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$L_\infty$-convergence of saddle-point approximations for second order problems

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Abstract — Let $u_0$ be the solution of the second second order boundary value problem
\[-\Delta u + qu = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,\]
with $\Omega$ bounded in $\mathbb{R}^2$. The solution will be denoted by $u_0$.

The basic idea of the mixed method is to characterize $(u_0, v_0)$ as the saddle-point of a quadratic functional and to approximate $(u_0, v_0)$ by elements of suitably chosen finite dimensional subspaces.

The construction of approximating finite element spaces and the $L_2$-error analysis for this problem was given by P. A. Raviart and J. M. Thomas [5]. Using the same subspaces our goal is to derive $L_\infty$-error estimates. The method of proof is based on weighted $L_2$-norms, similarly to the work of J. Nitsche [3], [4], and F. Natterer [2].

2. NOTATIONS, STATEMENT OF THE PROBLEM

If we define the operator
\[Tu := \text{grad } u\]
with $T : D(T) = \tilde{W}_2^1 \subseteq L_2 \rightarrow L_2 \times L_2$, then the dual operator is

$$T^* v = - \text{div} v$$

with $T^* : D(T^*) = W_2^1 \times W_2^1 \subseteq L_2 \times L_2 \rightarrow L_2$. (We omit the specification of the domain, if no confusion is possible.)

Later on we need the following assertions: $T$ is a closed operator and $R(T)$ is closed in $L_2 \times L_2$. Therefore by the Closed Range Theorem $L_2 \times L_2$ is the orthogonal sum of $R(T)$ and $N(T^*)$. (See, for instance, K. Yosida [7], p. 205.)

Further we define the operator $Q : L_2 \rightarrow L_2$ by $Qu := qu, u \in L_2$. Then equation (1) is equivalent to the system

$$T^* v + Qu = f$$

$$Tu - v = 0$$

with the solution $(u_0, v_0), v_0 := \text{grad} u_0$.

For convenience we assume that $\Omega$ is a bounded polygon. Suppose $\Gamma_h$ is a $\kappa$-regular triangulation of $\Omega$, $0 < h$, i.e. for any $\Delta \in \Gamma_h$ there are two circles $K$ and $\overline{K}$ with radii $\rho$ and $\overline{\rho}$ such that $K \subseteq \Delta \subseteq \overline{K}$ and

$$\kappa^{-1}h \leq \rho \leq \overline{\rho} \leq \kappa h.$$ 

In the following let $r \geq 1$ be a fixed integer.

By $(W_p \times W_p)' = (W_p \times W_p)'(\Gamma_h), 2 \leq p \leq \infty$, we denote those elements of $L_p \times L_p$, which fulfill the following conditions:

(i) $v \in W_p(\Delta) \times W_p(\Delta)$ for all $\Delta \in \Gamma_h$;

(ii) for all $u \in W_2^1(\Omega)$ we have

$$\int_{\Omega} v \cdot \text{grad} u \, ds + \int_{\Omega} u \text{div} v \, ds$$

$$= \int_{\partial \Omega} \overline{u} v \cdot v \, d\sigma.$$ 

($v$ is the exterior unit normal to $\partial \Omega$.)

Equation (3) holds if and only if for any pairs of adjacent triangles $\Delta_1, \Delta_2 \in \Gamma_h$ we have

$$v|_{\Delta_1} \cdot v_1 + v|_{\Delta_2} \cdot v_2 = 0 \quad \text{on} \quad \Delta_1 \cap \Delta_2,$$

where $v_i$ is the outward unit normal to the boundary of $\Delta_i, i = 1, 2$. (See P. A. Raviart-J. M. Thomas [5].)
We denote by \((\cdot, \cdot)\) the scalar product in \(L^2\) as well as in \(L^2 \times L^2\). We also write \(\|v\|_{W^p_r}^r\) instead of \(\|v\|_{W^p_r \times W^p_r}^r\) for \(v \in W^p_r \times W^p_r\). Finally we introduce in \((W^p_r \times \hat{W}^p_r)\) the norm
\[
\|v\|_{W^p_r}^r := \left\{ \sum_{\Delta \in \mathcal{T}_h} \|v\|_{W^p_r(\Delta)}^r \right\}^{1/p}, \quad 2 \leq p < \infty
\]
with the usual modification for \(p = \infty\).

Let us define the quadratic functional \(I: L^2 \times (W^1_2 \times W^1_2)' \to \mathbb{R}\) by
\[
I(u, v) := a(u, v) - \frac{1}{2} (v, v) - (f, u) + \frac{1}{2} (Q(u, u)
\]
with
\[
a(u, v) := - \int_\Omega u \text{ div } v \, ds.
\]
The equation
\[
I(u, v) - I(u_0, v_0) = - \frac{1}{2} (v - v_0, v - v_0) + \frac{1}{2} (Q(u - u_0), u - u_0)
\]
implies
\[
I(u_0, v) \leq I(u_0, v_0) \leq I(u, v_0) \quad (4)
\]
for all \(u \in L^2, \; v \in (W^1_2 \times W^1_2)', \) i.e. \((u_0, v_0)\) is a saddle-point of the functional \(I\).

Given finite dimensional subspaces \(U_h \subseteq L^2\) and \(V_h \subseteq (W^1_2 \times W^1_2)', \) we approximate \((u_0, v_0)\) by a saddle-point \((u_h, v_h)\) of \(I\) restricted to \(U_h \times V_h\). From the condition
\[
I(u_h, \eta) \leq I(u_h, v_h) \leq I(\xi, v_h)
\]
for all \(\xi \in U_h, \; \eta \in V_h\) we get
\[
a(\xi, v_h) + (\xi, Qu_h) = (f, \xi)
\]
\[
a(u_h, \eta) - (v_h, \eta) = 0 \quad (5)
\]
for all \(\xi \in U_h, \; \eta \in V_h\). (5) has a unique solution if \(U_h \subseteq \text{div } V_h\) holds.

Then the equation (5) can be written in the form
\[
a(\xi, v_0 - v_h) + (\xi, Q(u_0 - u_h)) = 0 \quad \text{for all } \xi \in U_h
\]
\[
a(u_0 - u_h, \eta) = (v_0 - v_h, \eta) \quad \text{for all } \eta \in V_h. \quad (5')
\]
Thus, the mapping \((u_0, v_0) \to (u_h, v_h)\) may be considered as a projection operator from \(L^2 \times (W^1_2 \times W^1_2)\) onto \(U_h \times V_h\).

For the sake of simplicity in the following we only regard the case \(Q = 0\).
3. CONSTRUCTION OF APPROXIMATING SUBSPACES, $L_2$-ERROR ESTIMATES

Given a $k$-regular triangulation $\Gamma_h$ and an integer $r \geq 1$, P. A. Raviart-J. M. Thomas [5] construct a linear subspace $V_h$ of $(W^{r+1}_p \times W^{r+1}_p)'(\Gamma_h)$ in the following way:

$\eta = (\eta_1, \eta_2)$ belongs to $V_h$ if in each triangle $\Delta \in \Gamma_h$ the functions $\eta_1$ and $\eta_2$ are special polynomials of degree $r + 1$, determined by the values of

$$\int_{K_i} \sigma^j \eta \cdot v \, d\sigma \quad , \quad 0 \leq j \leq r \quad , \quad i = 1, 2, 3$$

and

$$\int_{\Delta} s^k s^l \eta \, ds \quad , \quad 0 \leq k, l \quad , \quad k + l \leq r - 1,$$

where $K_i$ denotes the sides of $\Delta$. Furthermore in each triangle $\text{div} \eta$ is a polynomial of degree $r$.

Let $U_h$ be the space of finite elements of degree $r$ for the same triangulation $\Gamma_h$ (without any boundary or continuity conditions), and denote by $P_h : L_2 \rightarrow U_h$ the orthogonal projection from $L_2$ onto $U_h$.

Then $\text{div} \, V_h \subseteq U_h$ and the following assertion holds. (See [5], compare also P. G. Ciarlet-P. A. Raviart [1].)

**LEMMA 1**: There exists a linear projection operator $\Pi_h : (W^1_p \times W^1_p)' \rightarrow V_h$, $2 \leq p \leq \infty$, with the following properties:

(i) for all $v \in (W^1_p \times W^1_p)'$ the relation

$$\text{div} \, \Pi_h v = P_h \text{div} v$$

is valid;

(ii) for all $v \in (W^{r+1}_p \times W^{r+1}_p)'$ the estimate

$$\|v - \Pi_h v\|_{W^{k+1}_p} \leq C h^{l+1-k} \|v\|_{W^{l+1}_p} \quad , \quad 0 \leq k, l, k \leq l + 1 \leq r + 1$$

holds.

The following Lemma shows $U_h \subseteq \text{div} \, V_h$; therefore the equation (5) has a unique solution.

**LEMMA 2**: For each $\xi \in U_h$ there is an element $\eta \in V_h$ with $\text{div} \, \eta = \xi$.

**Proof**: For an arbitrary element $\xi \in U_h$ let $w$ be the element of $\hat{W}_2^1 \cap W^1_2$ with $\Delta w = \xi$. Defining $\eta := \Pi_h \text{grad} \, w$, relation (6) shows

$$\text{div} \, \eta = P_h \text{grad} \, w$$

$$= P_h \xi = \xi.$$
Now let \((u_h, v_h) \in U_h \times V_h\) be the saddle-point approximation of \((u_0, v_0)\) defined by (5). The following approximation theorem was obtained by P. A. Raviart-J. M. Thomas [5, Theorem 5]:

If \(u_0 \in \tilde{W}^2_0 \cap W^{r+2}_2\) and \(\Delta u_0 \in W^{r+1}_2\), then

\[
\|u_0 - u_h\|_{L_2} + \|v_0 - v_h\|_{L_2} + \|\text{div} (v_0 - v_h)\|_{L_2} \leq C h^{r+1} (\|u_0\|_{W^{r+2}_2} + \|\Delta u_0\|_{W^{r+1}_2}).
\]

For our purpose we need an "uncoupled" estimate.

**Lemma 3:** Suppose \(u_0 \in \tilde{W}^1_2 \cap W^{r+2}_2\). Then

\[
\|v_0 - v_h\|_{W^k_2} \leq C h^{r+1-k} \|u_0\|_{W^{r+2}_2}, \quad 0 \leq k \leq r + 1,
\]

where \(C\) is independent of \(u_0\) and \(h\).

**Proof:** Define \(\tilde{\xi}_h := P_h u_0\) and \(\eta_h := \Pi_h v_0\). Using (5') we find

\[
\|v_h - \eta_h\|_{L_2} = (v_h - \eta_h, v_h - \eta_h) - a(u_h - \xi_h, v_h - \eta_h) + a(u_h - \xi_h, v_h - \eta_h)
\]

\[
= (v_0 - \eta_h, v_h - \eta_h) - a(u_0 - \xi_h, v_h - \eta_h) + a(u_h - \xi_h, v_0 - \eta_h).
\]

From \(\text{div} V_h = U_h\) and relation (6) we obtain

\[
a(u_0 - \xi_h, v_h - \eta_h) = - (u_0 - \xi_h, \text{div} (v_h - \eta_h)) = 0
\]

and

\[
a(u_h - \xi_h, v_0 - \eta_h) = - (u_h - \xi_h, \text{div} v_0 - P_h \text{div} v_0) = 0.
\]

Therefore we get

\[
\|v_h - \eta_h\|_{L_2} \leq \|v_0 - \eta_h\|_{L_2},
\]

with the help of (7) the estimate (8) follows for the case \(k = 0\). For \(1 \leq k \leq r + 1\), (8) is obtained by inverse inequalities, obviously valid for the elements of \(V_h\).

**Remark:** If only \(u_0 \in \tilde{W}^1_2 \cap W^{r+1}_2\) is presumed for the solution of (1), with the same proof and by application of duality arguments (see R. Scholz [6]) we can show the error estimate

\[
\|u_0 - u_h\|_{L_2} + h \|v_0 - v_h\|_{L_2} \leq C h^{r+1} \|u_0\|_{W^{r+1}_2},
\]

\(C\) independent of \(u_0\) and \(h\).

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4. \( L_\infty \)-ERROR ESTIMATES

Our main result is the following theorem.

**Theorem**: Assume the solution \( u_0 \) of problem (1) fulfills the regularity condition \( u_0 \in \tilde{W}^{1,2} \cap W_r^{r+2} \cap W^{r+1}_\infty \), and let \( (u_h, v_h) \in \mathcal{U}_h \times \mathcal{V}_h \) be the saddle-point approximation of \( (u_0, v_0) \) defined by (5). Then the following error estimate holds:

\[
\|u_0 - u_h\|_{L_\infty} + h \|v_0 - v_h\|_{L_\infty} \leq C h^{r+1} \left\{ \|u_0\|_{W_r^{r+1}} + \|u_0\|_{W^{r+2}_\infty} \right\},
\]

where the constant \( C \) is independent of \( u_0 \) and \( h \).

In order to prove (9) we use "weighted" \( L_2 \)-norms.

Let \( s_0 \) be any point of \( \bar{\Omega} \). For \( \rho > 0 \) we define with \( \mu := \mu(s) := |s - s_0|^2 + \rho^2 \) for each \( \alpha \in \mathbb{R} \)

\[
\|u\|_\alpha := \|u - \mu\|_{L_2}, \quad u \in L_2.
\]

\(|s - s_0| \) denotes the Euclidean distance between the points \( s \) and \( s_0 \in \mathbb{R}^2 \).

Between \( L_\infty \)- and weighted \( L_2 \)-norms we have the following relations:

(i) if \( u \in L_\infty \) and \( \alpha > 1 \), then

\[
\|u\|_\alpha \leq C \rho^{-\alpha+1} \|u\|_{L_\infty};
\]

(ii) for \( \xi \in \mathcal{U}_h \) and the special choice of \( s_0 \in \bar{\Omega} \) such that \( |\xi(s_0)| = \|\xi\|_{L_\infty} \) we have

\[
\|\xi\|_{L_\infty} \leq C \gamma^{-\alpha} h^{-1} \|\xi\|_\alpha, \quad h = \gamma \rho.
\]

The constants \( C \) in (10) and (11) do not depend on \( \rho \) respectively \( h \) and the special point \( s_0 \in \bar{\Omega} \). For a proof see J. Nitsche [3], [4].

The weighted norms in \( L_2 \times L_2 \) are defined in an analogous manner.

**Proof of the Theorem**: For convenience we write \( u \) and \( v \) instead of \( u_0 \) and \( v_0 \). Since the operator \( (u, v) \rightarrow (u_h, v_h) \) is a projection, it suffices to prove

\[
\|u_h\|_{L_\infty} + h \|v_h\|_{L_\infty} \leq C \left\{ \|u\|_{L_\infty} + h \|v\|_{L_\infty} + \sum_{k=0}^{r+1} h^k \|v - v_h\|_{W^{r+1}_2} \right\}.
\]

First we show the estimate for \( u_h \). Let \( s_0 \in \bar{\Omega} \) be chosen such that

\[
|u_h(s_0)| = \|u_h\|_{L_\infty}.
\]

For \( \alpha > 1 \) we have

\[
\|u_h\|_\alpha^2 = (u_h, \mu^{-\alpha} u_h - \xi) - (u, \mu^{-\alpha} u_h - \xi) + (u, \mu^{-\alpha} u_h) - (u - u_h, \xi)
\]

with \( \xi := P_h \mu^{-\alpha} u_h \). With the same arguments as in J. Nitsche [3] we find for \( h = \gamma \rho, \gamma \) suitably chosen,

\[
\|u_h\|_\alpha^2 \leq C \left( \|u\|_\alpha^2 + |(u - u_h, \xi)| \right).
\]
Now let \( w \in \tilde{W}_2^1 \cap W_2^2 \) be the solution of the auxiliary problem
\[
- \Delta w = \xi \quad \text{in} \quad \Omega \\
w = 0 \quad \text{on} \quad \partial \Omega,
\]
and define \( \omega := \text{grad} \, w \). An easy computation gives \( \text{div} \, \Pi_h \omega = - \xi = \text{div} \, \omega \); hence we have \( \omega - \Pi_h \omega \in N(T^*) \). With the help of (5') therefore
\[
(u - u_h, \xi) = a(u - u_h, \omega) \\
= a(u - u_h, \Pi_h \omega) \\
= (v - v_h, \Pi_h \omega)
\]
(15)

From the Closed Range Theorem we get \( v - v_h = v - \tilde{v}_h - \hat{v}_h \) with \( v - \tilde{v}_h \in R(T) \) and \( - \hat{v}_h \in N(T^*) \). Using \( \omega \in R(T) \), \( \omega - \Pi_h \omega \in N(T^*) \), and (5'), we find
\[
|(v - \tilde{v}_h, \Pi_h \omega)| = |(v - v_h, \omega)| \\
= |(v - v_h, \omega)| \\
= |a(w, v - v_h)| \\
= |a(w - P_h w, v - v_h)| \\
\leq C h^2 \| w \|_{W_2^2} \| \text{div} \, (v - v_h) \|_{L_2} \\
\leq C h^2 \rho^{-\alpha} \| u_h \|_\alpha \| v - v_h \|_{W_2^2},
\]
and
\[
|(- (\hat{v}_h, \Pi_h \omega))| = |(\hat{v}_h, \omega - \Pi_h \omega)| \\
= |(v - v_h, \omega - \Pi_h \omega)| \\
\leq C h \| w \|_{W_2^2} \| v - v_h \|_{L_2} \\
\leq C h \rho^{-\alpha} \| u_h \|_\alpha \| v - v_h \|_{L_2}.
\]
Combining these inequalities with (13), (14), and (15) we get
\[
\| u_h \|_\alpha \leq C (\| u \|_\alpha + h \rho^{-\alpha} \| v - v_h \|_{L_2} + h^2 \rho^{-\alpha} \| v - v_h \|_{W_2^2}).
\]
Hence, the estimate (12) for \( u_h \) follows by (10) and (11).

Next let \( \xi_0 \in \overline{\Omega} \) be such that
\[
|v_{h,i}(\xi_0)| = \| v_{h,i} \|_{L_\infty} = \| v_h \|_{L_\infty},
\]
i = 1 or i = 2. We find for \( \alpha > 1 \)
\[
\| v - v_h \|_\alpha^2 = (v - v_h, (I - \Pi_h) \mu^{-\alpha} (v - v_h)) + (v - v_h, \Pi_h \mu^{-\alpha} (v - v_h)),
\]
where I denotes the identity. Using the approximation properties of the space \( V_h \) and
\[
|D^k \mu^{-\alpha}(s)| \leq C \rho^{-k} \mu^{-\alpha}(s), \quad k \geq 1,
\]
the first term can be estimated by
\[ |(v - v_h, (I - \Pi_h)\mu^{-a}(v - v_h))| \leq C h^{r+1} \|v - v_h\|_{L_2} \|\mu^{-a}(v - v_h)\|_{W^{r+1}} \]
\[ \leq C h^{r+1} \|v - v_h\|_{L_2} \sum_{k=0}^{r+1} \rho^{k-r-1-2a} \|v - v_h\|_{W^{\frac{r}{2}}} \quad (17) \]
\[ \leq C \rho^{-2a} \sum_{k=0}^{r+1} h^{2k} \|v - v_h\|_{W^{\frac{r}{2}}}. \]

Further, because of \((5)_2\) we can write
\[ |(v - v_h, \Pi_h\mu^{-a}(v - v_h))| = |a(u - u_h, \Pi_h\mu^{-a}(v - v_h))| \]
\[ = |(u - u_h, P_h \text{div} \mu^{-a}(v - v_h))| \]
\[ = |(P_h(u - u_h), \text{div} \mu^{-a}(v - v_h))| \]
\[ \leq \|P_h(u - u_h)\|_{a+1} \|\text{div} \mu^{-a}(v - v_h)\|_{-a-1} \]
\[ \leq C \|u - u_h\|_{a+1}^2 + \|\text{div} \mu^{-a}(v - v_h)\|_{-a-1}^2. \]

(Here we used the boundedness of \(P_h\) in weighted norms; see J. Nitsche [3].)

An elementary computation gives
\[ \|\text{div} \mu^{-a}(v - v_h)\|_{-a-1} \leq C (\rho^{-2(a-1)} \|\text{div} (v - v_h)\|_{L_2}^2 + \rho^{-2a} \|v - v_h\|_{L_2}^2). \]

Thus,
\[ \|v_h\|_a \leq \|v\|_a + \|v - v_h\|_a \]
\[ \leq C \left( \|u - u_h\|_{a+1} + \|v\|_a + \rho^{-a} \sum_{k=0}^{r+1} h^{2k} \|v - v_h\|_{W^{\frac{r}{2}}} \right). \]

Finally, using the relation \((10)\) and \((11)\) once more, we obtain the desired estimate \((12)\) for \(v_h\), and the proof is complete.

REFERENCES