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## ON RATIONAL APPROXIMATIONS TO THE EXPONENTIAL <sup>(1)</sup>

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Communiqué par P.-A. RAVIART

Abstract. — *We give in this paper a simple characterization of the A-acceptability property for a family of rational approximations to  $e^{-z}$ . This result, which is implicitly contained in Norsett [3] is obtained here in a direct way. As a corollary, we obtain the results of Ehle [1] on Pade approximations to  $e^{-z}$ .*

Let  $r(z) = \frac{1 + a_1 z + \dots + a_n z^n}{1 + b_1 z + \dots + b_n z^n}$ ,  $a_i, b_i \in \mathbf{R}$ , be a rational approximation to  $e^{-z}$ . The function  $r(z)$  is said to be *A-acceptable* iff

$$|r(z)| \leq 1 \quad \text{for } z \in \mathbf{C} \operatorname{Re} z \geq 0. \quad (1)$$

This paper is devoted to the proof of the following.

**THEOREM :** *Assume that the rational function  $r(z)$  satisfies*

$$r(z) = e^{-z} + o(z^{2n-1}) \quad (z \rightarrow 0) \quad (2)$$

*Then  $r(z)$  is A-acceptable iff*

$$|a_n| \leq b_n \quad (3)$$

or

$$r(z) \text{ is reducible.} \quad (4)$$

*Proof :* We set  $p(z) = 1 + a_1 z + \dots + a_n z^n$   
 $q(z) = 1 + b_1 z + \dots + b_n z^n$

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**1. THE CONDITION IS NECESSARY**

Assume that  $r(z)$  is  $A$ -acceptable. Then the poles of  $r(z)$  belong to the open half-plane  $\text{Re } z < 0$  and we have  $\lim_{|z| \rightarrow \infty} |r(z)| \leq 1$ . Moreover, if  $r = p/q$  is irreducible, we have  $b_n \geq 0$  (otherwise the polynomial  $q$  would have a positive root) and  $|a_n| \leq b_n$ .

**2. THE CONDITION IS SUFFICIENT**

Let us prove it by induction. We first notice that the result is clearly true for  $n = 1$ . Let us assume that the property is true for  $n - 1$ . We consider two cases :

*1<sup>st</sup> case  $r$  is reducible or  $a_n = b_n = 0$ .* Then it follows from (2) that  $r$  is the  $(n - 1/n - 1)$ -th Padé approximation to  $e^{-z}$  and, from Hummel and Seebeck [2] and Padé [4]  $r(z)$  may be written on the form  $\frac{A_{n-1}(-z)}{A_{n-1}(z)}$  with

$$A_{n-1}(z) = \sum_{k=0}^{n-1} \frac{(2n - 2 - k)! (n - 1)!}{(2n - 2)! k! (n - 1 - k)!} z^k.$$

Therefore  $r$  satisfies the theorem hypothesis for  $n - 1$  and it follows from the induction hypothesis that  $r$  is  $A$ -acceptable.

*2<sup>nd</sup> case  $b_n \neq 0$  and  $|a_n| \leq b_n$ .* We set  $(a, b) = (a_1, \dots, a_n, b_1, \dots, b_n)$  and  $E_n = \{ (a, b); r(z) = e^{-z} + 0(z^{2n-1}), |a_n| \leq b_n \text{ and } b_n \neq 0 \}$ .

The subset  $E_n$  of  $\mathbf{R}^{2n}$  is convex and therefore connected.

For all  $(a, b) \in E_n$  we have

$$|r(iy)|^2 = \frac{1 + \alpha_1 y^2 + \alpha_2 y^4 + \dots + \alpha_n y^{2n}}{1 + \beta_1 y^2 + \beta_2 y^4 + \dots + \beta_n y^{2n}} = 1 + 0(y^{2n-1})$$

and then we have  $\alpha_1 = \beta_1, \dots, \alpha_{n-1} = \beta_{n-1}$ . From the inequality  $|a_n| \leq b_n$  it follows that  $\alpha_n \leq \beta_n$  and therefore we have for all  $(a, b) \in E_n$  :

$$|r(iy)| \leq 1 \quad \text{for all } y \in \mathbf{R}. \tag{5}$$

And so  $r$  has no pole on the axis  $\text{Re } z = 0$ . Therefore  $q$  cannot have a root on this axis : indeed  $q(iy_0) = 0$  implies  $q(-iy_0) = 0$  and  $y_0 \neq 0$  for  $q(0) = 1$  ; since  $iy_0$  and  $-iy_0$  cannot be poles of  $r$ ,  $r$  may be written  $r(z) = p_1(z)/q_1(z)$  with  $d^0 p_1 \leq n - 2$  and  $d^0 q_1 \leq n - 2$ , which is incompatible with (2).

Now we set

$$F_n = \{ (a, b); (a, b) \in E_n \text{ and the roots of } q \text{ belong to the closed half plane } \text{Re } z \leq 0 \}$$

Since  $q$  has no root on the imaginary axis, we have

$$F_n = \{ (a, b); (a, b) \in E_n \text{ and the roots of } q \text{ belong to the open half plane } \operatorname{Re} z < 0 \}$$

We notice that  $F_n$  is not empty; indeed the  $(n - 1/n - 1) - th$  Padé approximation to  $e^{-z}$  is irreducible and it follows from the induction hypothesis that its denominator  $A_{n-1}$  has no pole in the half plane  $\operatorname{Re} z \geq 0$ . We set

$$p(z) = A_{n-1}(-z)(1+z) \quad \text{and} \quad q(z) = A_{n-1}(z)(1+z);$$

then we have  $(a, b) \in F_n$ .

From (6) and (7) it follows that  $F_n$  is an open and closed subset of  $E_n$  and then we have  $F_n = E_n$ . Therefore if  $(a, b) \in E_n$ ,  $r$  is analytic in a neighbourhood of the half plane  $\operatorname{Re} z \geq 0$ ; hence from the maximum principle and from (5) it follows that  $r$  is  $A$ -acceptable.

**COROLLARY 1 :** *The Padé approximations to  $e^{-z}$  of type  $(n/n)$ ,  $(n - 1/n)$ ,  $(n - 2/n)$  are  $A$ -acceptable.*

**COROLLARY 2 :** *The poles of Padé approximations to  $e^{-z}$  of type  $(n/n)$ ,  $(n - 1/n)$ ,  $(n - 2/n)$  belong to the open left half plane  $\operatorname{Re} z < 0$ .*

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