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L_∞-CONVERGENCE OF SADDLE-POINT APPROXIMATIONS FOR SECOND ORDER PROBLEMS ([1])

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Abstract – *Let u_0 be the solution of the second second order boundary value problem $-\Delta u + qu = f$ in Ω, $u = 0$ on $\partial\Omega$, with Ω bounded in $\mathbf{R}^2$ $(u_0$, grad $u_0)$ is characterized as the saddle-point of a quadratic functional and approximated by finite elements*

1. INTRODUCTION

Let $\Omega \subseteq \mathbf{R}^2$ be a bounded domain and $q \geq 0$ be a bounded and measurable function. We consider the second order model problem

$$\begin{aligned} -\Delta u + qu &= f \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

$f \in L_2(\Omega)$; the solution will be denoted by u_0.

The basic idea of the mixed method is to characterize (u_0, v_0), $v_0 := \text{grad } u_0$, as the saddle-point of a quadratic functional and to approximate (u_0, v_0) by elements of suitably chosen finite dimensional subspaces.

The construction of approximating finite element spaces and the L_2-error analysis for this problem was given by P. A. Raviart-J. M. Thomas [5]. Using the same subspaces our goal is to derive L_∞-error estimates. The method of proof is based on weighted L_2-norms, similarly to the work of J. Nitsche [3], [4], and F. Natterer [2].

2. NOTATIONS, STATEMENT OF THE PROBLEM

If we define the operator

$$Tu := \text{grad } u$$

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with $T: D(T) = \overset{\circ}{W}{}_2^1 \subseteq L_2 \to L_2 \times L_2$, then the dual operator is

$$T^* v = -\operatorname{div} v$$

with $T^* : D(T^*) = W_2^1 \times W_2^1 \subseteq L_2 \times L_2 \to L_2$. (We omit the specification of the domain, if no confusion is possible.)

Later on we need the following assertions : T is a closed operator and $R(T)$ is closed in $L_2 \times L_2$. Therefore by the Closed Range Theorem $L_2 \times L_2$ is the orthogonal sum of $R(T)$ and $N(T^*)$. (See, for instance, K. Yosida [7], p. 205.)

Further we define the operator $Q : L_2 \to L_2$ by $Qu := qu$, $u \in L_2$. Then equation (1) is equivalent to the system

$$\begin{aligned} T^* v + Qu &= f \\ Tu \quad - v &= 0 \end{aligned} \tag{2}$$

with the solution (u_0, v_0), $v_0 := \operatorname{grad} u_0$.

For convenience we assume that $\bar{\Omega}$ is a bounded polygon. Suppose Γ_h is a κ-regular triangulation of Ω, $0 < h$, i. e. for any $\Delta \in \Gamma_h$ there are two circles $\underline{K}$ and $\bar{K}$ with radii $\underline{\rho}$ and $\bar{\rho}$ such that $\underline{K} \subseteq \Delta \subseteq \bar{K}$ and

$$\kappa^{-1} h \leq \underline{\rho} \leq \bar{\rho} \leq \kappa h.$$

In the following let $r \geq 1$ be a fixed integer.

By $(W_p^r \times W_p^r)' = (W_p^r \times W_p^r)'(\Gamma_h)$, $2 \leq p \leq \infty$, we denote those elements of $L_p \times L_p$, which fulfill the following conditions :

(i) $v \in W_p^r(\Delta) \times W_p^r(\Delta)$ for all $\Delta \in \Gamma_h$;

(ii) for all $u \in W_2^1(\Omega)$ we have

$$\begin{aligned} \int_\Omega v \,.\, \operatorname{grad} u \, ds &+ \int_\Omega u \operatorname{div} v \, ds \\ &= \int_{\partial\Omega} uv \,.\, \nu \, d\sigma. \end{aligned} \tag{3}$$

(ν is the exterior unit normal to $\partial\Omega$.)

Equation (3) holds if and only if for any pairs of adjacent triangles $\Delta_1, \Delta_2 \in \Gamma_h$ we have

$$v|\Delta_1 \,.\, \nu_1 + v|\Delta_2 \,.\, \nu_2 = 0 \qquad \text{on} \quad \Delta_1 \cap \Delta_2,$$

where ν_i is the outward unit normal to the boundary of Δ_i, $i = 1,2$. (See P. A. Raviart-J. M. Thomas [5].)

We denote by $(.,.)$ the scalar product in L_2 as well as in $L_2 \times L_2$. We also write $\|v\|_{W_p^r}$ instead of $\|v\|_{W_p^r \times W_p^r}$ for $v \in W_p^r \times W_p^r$. Finally we introduce in $(W_p^r \times W_p^r)'$ the norm

$$\|v\|'_{W_p^r} := \left\{ \sum_{\Delta \in \Gamma_h} \|v\|^p_{W_p^r(\Delta)} \right\}^{1/p}, \quad 2 \leq p < \infty$$

with the usual modification for $p = \infty$.

Let us define the quadratic functional $I : L_2 \times (W_2^1 \times W_2^1)' \to \mathbb{R}$ by

$$I(u, v) := a(u, v) - \frac{1}{2}(v, v) - (f, u) + \frac{1}{2}(Qu, u)$$

with

$$a(u, v) := -\int_\Omega u \operatorname{div} v \, ds.$$

The equation

$$I(u, v) - I(u_0, v_0) = -\frac{1}{2}(v - v_0, v - v_0) + \frac{1}{2}(Q(u - u_0), u - u_0) + a(u - u_0, v - v_0)$$

implies

$$I(u_0, v) \leq I(u_0, v_0) \leq I(u, v_0) \tag{4}$$

for all $u \in L_2$, $v \in (W_2^1 \times W_2^1)'$, i. e. (u_0, v_0) is a saddle-point of the functional I.

Given finite dimensional subspaces $U_h \subseteq L_2$ and $V_h \subseteq (W_2^1 \times W_2^1)'$, we approximate (u_0, v_0) by a saddle-point (u_h, v_h) of I restricted to $U_h \times V_h$. From the condition

$$I(u_h, \eta) \leqq I(u_h, v_h) \leqq I(\xi, v_h)$$

for all $\xi \in U_h$, $\eta \in V_h$ we get

$$\begin{aligned} a(\xi, v_h) + (\xi, Qu_h) &= (f, \xi) \\ a(u_h, \eta) - (v_h, \eta) &= 0 \end{aligned} \tag{5}$$

for all $\xi \in U_h$, $\eta \in V_h$. (5) has a unique solution if $U_h \subseteq \operatorname{div} V_h$ holds.

Then the equation (5) can be written in the form

$$\begin{aligned} a(\xi, v_0 - v_h) + (\xi, Q(u_0 - u_h)) &= 0 \qquad \text{for all } \xi \in U_h \\ a(u_0 - u_h, \eta) &= (v_0 - v_h, \eta) \qquad \text{for all } \eta \in V_h. \end{aligned} \tag{5'}$$

Thus, the mapping $(u_0, v_0) \to (u_h, v_h)$ may be considered as a projection operator from $L_2 \times (W_2^1 \times W_2^1)'$ onto $U_h \times V_h$.

For the sake of simplicity in the following we only regard the case $Q = 0$.

3. CONSTRUCTION OF APPROXIMATING SUBSPACES, L_2-ERROR ESTIMATES

Given a κ-regular triangulation Γ_h and an integer $r \geq 1$, P. A. Raviart-J. M. Thomas [5] construct a linear subspace V_h of $(W_p^{r+1} \times W_p^{r+1})'(\Gamma_h)$ in the following way:

$\eta = (\eta_1, \eta_2)$ belongs to V_h if in each triangle $\Delta \in \Gamma_h$ the functions η_1 and η_2 are special polynomials of degree $r + 1$, determined by the values of

$$\int_{K_i} \sigma^j \eta \cdot \nu \, d\sigma \quad , \quad 0 \leq j \leq r \quad , \quad i = 1, 2, 3$$

and

$$\int_{\Delta} s_1^k s_2^l \eta \, ds \quad , \quad 0 \leq k, l \quad , \quad k + l \leq r - 1,$$

where K_i denotes the sides of Δ. Furthermore in each triangle $\operatorname{div} \eta$ is a polynomial of degree r.

Let U_h be the space of finite elements of degree r for the same triangulation Γ_h (without any boundary or continuity conditions), and denote by $P_h : L_2 \to U_h$ the orthogonal projection from L_2 onto U_h.

Then $\operatorname{div} V_h \subseteq U_h$ and the following assertion holds. (See [5], compare also P. G. Ciarlet-P. A. Raviart [1].)

LEMMA 1 : *There exists a linear projection operator* $\Pi_h : (W_p^1 \times W_p^1)' \to V_h$, $2 \leq p \leq \infty$, with the following properties :

(i) *for all* $v \in (W_p^1 \times W_p^1)'$ *the relation*

$$div\, \Pi_h v = P_h \, div\, v \tag{6}$$

is valid;

(ii) *for all* $v \in (W_p^{l+1}) \times W_p^{l+1})'$ *the estimate*

$$\|v - \Pi_h v\|'_{W_p^k} \leq C h^{l+1-k} \|v\|'_{W_p^{l+1}}, \quad 0 \leq k, l, \; k \leq l + 1 \leq r + 1 \tag{7}$$

holds.

The following Lemma shows $U_h \subseteq \operatorname{div} V_h$; therefore the equation (5) has a unique solution.

LEMMA 2 : *For each* $\xi \in U_h$ *there is an element* $\eta \in V_h$ *with* $\operatorname{div} \eta = \xi$.

Proof: For an arbitrary element $\xi \in U_h$ let w be the element of $\mathring{W}_2^1 \cap W_2^2$ with $\Delta w = \xi$. Defining $\eta := \Pi_h \operatorname{grad} w$, relation (6) shows

$$\begin{aligned} \operatorname{div} \eta &= P_h \operatorname{div} \operatorname{grad} w \\ &= P_h \xi = \xi . \end{aligned}$$

Now let $(u_h, v_h) \in U_h \times V_h$ be the saddle-point approximation of (u_0, v_0) defined by (5). The following approximation theorem was obtained by P. A. Raviart-J. M. Thomas [5, Theorem 5]:

If $u_0 \in \overset{\circ}{W}{}_2^1 \cap W_2^{r+2}$ and $\Delta u_0 \in W_2^{r+1}$, then

$$\|u_0 - u_h\|_{L_2} + \|v_0 - v_h\|_{L_2} + \|\operatorname{div}(v_0 - v_h)\|_{L_2} \leq C h^{r+1} (\|u_0\|_{W_2^{r+2}} + \|\Delta u_0\|_{W_2^{r+1}}).$$

For our purpose we need an "uncoupled" estimate.

LEMMA 3 : *Suppose* $u_0 \in \overset{\circ}{W}{}_2^1 \cap W_2^{r+2}$. *Then*

$$\|v_0 - v_h\|'_{W_2^k} \leq C h^{r+1-k} \|u_0\|_{W_2^{r+2}}, \qquad 0 \leq k \leq r+1, \tag{8}$$

where C *is independent of* u_0 *and* h.

Proof : Define $\xi_h := P_h u_0$ and $\eta_h := \Pi_h v_0$. Using (5′) we find

$$\begin{aligned}\|v_h - \eta_h\|_{L_2}^2 &= (v_h - \eta_h, v_h - \eta_h) - a(u_h - \xi_h, v_h - \eta_h) + a(u_h - \xi_h, v_h - \eta_h) \\ &= (v_0 - \eta_h, v_h - \eta_h) - a(u_0 - \xi_h, v_h - \eta_h) + a(u_h - \xi_h, v_0 - \eta_h).\end{aligned}$$

From $\operatorname{div} V_h = U_h$ and relation (6) we obtain

$$\begin{aligned} a(u_0 - \xi_h, v_h - \eta_h) &= -(u_0 - \xi_h, \operatorname{div}(v_h - \eta_h)) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} a(u_h - \xi_h, v_0 - \eta_h) &= -(u_h - \xi_h, \operatorname{div} v_0 - P_h \operatorname{div} v_0) \\ &= 0. \end{aligned}$$

Therefore we get

$$\|v_h - \eta_h\|_{L_2}^2 \leqq \|v_0 - \eta_h\|_{L_2} \|v_h - \eta_h\|_{L_2};$$

with the help of (7) the estimate (8) follows for the case $k = 0$. For $1 \leq k \leq r+1$, (8) is obtained by inverse inequalities, obviously valid for the elements of V_h.

REMARK : If only $u_0 \in \overset{\circ}{W}{}_2^1 \cap W_2^{r+1}$ is presumed for the solution of (1), with the same proof and by application of duality arguments (see R. Scholz [6]) we can show the error estimate

$$\|u_0 - u_h\|_{L_2} + h\|v_0 - v_h\|_{L_2} \leq C h^{r+1} \|u_0\|_{W_2^{r+1}},$$

C independent of u_0 and h.

4. L_∞-ERROR ESTIMATES

Our main result is the following theorem.

THEOREM : *Assume the solution u_0 of problem (1) fulfills the regularity condition $u_0 \in \overset{\circ}{W}{}_2^1 \cap W_2^{r+2} \cap W_\infty^{r+1}$, and let $(u_h, v_h) \in U_h \times V_h$ be the saddle-point approximation of (u_0, v_0) defined by (5). Then the following error estimate holds :*

$$\|u_0 - u_h\|_{L_\infty} + h\,\|v_0 - v_h\|_{L_\infty} \leq C h^{r+1}\left\{\|u_0\|_{W_\infty^{r+1}} + \|u_0\|_{W_2^{r+2}}\right\}, \quad (9)$$

where the constant C is independent of u_0 and h.

In order to prove (9) we use "weighted" L_2-norms.

Let s_0 be any point of $\overline{\Omega}$. For $\rho > 0$ we define with $\mu := \mu(s) := |s - s_0|^2 + \rho^2$ for each $\alpha \in \mathbb{R}$

$$\|u\|_\alpha := \|\mu^{-\alpha/2} u\|_{L_2}, \qquad u \in L_2.$$

($|s - s_0|$ denotes the Euclidean distance between the points s and $s_0 \in \mathbb{R}^2$.) Between L_∞- and weighted L_2-norms we have the following relations :

(i) if $u \in L_\infty$ and $\alpha > 1$, then

$$\|u\|_\alpha \leq C \rho^{-\alpha+1}\,\|u\|_{L_\infty}; \quad (10)$$

(ii) for $\xi \in U_h$ and the special choice of $s_0 \in \overline{\Omega}$ such that $|\xi(s_0)| = \|\xi\|_{L_\infty}$ we have

$$\|\xi\|_{L_\infty} \leq C \gamma^{-\alpha} h^{\alpha-1}\,\|\xi\|_\alpha, \qquad h = \gamma\rho. \quad (11)$$

The constants C in (10) and (11) do not depend on ρ respectively h and the special point $s_0 \in \overline{\Omega}$. For a proof see J. Nitsche [3], [4]

The weighted norms in $L_2 \times L_2$ are defined in an analogous manner.

Proof of the Theorem : For convenience we write u and v instead of u_0 and v_0. Since the operator $(u, v) \to (u_h, v_h)$ is a projection, it suffices to prove

$$\|u_h\|_{L_\infty} + h\,\|v_h\|_{L_\infty} \leq C\left\{\|u\|_{L_\infty} + h\,\|v\|_{L_\infty} + \sum_{k=0}^{r+1} h^k\,\|v - v_h\|'_{W_2^k}\right\}. \quad (12)$$

First we show the estimate for u_h. Let $s_0 \in \overline{\Omega}$ be chosen such that $|u_h(s_0)| = \|u_h\|_{L_\infty}$. For $\alpha > 1$ we have

$$\|u_h\|_\alpha^2 = (u_h, \mu^{-\alpha} u_h - \xi) - (u, \mu^{-\alpha} u_h - \xi) + (u, \mu^{-\alpha} u_h) - (u - u_h, \xi) \quad (13)$$

with $\xi := P_h \mu^{-\alpha} u_h$. With the same arguments as in J. Nitsche [3] we find for $h = \gamma\rho$, γ suitably chosen,

$$\|u_h\|_\alpha^2 \leq C(\|u\|_\alpha^2 + |(u - u_h, \xi)|). \quad (14)$$

Now let $w \in \mathring{W}_2^1 \cap W_2^2$ be the solution of the auxiliary problem

$$\begin{aligned} -\Delta w &= \xi \quad \text{in} \quad \Omega \\ w &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

and define $\omega := \operatorname{grad} w$. An easy computation gives $\operatorname{div} \Pi_h \omega = -\xi = \operatorname{div} \omega$; hence we have $\omega - \Pi_h \omega \in N(T^*)$. With the help of $(5_2')$ therefore

$$\begin{aligned} (u - u_h, \xi) &= a(u - u_h, \omega) \\ &= a(u - u_h, \Pi_h \omega) \\ &= (v - v_h, \Pi_h \omega) \end{aligned} \tag{15}$$

From the Closed Range Theorem we get $v - v_h = v - \bar{v}_h - \hat{v}_h$ with $v - \bar{v}_h \in R(T)$ and $-\hat{v}_h \in N(T^*)$. Using $\omega \in R(T)$, $\omega - \Pi_h \omega \in N(T^*)$, and $(5_1')$, we find

$$\begin{aligned} |(v - \bar{v}_h, \Pi_h \omega)| &= |(v - \bar{v}_h, \omega)| \\ &= |(v - v_h, \omega)| \\ &= |a(w, v - v_h)| \\ &= |a(w - P_h w, v - v_h)| \\ &\leq C h^2 \|w\|_{W_2^2} \|\operatorname{div}(v - v_h)\|_{L_2} \\ &\leq C h^2 \rho^{-\alpha} \|u_h\|_\alpha \|v - v_h\|'_{W_2^1}, \end{aligned}$$

and

$$\begin{aligned} |-(\hat{v}_h, \Pi_h \omega)| &= |(\hat{v}_h, \omega - \Pi_h \omega)| \\ &= |(v - v_h, \omega - \Pi_h \omega)| \\ &\leq C h \|w\|_{W_2^2} \|v - v_h\|_{L_2} \\ &\leq C h \rho^{-\alpha} \|u_h\|_\alpha \|v - v_h\|_{L_2}. \end{aligned}$$

Combining these inequalities with (13), (14), and (15) we get

$$\|u_h\|_\alpha \leq C(\|u\|_\alpha + h\rho^{-\alpha} \|v - v_h\|_{L_2} + h^2 \rho^{-\alpha} \|v - v_h\|'_{W_2^1}).$$

Hence, the estimate (12) for u_h follows by (10) and (11).

Next let $s_0 \in \bar{\Omega}$ be such that

$$|v_{h.i}(s_0)| = \|v_{h.i}\|_{L\infty} = \|v_h\|_{L\infty},$$

$i = 1$ or $i = 2$. We find for $\alpha > 1$

$$\|v - v_h\|_\alpha^2 = (v - v_h, (I - \Pi_h)\mu^{-\alpha}(v - v_h)) + (v - v_h, \Pi_h \mu^{-\alpha}(v - v_h)),$$

where I denotes the identity. Using the approximation properties of the space V_h and

$$|D^k \mu^{-\alpha}(s)| \leq C \rho^{-k} \mu^{-\alpha}(s), \qquad k \geq 1,$$

the first term can be estimated by

$$\begin{aligned}
|(v - v_h, (I - \Pi_h)\mu^{-\alpha}(v - v_h))| \\
&\leq Ch^{r+1} \|v - v_h\|_{L_2} \|\mu^{-\alpha}(v - v_h)\|'_{W_2^{r+1}} \\
&\leq Ch^{r+1} \|v - v_h\|_{L_2} \sum_{k=0}^{r+1} \rho^{k-r-1-2\alpha} \|v - v_h\|'_{W_2^k} \quad (17) \\
&\leq C\rho^{-2\alpha} \sum_{k=0}^{r+1} h^{2k} \|v - v_h\|'^2_{W_2^k}.
\end{aligned}$$

Further, because of $(5'_2)$ we can write

$$\begin{aligned}
|(v - v_h, \Pi_h \mu^{-\alpha}(v - v_h)| &= |a(u - u_h, \Pi_h \mu^{-\alpha}(v - v_h))| \\
&= |(u - u_h, P_h \operatorname{div} \mu^{-\alpha}(v - v_h))| \\
&= |(P_h(u - u_h), \operatorname{div} \mu^{-\alpha}(v - v_h))| \\
&\leq \|P_h(u - u_h)\|_{\alpha+1} \|\operatorname{div} \mu^{-\alpha}(v - v_h)\|_{-\alpha-1} \\
&\leq C \|u - u_h\|^2_{\alpha+1} + \|\operatorname{div} \mu^{-\alpha}(v - v_h)\|^2_{-\alpha-1}.
\end{aligned}$$

(Here we used the boundedness of P_h in weighted norms; see J. Nitsche [3].)

An elementary computation gives

$$\|\operatorname{div} \mu^{-\alpha}(v - v_h)\|^2_{-\alpha-1} \leq C(\rho^{-2(\alpha-1)} \|\operatorname{div}(v - v_h)\|^2_{L_2} + \rho^{-2\alpha} \|v - v_h\|^2_{L_2}).$$

Thus,

$$\begin{aligned}
\|v_h\|_\alpha &\leq \|v\|_\alpha + \|v - v_h\|_\alpha \\
&\leq C\left(\|u - u_h\|_{\alpha+1} + \|v\|_\alpha + \rho^{-\alpha} \sum_{k=0}^{r+1} h^k \|v - v_h\|'_{W_2^k} \right).
\end{aligned}$$

Finally, using the relation (10) and (11) once more, we obtain the desired estimate (12) for v_h, and the proof is complete.

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