a_0(x) \in L^\infty(\Omega)$

let us suppose:

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \alpha_0 |\xi|^2 \quad \forall \xi \in \mathbf{R}^2 \quad , \quad \alpha_0 > 0 \qquad \text{and} \qquad a_0(x) \geq c > 0$$

so that $a(.,.)$ is a bilinear, continuous, coercive form on $H^1(\Omega) \times H^1(\Omega)$ i. e. $a(u, u) \geq \alpha \|u\|_{1,\Omega}^2$;

$$|a(u, v)| \leq M \|u\|_{1,\Omega} \cdot \|v\|_{1,\Omega}, \alpha, M \in \mathbf{R}; \qquad \alpha > 0, M > 0$$

We shall consider the problem

$$u \in K : a(u, v - u) \geqq \langle f, v - u \rangle \qquad \forall v \in K \tag{2}$$

where K is a convex set:

$$K = \{ v \mid v \in H^1(\Omega) : v \geq \psi \text{ on } \Gamma \}$$

$\langle , \rangle$ denotes the pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$, $f \in L^2(\Omega)$, $\psi \in H^2(\Omega)$.

In these conditions (see e. g. [3], [14]) there is one and only one solution u of the problem (2) and $u \in H^2(\Omega)$.

In what follows we shall use the notations:

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u, \qquad \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_i}\cos(n, x_j)$$

where n is the outer normal to Γ.

Let us recall (see [14]) the problem (2) is equivalent to the following system:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u \geq \psi & \text{on } \Gamma \\ \dfrac{\partial u}{\partial \nu} \geq 0 & \text{on } \Gamma \\ (u - \psi)\dfrac{\partial u}{\partial \nu} = & \text{on } \Gamma \end{cases} \tag{3}$$

REMARK 3.1.

If $a_{ij} = a_{ji}$ the problem (2) can be formulated as follows:

find $u \in K$ such that

$$J(u) = \inf_{v \in K} J(v)$$

where $J(v) = \frac{1}{2} a(v, v) - \langle f, v \rangle$ is a weakly lower semicontinuous, strictly convex, differentiable functional.

4. A DISCRETIZATION OF THI PROBLEM BY THE FINITE ELEMENT METHOD

We shall sketch here an approximation of problem (2) by means of triangular affine elements.

We shall suppose that Ω is a bounded convex open subset of $\mathbf{R}^2$, with a smooth boundary Γ (e. g. C^2).

Given h, $0 < h < 1$, we first inscribe a polygon Ω_h in Ω whose vertices belong to Γ and whose sides have a length which does not exceed h.

We then decompose Ω_h into triangles in such a way that:

$$0 < l \leq h, \qquad l'/l'' \leq \beta \;, \quad 0 < \vartheta_0 < \vartheta \leqslant \frac{\pi}{2}$$

where β and $\vartheta_0 < \frac{\pi}{2}$ are given positive constants l, l', l'' are lenghts of arbitrary sides of the triangulation, ϑ an arbitrary angle of our triangles.

(As always, the triangulation is not permitted to place a vertex of one triangle along the edge of another, each pair of triangles shares a vertex, a whole edge, or nothing.)

We call "*regular*" (see [2], [11]) such a triangulation.

We shall denote by $I_0 = \{ 1, \ldots, N_0 \}$ the set of all indices i associated with the internal nodes x_i of the triangulation $(x_i \in \Omega)$ and we shall denote by $I_1 = \{ N_0 + 1, \ldots, N \}$ the set of all indices i associated with the boundary nodes of the triangulation $(x_i \in \partial \Omega)$: and let be $I = I_0 \cup I_1$.

We number the vertices in such a way that $x_{N_0+1}, \ldots, x_N$ are the boundary vertices, numbered consecutively counterclockwise around Γ: the curved side (lying on Γ) with end points x_i, x_{i+1} is Γ_i (for sake of notation we shall make the identification $x_{N+1} = x_{N_0+1}$) we shall denote by Σ_i the zone between the curved side Γ_i and the right side with endpoints x_i, x_{i+1}, and by T_i the triangle corresponding to the $[x_{i+1}, x_i]$ side: finally we shall call *curved elements* the following subdomains:

$$T_i \cup \Sigma_i \quad , \quad i \in I_1 .$$

For each $i \in I$, we shall consider the continuous function

$$\varphi_i^h(x) \quad , \quad x \in \Omega$$

which is affine in each triangular or curved (see the above position) element of the decomposition, is $= 1$ at x_i and $= 0$ in all $x_j \neq x_i$, $j \in I$.

We shall now consider the piecewise affine function $v_h(x)$ defined by

$$v_h(x) = \sum_{i \in I} v_i^h \varphi_i^h(x) \quad , \quad \{ v_i^h \}_{i \in I} \in \mathbf{R}^N \tag{4}$$

and the space:

$$H_h^1(\Omega) = \left\{ v_h : v_h = \sum_{i \in I} v_i^h \varphi_i^h(x) \right\}$$

(trivially $H_h^1(\Omega)$ is contained in the space $H^1(\Omega)$).

In [2] it was shown that :

$$\|u - u_I\|_{r,\Omega_h} \leq c h^{2-r} |u|_{2,\Omega_h} \quad , \quad r = 0,1, \quad \forall u \in H^2(\Omega)$$

where $u_I(x)$ is the piecewise affine function which interpolates u at every vertex

$$\text{i.e.} \qquad u_I(x) = \sum_{i \in I} u(x_i) \varphi_i^h(x)$$

but we can modify the process of the quoted paper and using the regularity conditions of the decomposition and a continuous extension theorem in the seminorms (see [26]) we can obtain:

$$\|u - u_I\|_{r,\Omega} \leq c h^{2-r} |u|_{2,\Omega} \quad , \quad r = 0,1. \tag{6}$$

We shall consider the convex

$$K_h = \{ v_h(x) : v_h \in H_h^1(\Omega)/v_h(x_i) \geq \psi(x_i) \, \forall i \in I_1 \}$$

The approximate problem is obtained by replacing K with K_h in problem (2)

$$u_h \in K_h : a(u_h, u_h - v_h) \leq \langle f, u_h - v_h \rangle \qquad \forall v_h \in K_h \tag{7}$$

Let us write the discrete problem by replacing the expression (4) of $u_h(x)$ that is:

$$u_h(x) = \sum_{i \in I} U_i \varphi_i^h(x) \tag{8}$$

PROPOSITION 4.1. The problem (7) is equivalent to the following discrete system.

$$\begin{cases} M_{I_0} = A_{I_0 I} U_I - b_{I_0} = 0 \\ U_{I_1} - \Psi_{I_1} \geq 0 \\ M_{I_1} = A_{I_1 I} U_I - b_{I_1} \geq 0 \\ M_{I_1}(U_{I_1} - \Psi_{I_1}) = 0 \end{cases} \tag{9}$$

where $A = A_{II} = \{ a_{ij} \}_{i,j \in I}, a_{ij} = a(\varphi_j^h, \varphi_i^h); b_I = \{ b_i \}_{i \in I}$

$$b_i = (f, \varphi_i^h); \quad \Psi_{I_1} = \{ \Psi_i \}_{i \in I_1} \quad , \quad \Psi_i = \Psi(x_i)$$

PROOF. By choosing $v_h = u_h \pm \varphi_i^h$ for every $i \in I_0$ we find the first equation

i$_h$) $$M_i = \sum_{j \in I} a_{ij} U_j - b_i = 0 \qquad i \in I_0.$$

The second condition in (9) is the definition of K_h

ii$_h$) $$U_i \geq \Psi_i \quad i \in I_1$$

By putting $v_h = u_h + \varphi_i^h$ $i \in I_1$ we have

iii$_h$) $$M_i = \sum_{j \in I} a_{ij} U_j - b_i \geq 0 \qquad i \in I_1.$$

The last condition in (9) is obtained by choosing

$$v_h(x_i) = \varepsilon u_h(x_i) \qquad i \in I_0$$
$$v_h(x_i) = \psi(x_i) + \varepsilon(u_h - \psi)(x_i) \quad i \in I_1$$

in fact i$_h$) gives

$$(\varepsilon - 1) \sum_{i \in I_1} (U_i - \Psi_i) \left\{ \sum_{j \in I} a_{ij} U_j - b_i \right\} \geq 0$$

and with choices: $\varepsilon > 1$, $1 > \varepsilon > 0$ due to ii$_h$) and iii$_h$) we find:

iv$_h$) $$M_{I_1}(U_{I_1} - \Psi_{I_1}) = 0.$$

It is easy to check, in turn, that if we take the coefficients U_j of the function (8) to be the solution of (9) then $u_h(x)$ is the solution of (7).

REMARK 4.1. The matrix A belongs to the class (P) that is to say, all principal minors $A_{JJ} = (a_{h,k})_{h,k\in J}$, $J \subset I$ have a positive determinant; therefore the existence and uniqueness of the solution U_I of (9) is a well known result; if A belongs, also, to the class (Z), that is $a_{ij} \leq 0$ $(i \neq j)$ (*) then a monotone algorithm for solving system (9) can be found, (see e. g. [12], [17], [22] for other unilateral problems).

5. ERROR ESTIMATES

In order to estimate the distance between the solutions u of (2) and u_h of (7) we shall follow the procedure described in [9] for another unilateral problem; of course our problem demands some modifications of the Falk's method for the "inability" of piecewise polinomials to satisfy the conditions on a curved boundary.

Our main result is this: the error is of order $h^{\frac{3}{4}}$ in the energy norm:

THEOREM I. We have

$$\|u - u_h\|_{1,\Omega} \leq Ch^{\frac{3}{4}} \text{ (**)} \tag{10}$$

Proof. Let us now write the inequalities (2) and (7) by choosing $v = \Psi$ in (2) and $v = u_I$ in (7)

$$a(u, u - \psi) \leq \langle f, u - \psi \rangle$$
$$a(u_h, u_h - u_I) \leq \langle f, u_h - u_I \rangle$$

We find

$$a(u, u) + a(u_h, u_h) \leq \langle f, u - u_I + u_h - \psi \rangle + a(u, \psi) + a(u_h, u_I)$$

and, by subtracting $a(u, u_h) + a(u_h, u)$, also

$$a(u - u_h, u - u_h) \leq \langle f, u - u_I + u_h - \psi \rangle - a(u, u - u_I + u_h - \psi) + a(u - u_h, u - u_I).$$

Let us write the inequality using Green's formula

$$a(u - u_h, u - u_h) \leq (f - Lu, u - u_I + u_h - \psi) + a(u - u_h, u - u_I) - + \int_\Gamma (u - u_I + u_h - \psi)\frac{\partial u}{\partial \nu}\, d\Gamma$$

(*) This is true, for example, for $Lu = -\Delta u + a_0 u$.

(**) In the sequel C will denote a generic constant not necessarily the same in any two places.

and by the first equation of the system (3)

$$a(u - u_h, u - u_h) \leq a(u - u_h, u - u_I) - \int_\Gamma (u - u_I + u_h - \psi)\frac{\partial u}{\partial \nu} d\Gamma \quad (11)$$

We shall split the last term into two integrals i. e.

$$- \int_\Gamma (u_h - \psi_I)\frac{\partial u}{\partial \nu} d\Gamma - \int_\Gamma ((u - \psi) - (u - \psi)_I)\frac{\partial u}{\partial \nu} d\Gamma$$

The estimation of $\|u - u_h\|_{1,\Omega}$ is therefore reduced to study the convergence of the boundary integrals.

We begin by remarking that $g = u_h - \psi_I$ is nonnegative at the boundary nodes, if also $u_h - \psi_I \geqslant 0$ on Γ we could eliminate the integral $- \int_\Gamma \frac{\partial u}{\partial \nu}(u_h - \psi_I)$ and so the inequality would increase (cf. systems (3) (9)); if this is not the case. let us denote by $\tilde{\Gamma}$ the subset of Γ in which.

$$g < 0$$

and let us split $\tilde{\Gamma}$ into a finite number of curved sides $\tilde{\Gamma}_i$ with endpoints z_i, z_{i+1} such that:

$$\tilde{\Gamma} = \bigcup_{i \in I_1} \tilde{\Gamma}_i \quad , \quad \tilde{\Gamma}_i \subset \Gamma_i \quad , \quad g(z_i) = g(z_{i+1}) = 0\,(^1).$$

We can now prove the following:

PROPOSITION 5.1.: We have:

$$- \int_\Gamma g \frac{\partial u}{\partial \nu} d\Gamma \leq \mathrm{C}h^{\frac{3}{2}} \|u\|_{2,\Omega} \quad (12)$$

Proof. By the third inequality of (3) we find:

$$- \int_\Gamma g \frac{\partial u}{\partial \nu} d\Gamma \leq - \int_{\tilde{\Gamma}} g \frac{\partial u}{\partial \nu} d\Gamma \leq \|g\|_{0,\tilde{\Gamma}} \left\| \frac{\partial u}{\partial \nu} \right\|_{\frac{1}{2},\tilde{\Gamma}}$$

We may increase the inequality by replacing $\left\| \frac{\partial u}{\partial \nu} \right\|_{\frac{1}{2},\tilde{\Gamma}}$ with $\left\| \frac{\partial u}{\partial \nu} \right\|_{\frac{1}{2},\Gamma}$ and by a trace theorem (see e. g. [15] [20]) with $C \|u\|_{2,\Omega}$. We denote by z the intersection between the side $[z_i, z_{i+1}]$ and its normal from x; by $\tilde{\Sigma}_i$ the zone between the curved side $\tilde{\Gamma}_i$ and the right side $[z_{i+1}, z_i]$ (see the figure 1)

(1) $\tilde{\Gamma}_i$ may be empty for some i.

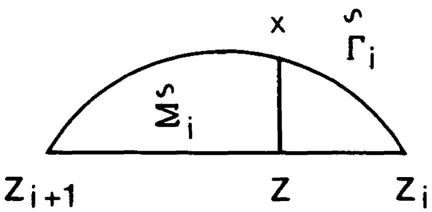

Figure 1.

The following inequalities can be obtained using Schwartz inequality and the smoothness of the boundary (*cfr.* [26])

$$\begin{cases} \displaystyle\int_{\tilde{\Gamma}_i} |g(x)|^2 \, d\Gamma \leq \int_{\tilde{\Gamma}_i} \left| \int_z^x \left| \frac{\partial g}{\partial n} \right| dn \right|^2 d\Gamma \\ \qquad\qquad \displaystyle\leq \int_{\tilde{\Gamma}_i} |z - x| \int_z^x \left| \frac{\partial g}{\partial n} \right|^2 dn \, d\Gamma \leq \mathrm{Ch}^2 \, |g|^2_{1,\tilde{\Sigma}_i} \end{cases} \tag{13}$$

finally we apply Berger's ideas (see [26]) for piecewise polynomial functions: i. e.

$$\int_{\tilde{\Sigma}_i} |D^\alpha g|^2 \, dx \leq \mathrm{Ch} \int_{T_i} |D^\alpha g|^2 \, dx \qquad \forall \, |\alpha| \leq 1$$

and combining the relations obtained above we have (12). Now we return to (11) and we prove the:

PROPOSITION 5.2. We have

$$- \int_\Gamma ((u - \psi) - (u - \psi)_I) \frac{\partial u}{\partial v} \, d\Gamma \leq \mathrm{Ch}^{\frac{3}{2}} \tag{14}$$

Proof.

Let us write (see proposition 5.1 and figure 2) for $v = (u - \psi) - (u - \psi)_I$

$$- \int_\Gamma v \frac{\partial u}{\partial \mathrm{v}} \, d\Gamma \leq C \, \|u\|_{2,\Omega} \, \|v\|_{0,\Gamma} \qquad \text{and} \qquad \|v\|^2_{0,\Gamma} = \sum_{1 = N_0 + 1}^{N-1} \|v\|^2_{0,\Gamma_i}$$

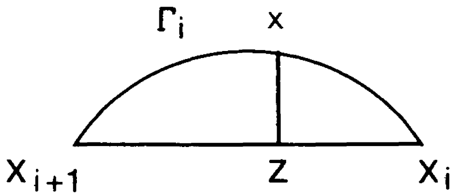

Figure 2.

it is now easy to prove the following inequalities

$$\begin{cases} \displaystyle\int_{\Gamma_i} |v(x)|^2 \, d\Gamma = \int_{\Gamma_i} \left[\int_{x_i}^{z} \frac{\partial v}{\partial \lambda} d\lambda + \int_{z}^{x} \frac{\partial v}{\partial n} dn \right]^2 d\Gamma \\ \displaystyle \qquad \le \int_{\Gamma_i} 2 \left[|z - x_i| \int_{x_i}^{x_{i+1}} \left| \frac{\partial v}{\partial \lambda} \right|^2 d\lambda + 2 \, |x - z| \int_{z}^{x} |\operatorname{grad} v|^2 \, dn \right] d\Gamma . \end{cases} \quad (15)$$

We shall apply Bramble and Zlamal's method (*cfr.* [2]) and use the trace theorems (see [20]) and the regularity assumptions, to yield (14).

We are finally ready to prove our theorem I (*): we shall replace in (11) the results of the propositions 5.1 and 5.2 and use the coerciviness and continuity assumption.

6. REMARKS

Zlámal introduced in the finite element method the "*curved elements*" (see e. g. [29]), by introducing *Zlámal curved elements* to our problem, we find again the same rate of convergence for the approximation error.

Namely we shall consider a triangulation of the given domain Ω into triangles completed along the boundary Γ by curved elements, so that the union of their closures is $\bar{\Omega}$ (the usual regularity conditions are supposed satisfied): we construct a finite-dimensional subspace (V_h) of trial functions belonging to $H^1(\Omega)$:

$$v_h(x^1, x^2) = r_h(\xi(x^1, x^2), \eta(x^1, x^2)) \quad (16)$$

When $\xi(x^1, x^2)$, $\eta(x^1, x^2)$ is the inverse mapping of

$$\begin{cases} x^1 = x^1(\xi, \eta) \\ x^2 = x^2(\xi, \eta) \end{cases} \quad (17)$$

which maps the unit triangle T_1 with vertices $R_1 \equiv (0, 0)$, $R_2 = (1, 0)$, $R_3 = (0, 1)$ in the ξ-η-plane one-to-one into the triangle T (which may be a curved one) with vertices (x_i^1, x_i^2), (x_{i+1}^1, x_{i+1}^2), (x_j^1, x_j^2) (see [29]) and $r(\xi, \eta)$

(*) In order to obtain the optimal error estimate :

i.e. $\|u - u_h\|_{1,\Omega} \le ch$

we should need the following results

$$(+) \quad \|g\|_{-\frac{1}{2}, \tilde{\Gamma}} \le ch^{1/2} \|g\|_{0, \tilde{\Gamma}} \quad \text{where} \quad g = u_h - \psi_I$$

$$(++) \quad \|v\|_{-\frac{1}{2}, \tilde{\Gamma}} \le ch^{1/2} \|v\|_{0, \tilde{\Gamma}} \quad \text{where} \quad v = (u - \psi) - (u - \psi)_I$$

which at the moment we are not able to prove.

is an affine function in T_1, We shall also use the Zlámal theorem 2 to estimate the difference $u - \tilde{u}_I$ where $\tilde{u}_I$ is the "*interpolate of u*" i. e.: the function from V_h such that

$$\tilde{u}_I(x_i) = u(x_i) \qquad \forall i \in I.$$

We shall choose $\tilde{K}_h$ as the convex set of the all function of V_h such that:

$$v_h(x_i) \geq \psi(x_i) \qquad \forall i \in I_1$$

(and then also $v_h(x) \geq \psi(x)$ $\forall x \in \Gamma$).

$\tilde{u}_h$ denotes the solution of the discrete problem (7) corresponding to the convex set $\tilde{K}_h$. We can now repeat our theorem I replacing u_h with $\tilde{u}_h$ to obtain the error bounds

$$\|u - \tilde{u}_h\|_1 \leq Ch^{\frac{3}{4}}.$$

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