RAIRO. ANALYSE NUMÉRIQUE

MICHEL FORTIN

An analysis of the convergence of mixed finite element methods

RAIRO. Analyse numérique, tome 11, nº 4 (1977), p. 341-354

<http://www.numdam.org/item?id=M2AN_1977__11_4_341_0>

R.A.I.R.O Analyse numérique/Numerical Analysis
(vol. 11, n° 4, 1977, p. 341 à 354)

AN ANALYSIS OF THE CONVERGENCE OF MIXED FINITE ELEMENT METHODS (*) (1)

par Michel FORTIN (2)
Communiqué par P.-A. RAVIART

Abstract. — *This paper deals with convergence proofs in mixed finite element methods. After recalling abstract conditions of Brezzi, one shows that these conditions are, in some cases, equivalent to the possibility of building an uniformly continuous operator Π_h from V into V_h. Moreover some properties of discrete operators involved in the approximation are characterized. Two examples show that building the operator Π_h can be done through an interpolation operator. A third example presents a case which is still out of reach of present techniques.*

1. INTRODUCTION

The aim of this paper, is to study, in a rather general setting, the convergence properties of approximations, by finite elements, of saddle-point problems related to the minimization of convex functionals under a linear constraint. Applications are, of course, mixed finite elements methods and hybrid methods, but the results given here are mainly adapted to the case of mixed methods.

The problem we consider has already been treated in Brezzi [1] and Brezzi-Raviart [2], among others. The case we consider is slightly more general in a sense to be precised later. However the main result will be to give sufficient (and in some cases necessary) conditions to verify the abstract "stability" condition of [1]. These new conditions can, in many cases, be quite easily verified, thus simplifying, in a considerable way, convergence proofs. Although it would be too long to present a full account of the previous works on the subject, the ready may refer, apart from the above cited papers, to Oden [6-7], and Johnson [5] for a more complete view of the problem.

The exposition will proceed as follows. In No. 2, we study the abstract continuous problem and give an existence and uniqueness theorem. In No. 3 we recall the general abstract condition of [1] for the convergence of approximations. In No. 4, we present a few lemmas characterizing the Kernels and Images of some operators appearing in the problem and we use these results to give practical convergence conditions. Finally in No. 5, we give some examples of application of these results.

(*) Manuscrit reçu le 22 avril 1977.
(1) This work was supported in part by NRCC Grant A 8195 and in part by a FCAC grant from the Department of Education of the Province of Québec.
(2) Faculté des Sciences et de Génie, Université Laval, Québec (Canada).

2. THE GENERAL PROBLEM

Let V and W be two real Hilbert spaces whose norms and scalar products are respectively denoted $|.|_V$, $(.,.)_V$, $|.|_W$ and $(.,.)_W$.

We give on $V \times V$ a continuous, symmetrical, bilinear form $a(u, v)$ and on $V \times W$ a continuous bilinear form $b(v, \varphi)$. Continuity of b implies that there exist a constant, denoted $\|b\|$, such that,

$$|b(v, \varphi)| \leqq \|b\| \, |v|_V \, |\varphi|_W, \qquad \forall v \in V, \qquad \forall \varphi \in W. \tag{2.1}$$

In the same way, the norm of a as a bilinear form on $V \times V$ will be denoted $\|a\|_V$. Let $f \in V'$ and $g \in W'$ be given. The brackets $\langle .,. \rangle$ will denote duality between both V' and V and W' and W, no ambiguity being possible. We consider the functional, on $V \times W$,

$$L(v, \varphi) = \frac{1}{2} a(v, v) - \langle f, v \rangle + b(v, \varphi) - \langle g, \varphi \rangle, \tag{2.2}$$

and we want to find a pair $(u, \lambda) \in V \times W$, saddle-point of $L(v, \varphi)$ and $V \times W$, that is,

$$L(u, \varphi) \leqq L(u, \lambda) \leqq L(v, \lambda), \qquad \forall v \in V, \qquad \forall \varphi \in W. \tag{2.3}$$

This is, of course, equivalent to solving, the linearly constrained, quadratic problem,

$$\frac{1}{2} a(u, u) - \langle f, u \rangle \leqq \frac{1}{2} a(v, v) - \langle f, v \rangle, \qquad \forall v \in Z(g), \qquad u \in Z(g); \tag{2.4}$$

$$Z(g) = \{ v \mid v \in V, \, b(v, \varphi) = \langle g, \varphi \rangle, \forall \varphi \in W \}. \tag{2.5}$$

The saddle-point (u, λ) is then also solution of the system,

$$a(u, v) + b(v, \lambda) = \langle f, v \rangle, \qquad \forall v \in V, \tag{2.6}$$

$$b(u, \varphi) = \langle g, \varphi \rangle, \qquad \forall \varphi \in W, \tag{2.7}$$

$$u \in V, \qquad \lambda \in W. \tag{2.8}$$

We remind that under some hypotheses, this saddle-point problem has a solution, eventually a unique solution. We first recall a few classical notations. First let us remind that the continuous bilinear form $b(v, \varphi)$ defines a continuous linear operator B from V into W', precisely,

$$\langle Bv, \varphi \rangle = b(v, \varphi), \qquad \forall \varphi \in W. \tag{2.9}$$

In the same way, the transpose B^* of B, from W into V' is defined by

$$\langle v, B^* \varphi \rangle = b(v, \varphi), \qquad \forall v \in V. \tag{2.10}$$

Condition (2.7) is thus clearly equivalent to

$$Bu = g \tag{2.11}$$

and we also have, according to (2.5):

$$Z(g) = \{ v \mid v \in V, Bv = g \}. \tag{2.12}$$

A necessary condition for the existence of a solution u to (2.11) is, of course, $g \in \operatorname{Im} B$. We shall assume that this is fulfilled. Let then v_g be any element of $Z(g)$. Our problem may then be written, writing $u = u_0 + v_g$ in the equivalent form,

$$a(u_0, v_0) = \langle f, v_0 \rangle - a(v_g, v_0), \qquad \forall v_0 \in \operatorname{Ker} B; \tag{2.13}$$

$$u_0 \in \operatorname{Ker} B. \tag{2.14}$$

According to the Lax-Milgram theorem, we have for the existence of u_0 (and then of $u = u_0 + v_g$) the classical coerciveness condition:

$$a(v_0, v_0) \geqq \alpha \| v_0 \|_v^2, \qquad \forall v_0 \in \operatorname{Ker} B. \tag{2.15}$$

This condition implies the existence of a unique solution u to (2.4)-(2.5) i. e. to the *primal problem.*

To prove the existence of a saddle-point, we must show the existence of a Lagrange multiplier for the linear constraint (2.11).

Before doing so, we recall, some facts about the properties of B and B^*.

LEMMA 2.1: *The following statements are equivalent:*

The range $\operatorname{Im} B$ *is closed in* W', (2.16)

$$\sup_{v \in V} \frac{|b(v, \varphi)|}{|v|_V} \geqq k \inf_{\varphi_0 \in \operatorname{Ker} B^*} |\varphi + \varphi_0|_W \tag{2.17}$$

$$|B^* \varphi|_{V'} \geqq k |\varphi|_{W/\operatorname{Ker} B^*}, \tag{2.18}$$

$$|Bu|_{W'} \geqq k |u|_{V/\operatorname{Ker} B}, \tag{2.19}$$

B admits a continuous lifting from W' into V. (2.20)

Proof: This a restatement of the closed range theorem (*cf.* e. g. Yosida [9]). ■
We then have the following result:

PROPOSITION 2.1: *Let* $\operatorname{Im} B$ *be closed in* W' *and let* (2.15) *be satisfied. Then the saddle-point problem* (2.3) *has a unique solution* (u, λ) *in* $V \times W/\operatorname{Ker} B^*$. *The Lagrange multiplier* λ *is thus unique up to the addition of any element of* $\operatorname{Ker} B^*$.

Proof: See Brezzi [1]. ■

3. ABSTRACT CONVERGENCE RESULTS

We approximate here the saddle-point problem (2.3) by internal approximation which in practice will be realized by finite elements. We consider two finite dimensional spaces,

$$V_h \subset V; \qquad W_h \subset W \tag{3.1}$$

with the topology induced by V and W respectively.

We now consider a discrete saddle-point problem,

$$L(u_h, \varphi_h) \leqq L(u_h, \lambda_h) \leqq L(v_h, \lambda_h), \qquad \forall v_h \in V_h, \qquad \forall \varphi_h \in W_h, \tag{3.2}$$

which is characterized by the optimality conditions,

$$a(u_h, v_h) + b(v_h, \lambda_h) = \langle f, v_h \rangle, \qquad \forall v_h \in V_h, \tag{3.3}$$

$$b(u_h, \varphi_h) = \langle g, \varphi_h \rangle, \qquad \forall \varphi_h \in W_h, \tag{3.4}$$

$$u_h \in V_h, \qquad \lambda_h \in W_h. \tag{3.5}$$

The continuous bilinear form $b(.,\ .)$ still defines here a continuous operator, $B_h : V_h \to W_h'$, and its transpose $B_h^* : W_h \to V_h'$. In general one cannot identify B_h as the restriction of B to V_h but one has

$$B_h v_h = P_{W_h'}(B v_h), \tag{3.6}$$

where $P_{W_h'}$ is the projection operator from W' to W_h'. Let $\tilde{g}_h = P_{W_h'}(g)$ (that is $\langle \tilde{g}_h, \varphi_h \rangle = \langle g, \varphi_h \rangle$, $\forall\ \varphi_h\ W_h$).

Then (3.4) can be written as

$$B_h u_h = \tilde{g}_h \tag{3.7}$$

and a necessary condition for existence is of course:

$$\tilde{g}_h \in \operatorname{Im} B_h. \tag{3.8}$$

Under a proper coerciveness condition, for instance:

$$a(v_h, v_h) \geqq \alpha \left| v_h \right|^2_{V_h/\operatorname{Ker} B_h}, \tag{3.9}$$

Proposition 2.1, implies the existence of a discrete saddle-point (u_h, λ_h), for in this finite dimensional case, $\operatorname{Im} B_h$ is always closed. We would then want to know if (u_h, λ_h) is an approximation to (u, λ). In order to solve this problem, we first present abstracts results, extending to the case where B is not surjective, the results of Brezzi [1].

It is clear Lemma 2.1 is still valid, and even trivial in finite dimensional spaces. However, the various constants C, and k generally depend on h. Convergence proofs will rely heavily on the following definition.

DEFINITION 3.1: We say that B_h satisfies the *uniformly continuous lifting property* (UCLP) if the following equivalent conditions hold with k, *and* c *independant of* h

$$\sup_{v_h \in V_h} \frac{|b(v_h, \varphi_h)|}{|v_h|_{V_h}} \geqq k \inf_{\varphi_{0h} \in \operatorname{Ker} B_h^*} |\varphi_h + \varphi_{0h}|_{W_h}, \tag{3.10}$$

$$|B_h^* \varphi_h|_{V_h'} \geqq k |\varphi_h|_{W_h/\operatorname{Ker} B_h^*}, \tag{3.11}$$

$$|B_h u_h|_{W_h'} \geqq k |u_h|_{V_h/\operatorname{Ker} B_h}. \tag{3.12}$$

$$\text{For any } g_h \in \operatorname{Im} B_h, \quad \text{there exists } u_h \in V_h \quad \text{such that} \quad B_h u_h = g_h, \quad |u_h|_{V_h} < c |g_h|_{W_h'}. \quad \blacksquare \tag{3.13}$$

We now define

$$Z_h(g) = \{ v_h \in V_h \mid b(v_h, \varphi_h) = \langle g, \varphi_h \rangle, \forall \varphi_h \in W_h \}, \tag{3.14}$$

or equivalently

$$Z_h(g) = \{ v_h \in V_h \mid B_h v_h = \tilde{g}_h \}. \tag{3.15}$$

We now recall the following classical results of Brezzi [1] and Brezzi-Raviart [2].

PROPOSITION 3.1: *Let* (3.8) *hold and let* $a(.,.)$ *be V-coercive, that is*

$$a(v, v) \geqq \alpha |v|_V^2. \tag{3.16}$$

Then there exists a constant C independant of h such that

$$|u - u_h|_V \leqq C(\inf_{v_h \in Z_h(g)} |u - v_h|_V + \inf_{\varphi_h \in W_h} |\lambda - \varphi_h|_W). \tag{3.17}$$

Moreover, if B_h satisfies the UCLP *condition and if we denote λ and λ_h the minimal norm Lagrange multipliers (that is with zero component in* Ker B^* *and* Ker B_h^* *respectively), then there exists a constant C independant of h such that:*

$$|u - u_h|_V + |\lambda - \lambda_h|_W \leqq C \{ \inf_{w_h \in V_h} |u - v_h|_V + \inf_{\varphi_h \in \operatorname{Im} B_h} |\lambda - \varphi_h|_W \}. \quad \blacksquare \tag{3.18}$$

Another special case is of special interest: Let us suppose that we have $V \subsetneq H$, where H is a Hilbert space, and that a is H-coercive, that is

$$a(v, v) \geqq \alpha |v|_H^2, \tag{3.19}$$

but not V-coercive [i. e. (3.16)].

The following result can then be proved:

PROPOSITION 3.2: *Let* (3.8)-(3.19) *and the* UCLP *condition hold. Then if one has*

$$\operatorname{Ker} B_h \subset \operatorname{Ker} B, \tag{3.20}$$

There exists a constant C independant of h such that

$$|u-u_h|_H+|\lambda-\lambda_h|_W \leqq C\{ \inf_{w_h \in V_h} |u-w_h|_V + \inf_{\varphi_h \in \operatorname{Im} B_h} |\lambda-\varphi_h|_W\}. \quad \blacksquare \tag{3.21}$$

It must be noted that we have not stated the results of [2] in their most general form. In order to use them in practice the following questions must be answered:

Q 3.1: When is, in general, (3.8) satisfied?
Q 3.2: When is the UCLP condition satisfied?
Q 3.3: When is (3.20) satisfied?
Q 3.4: Can we replace in (3.21), $\inf_{\varphi_h \in \operatorname{Im} B_h} |\lambda-\varphi_h|_W$ by an infimum over all $\varphi_h \in W_h$?

We shall try in the following section to give equivalent or sufficient conditions for the answers to be positive.

4. EQUIVALENT FORMS FOR CONVERGENCE CONDITIONS

When trying to apply the abstract results of No. 3 to a precise case, the main problem lies in the verification of the continuous lifting property or the condition on Kernels (3.16). We shall first give some algebraic lemmas that will clarify the relations between the ranges and kernels of B and B_h. As may be expected, the continuous lifting property will then be splitted in a consistency and a stability condition and we shall give sufficient conditions for the stability to hold. We restrict ourselves, to simplify the proofs to the case where W and W_h are identified to their dual spaces. We first have.

LEMMA 4.1: *The following statements are equivalent:*

$$\left.\begin{array}{c}\text{For any } u \in V, \text{ there exists } \hat{u}_h = \Pi_h u \in V_h, \text{ such that,} \\ b(u-\hat{u}_h, \varphi_h) = 0, \ \forall \varphi_h \in W_h, \text{ or equivalently } \hat{u}_h \in Z_h(Bu);\end{array}\right\} \tag{4.1}$$

$$\operatorname{Im} B_h = P_{W_h}(\operatorname{Im} B), \tag{4.2}$$

P_{W_h} being the projection operator of W on W_h.

$$\operatorname{Ker} B_h^* = \operatorname{Ker} B^* \cap W_h \subset \operatorname{Ker} B^*. \tag{4.3}$$

Proof: The equivalence of (4.1) and (4.2) is trivial: by definition, one always has

$$\operatorname{Im} B_h = P_{W_h}(B V_h) \subset P_{W_h}(\operatorname{Im} B);$$

it is therefore sufficient to consider the reverse inclusion which is nothing that another statement of (4.1). To show the equivalence of (4.1) or (4.2) with (4.3), let us suppose that (4.1) is satisfied and let $\bar{\varphi}_h$ be given in Ker B_h^*, i. e.,

$$b(v_h, \bar{\varphi}_h) = 0, \qquad \forall v_h \in V_h.$$

We have to show that $b(v, \bar{\varphi}_h) = 0$, for any $v \in V$, which implies $\bar{\varphi}_h \in \operatorname{Ker} B^*$. But for $v \in V$, there exists by (4.1), $\hat{v}_h \in V_h$ such that

$$b(v, \varphi_h) = b(\hat{v}_h, \varphi_h), \qquad \varphi_h \in W_h.$$

In particular this is true for $\bar{\varphi}_h$ so $b(v, \bar{\varphi}_h) = b(\hat{v}_h, \bar{\varphi}_h) = 0$.

Conversely, let $u \in V$ and consider $P_{W_h}(Bu)$.

We then have, by definition of the projection operator,

$$(\varphi_h, Bu - P_{W_h}(Bu))_W = 0, \qquad \varphi_h \in W_h. \tag{4.4}$$

We want to show that $P_{W_h}(Bu) \subset \operatorname{Im} B_h$, or equivalently

$$P_{W_h}(Bu) \in (\operatorname{Ker} B_h^*)^{\perp}. \tag{4.5}$$

Let then $\bar{\varphi}_h \in \operatorname{Ker} B_h^* \subset \operatorname{Ker} B^*$ be given, and take $\varphi_h = \bar{\varphi}_h$ in (4.4), we obtain,

$$(\bar{\varphi}_h, P_{W_h}(Bu)) = (\bar{\varphi}_h, Bu) = (B^* \bar{\varphi}_h, u) = 0. \quad \blacksquare \tag{4.6}$$

Remark 4.1: The previous proof shows, in fact, that (4.1) is equivalent to,

$$\operatorname{Ker} B_h^* \subset \operatorname{Ker} B^* \cap W_h, \tag{4.7}$$

the reverse inclusion always being true. Moreover the identification of W to W' is not essential for the proof, it is sufficient to restrict the analysis to $W_h' \cap W' = \tilde{W}_h'$. By definition of B_h, one has immediately $B_h v_h \in \tilde{W}_h'$, $v_h \in V_h$ and the previous proof can be extended, with a few technical subtilities. ■

Merely exchanging the roles of V and W, and taking into acount the preceeding remark, we have thus shown:

LEMMA 4.2: *The following statements are equivalent:*

$$\begin{array}{l} \textit{For any } \varphi \in W, \textit{ there exists } \hat{\varphi}_h \in W_h \textit{ such that,} \\ \quad b(v_h, \varphi - \hat{\varphi}_h) = 0, \qquad \forall v_h \in V_h, \end{array} \tag{4.8}$$

$$\operatorname{Im} B_h^* = P_{V_h' \cap V'}(\operatorname{Im} B^*). \tag{4.9}$$

$$\operatorname{Ker} B_h = \operatorname{Ker} B \cap V_h \subset \operatorname{Ker} B. \quad \blacksquare \tag{4.10}$$

We have thus obtained a characterization of condition (3.20) $\operatorname{Ker} B_h \subset \operatorname{Ker} B$ and given a partial answer to Q 3.3. ■

Finally to conclude this analysis, we prove.

LEMMA 4.3: *The following statements are equivalent:*

$$\operatorname{Im} B_h \subset (\operatorname{Im} B) \cap W_h, \tag{4.11}$$

$$P_{W_h}(\operatorname{Ker} B^*) \subset \operatorname{Ker} B_h^*. \tag{4.12}$$

Proof: Let (4.11) hold and $\varphi \in \operatorname{Ker} B^*$; we want to show that

$$\bar{\varphi}_h = P_{W_h}(\varphi) \in \operatorname{Ker} B_h^*.$$

But, by definition, we have

$$(\varphi, \lambda_h) = (\varphi_h, \lambda_h), \qquad \forall \lambda_h \in W. \tag{4.13}$$

We have to show that $(\varphi_h, \lambda_h) = 0$, if $\lambda_h \in \operatorname{Im} B_h$. However in this case we have $\lambda_h \in \operatorname{Im} B$ and $(\varphi, \lambda_h) = 0$, for $\varphi \in \operatorname{Ker} B^*$.

Conversely, let (4.12) hold and $\lambda_h \in \operatorname{Im} B_h$; we want to show that $\lambda_h \in \operatorname{Im} B$, that is λ_h is orthogonal to $\operatorname{Ker} B^*$. But for any $\varphi \in \operatorname{Ker} B^*$, one has

$$(\varphi, \lambda_h) = (\varphi_h, \lambda_h) = 0, \text{ for } \varphi_h = P_{W_h} \varphi \in \operatorname{Ker} B_h^*. \quad \blacksquare$$

Remark 4.1: (4.12) is of course satisfied if $\operatorname{Ker} B_h^* = \operatorname{Ker} B^*$, in particular if B and B_h are surjective. *Moreover* if $\operatorname{Ker} B_h^* \subset \operatorname{Ker} B^*$ inclusions in (4.11) and (4.12) may be replaced by equalities. ■

Remark 4.2: In proposition 3.1, we supposed that $\tilde{g}_h = P_{W_h}(g)$ belonged to $\operatorname{Im} B_h$. From Lemma 4.1, we deduce that this will be the case in general, if and only if $\operatorname{Ker} B_h^* \subset \operatorname{Ker} B^*$. An important case is however $g = \tilde{g}_h = 0$ where this last condition needs not be satisfied. This answers in part Q.3.1. ■

Remark 4.3: In the same way, let $\operatorname{Ker} B_h^* \subset \operatorname{Ker} B^*$. Then by (4.2), we have as $\lambda \in \operatorname{Im} B$

$$\inf_{\varphi_h \in \operatorname{Im} B_h} |\bar{\lambda} - \varphi_h| = \inf_{\varphi_h \in W_h} |\bar{\lambda} - \varphi_h|. \tag{4.14}$$

This answers Q. 3.4. ■

The main problem that remains is to characterize the UCLP condition. This will be done in two steps. We first prove.

PROPOSITION 4.1: *Let* Im B *be closed, and let* (4.1)-(4.3) *be satisfied*, the linear operator $\Pi_h : V \longrightarrow V_h$ *being uniformly continuous, that is, there exists a constant* c, *independant of* h *such that*

$$|\hat{u}_h|_{V_h} = |\Pi_h u|_{V_h} \leqq c\,|u|_V, \tag{4.15}$$

Then there exists a constant k, *independant of* h, *such that*

$$|B_h^* \varphi_h|_{V_h'} \geqq k \inf_{\varphi_0 \in \operatorname{Ker} B^*} |\varphi_0 + \varphi_h|_W. \tag{4.16}$$

Proof: We clearly have

$$|B_h^* \varphi_h|_{V_h'} = \sup_{v_h} \frac{|b(v_h, \varphi_h)|}{|v_h|_{V_h}} \geqq \sup_{v \in V} \frac{|b(\Pi_h v, \varphi_h)|}{|\Pi_h v|_{V_h}}. \tag{4.17}$$

By (4.1) and (4.15) we have

$$\sup_{v \in V} \frac{|b(\Pi_h v, \varphi_h|}{|\Pi_h(v)|_{V_h}} \geqq \sup_{v \in V} \frac{1}{c} \frac{|b(v, \varphi_h)|}{|v|_V}, \tag{4.18}$$

and Im B, being closed,

$$\sup_{v \in V} \frac{|b(v, \varphi_h)|}{|v|_V} \geqq k \inf_{\varphi_0 \in \operatorname{Ker} B^*} |\varphi_0 + \varphi_h|. \quad \blacksquare \tag{4.19}$$

Remark 4.4: We have thus shown, that if Ker $B_h^* \subset$ Ker B^* then building an uniformly continuous operator Π_h implies (4.16) which is *almost* the UCLP condition. Moreover from Remark 4.2, we see that Ker $B_h^* \subset$ Ker B^* is also, in general, a condition for the existence of a discrete solution. We shall then, try to see what must be added to get the uniform lifting which is necessary for convergence proofs. ■

We now prove

PROPOSITION 4.2: *If* (4.11)-(4.12) *hold, that is* Im $B_h \subset$ Im B, *then* (4.16) *implies the* UCLP *condition. If moreover* Ker $B_h^* \subset$ Ker B^* [*cf.* (4.1)-(4.3)] *then* (4.16) *and the* UCLP *condition are equivalent.*

Proof: Let (4.16) be satisfied, and consider $\varphi_0 \in$ Ker B^*. If Im $B_h \subset$ Im $B \cap W_h$, then by Lemma 4.3, $P_{W_h} \varphi_0 \in$ Ker B_h^*. Moreover, one has

$$P_{W_h}(\varphi_h + \varphi_0) = \varphi_h + P_{W_h} \varphi_0,$$

and

$$|\varphi_h + P_{W_h} \varphi_0|_W = |P_{W_h}(\varphi_h + \varphi_0)|_W \leqq |\varphi_h + \varphi_0|_W. \tag{4.20}$$

Thus

$$\inf_{\varphi_{0h} \in \operatorname{Ker} B_h^*} |\varphi_h + \varphi_{0h}| \leqq \inf_{\varphi_0 \in \operatorname{Ker} B^*} |\varphi_h + \varphi_0| \leqq 1/k \, |B_h^* \varphi_h|_{V_h'}. \tag{4.30}$$

Conversely, if Ker $B_h^* \subset$ Ker B^*, one has

$$\inf_{\varphi_0 \in \operatorname{Ker} B^*} |\varphi_h + \varphi_0| \leqq \inf_{\varphi_{0h} \in \operatorname{Ker} B^*} |\varphi_h + \varphi_0|, \tag{4.31}$$

so that (3.25) implies (3.14). ■

Remark 4.5: Let us suppose that (4.3) and (4.11) are both satisfied. That is

$$\operatorname{Ker} B_h^* = (\operatorname{Ker} B^*) \cap W_h \qquad \text{and} \qquad \operatorname{Im} B_h = (\operatorname{Im} B) \cap W_h.$$

The decomposition,

$$W_h = (\operatorname{Im} B_h) \oplus (\operatorname{Ker} B_h^*), \tag{4.29}$$

is then merely the restriction to W_h of the decomposition $W = (\operatorname{Im} B) \oplus (\operatorname{Ker} B^*)$, if $\varphi_h \in W_h$ is written as $\varphi_h = \varphi_h^I + \varphi_h^k$ with $\varphi_h^I \in$ Im B_h and $\varphi_h^k \in$ Ker B_h^*, then $\varphi_h^I \in$ Im B and $\varphi_h^k \in$ Ker B_h^*.

The preceeding result shows that this (hard to realize) case has a special importance. ■

Remark 4.6: The hypotheses of Proposition 4.1 and 4.2 are readily satisfied if one has Ker $B_h^* = \text{Ker } B^*$, in particular if B and B_h are both surjective. This situation will hold in most practical cases, so that we have reduced the verification of UCLP to building an uniformly continuous operator Π_h. We shall give examples in the next section showing how our results may be applied. ■

5. SOME EXAMPLES OF APPLICATIONS

The main result of this paper is that the abstract convergence condition of Brezzi [1], may be checked through the construction of an operator Π_h satisfying (4.15). We want to give rapidly here two examples where this operator may be explicitly built. Let us also refer to Brezzi-Raviart [2], where to our suggestion, Proposition 4.1 has been used to prove the convergence of the Hermann-Johnson's scheme for the biharmonic problem. Our first example treats of the approximation of Stokes' creeping flow problem in fluid mechanics and the second one to the approximation of Dirichlet's problem by mixed finite elements. Finally we give an example of a case where Ker $B_h^* \not\subset \text{Ker } B^*$.

Example 5.1: We consider in a domain $\Omega \subset \mathbf{R}^2$, with polygonal boundary, the Stokes problem. Let $\vec{u} = (u_1, u_2)$ the velocity of the fluid, p the pressure, we have to solve:

$$-\Delta \vec{u} + \text{grad}\, p = \vec{f}, \tag{5.1}$$

$$\text{div}\, \vec{u} = 0, \tag{5.2}$$

$$\vec{u} \in (H_0^1(\Omega))^2,\ p \in L^2(\Omega). \tag{5.3}$$

We so have,

$$V = (H_0^1(\Omega))^2, \qquad W = L^2(\Omega),\ B = \text{div}.$$

Let us note that B is not surjective and that Ker B^* is formed by constants. Let us consider an approximation of $H_0^1(\Omega)$ by quadratic conforming finite elements. The domain is triangulated and on each triangle, a function of V_h is defined by twelve degrees of freedom, which are the values of u_1 and u_2 at the vertices and at the midpoint of the sides. (*Fig.* 5.1).

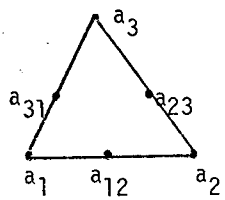

Figure 5.1.

These nodes are numbered as on the Figure.

We norm consider, as in Fortin [4], and Crouzeix-Raviart [3], an approximation W_h of $L^2(\Omega)$ by functions which are piecewise constants on the triangles. The operator B_h then associates to $\vec{v}_h \in V_h$ its average divergence on each triangle.

Given $\vec{u} \in V$, it would be natural to define $\hat{u}_h = \Pi_h u$, taking,

$$\hat{u}_{kh}(a_i) = u_k(a_i), \qquad i = 1, 2, 3; \qquad k = 1, 2, \tag{5.4}$$

$$u_{kh}(a_{ij}) = \frac{1}{|a_i a_j|} \int_{a_i}^{a_j} u_k \, d\sigma, \qquad i, j = 1, 2, 3, \qquad k = 1, 2. \tag{5.5}$$

It is then easy to check that

$$b(u - u_h, \varphi_h) = \int_\Omega \operatorname{div}(u - u_h) \varphi_h \, dx = 0, \qquad \varphi_h \in W_h. \tag{5.6}$$

However this definition is not possible as the functions of V are not smooth enough to define a point value $u(a_i)$.

Crouzeix and Raviart have shown that it is possible to build Π_h in an indirect way, first taking,

$$\tilde{u}_h = P_{V_h}(\vec{u}), \tag{5.7}$$

and then

$$\hat{u}_{kh}(a_i) = \tilde{u}_{kh}(a_i), \qquad i = 1, 2, 3; \qquad k = 1, 2. \tag{5.8}$$

$$\hat{u}_{kh}(a_{ij}) = \frac{1}{|a_i - a_j|} \int_{a_i}^{a_j} \tilde{u}_{kh} \, d\sigma, \qquad i, j = 1, 2, 3; \qquad k = 1, 2. \tag{5.9}$$

UCLP condition is then proved in [3], for a slightly more general case. By Lemma 4.1, this proves that Ker B_h^* = Ker B^* and is therefore formed of constant functions. ■

Example 5.2: Raviart and Thomas [8], introduce a mixed approximation for Dirichlet's problem in $\mathbf{R}^2$, using the following functions spaces

$$V = H(\operatorname{div}; \Omega) = \{ p \mid p = (p_1, p_2) \in (L^2(\Omega))^2, \operatorname{div} p \in L^2(\Omega) \}. \tag{5.10}$$

$$W = \{ v \mid v \in L^2(\Omega) \}. \tag{5.11}$$

They then solve for $f \in L^2(\Omega)$,

$$(p, q)_{(L^2(\Omega))^2} + (\operatorname{div} q, u) = 0, \qquad \forall q \in V, \tag{5.12}$$

$$(\operatorname{div} p, v) = (f, v), \qquad \forall v \in W. \tag{5.13}$$

We thus have for $q \in V$, $v \in W$:

$$b(q, v) = (\operatorname{div} q, v), \tag{5.14}$$

that is

$$Bq = \operatorname{div} q. \tag{5.15}$$

Let u be the unique solution of

$$\left.\begin{aligned} -\Delta u &= f, \\ u|_\Gamma &= 0. \end{aligned}\right\} \tag{5.16}$$

Then (-grad u, u) is the unique solution to (5.12)-(5.13).

Following [8], we define $V_h \subset V$, using piecewise-polynomials of degree $k+1$ on a triangulation $\mathcal{T}_h$ of Ω. It is required that on any triangle boundary, the normal trace $q_h . \nu$ of $q_h \in V$, be a polynomial of degree k and that this normal trace be continuous from one triangle to another. We define W_h using piecewise polynomials of degree $\leqq k$ on each triangle, without any continuity condition. With respect to V_h, Raviart and Thomas show that such a space can be built and that the degrees of freedom on each triangle K can be chosen as

$$\text{the moments of order} \leqq k \text{ of } q_h . \nu \text{ on } \partial K, \tag{5.17}$$

$$\text{the moments of order} \leqq k-1 \text{ of } q_h \text{ on } K. \tag{5.18}$$

The degrees of freedom (5.17) indeed insure the continuity of $q_h . \nu$ on interfaces. We now show that we can use the results of section 4 to prove the convergence of this approximation. In order to do so, we have to build an build an uniformly continuous linear operator Π_h from V into V_h such that $b(q - \Pi_h q, v_h) = 0$, $\forall v_h \in W$, or more precisely

$$\int_\Omega (\operatorname{div} q - \operatorname{div} \Pi_h q) v_h \, dx = 0, \qquad \forall v_h \in W_h . \tag{5.19}$$

Integrating by parts on each triangle K, this becomes,

$$-\sum_K \int_K \operatorname{grad} v_h . (q - \Pi_h q) \, dx + \int_{\partial K} v_h (q - \Pi_h q) . \nu \, d\sigma = 0. \tag{5.20}$$

Let us define tentatively Π_h as the interpolation operator on the degrees of freedom of V_h, that is on each triangle K, and for any side K' of K.

$$\int_{K'} (q - \Pi_h q) . \nu \varphi \, d\sigma = 0, \qquad \forall \varphi \in P_k(K'), \tag{5.21}$$

$$\int_K (q - \Pi_h q) \varphi \, dx = 0, \qquad \forall \varphi \in P_{k-1}(K). \tag{5.22}$$

Then, as $\operatorname{grad} v_h |_K \in P_{k-1}(K)$ and $v_h |_{K'} \in P_k(K')$, for any $v_h \in W_h$, condition (5.19) is evidently satisfied.

A problem however arises, as for $q \in V$, the moments on the sides may not be defined due to a lack of regularity. If however we can take $q \in (\mathrm{H}^1(\Omega))^2$, we can use (5.21) and (5.22) and moreover we have.

$$| \tilde{q} - \Pi_h \tilde{q} |_{H(\operatorname{div}, \Omega)} \leqq Ch | \tilde{q} |_{1, \Omega}, \tag{5.23}$$

which is indeed stronger than the uniform continuity requirement.

We now show that we can deduce the result for $q \in H(\operatorname{div}; \Omega)$ from the result for $q \in (H^1(\Omega))^2$. To do so, we build for any $q \in H(\operatorname{div}, \Omega)$, a $\tilde{q} \in (H^1(\Omega))^2$

such that

$$\left.\begin{aligned} \operatorname{div} q &= \operatorname{div} \tilde{q}, \\ |\tilde{q}|_{(H^1(\Omega))^2} &\leqq C |q|_{H(\operatorname{div}, \Omega)}. \end{aligned}\right\} \tag{5.24}$$

This is easily done, if the boundary Γ of Ω is smooth enough, by solving

$$\left.\begin{aligned} -\Delta\varphi &= \operatorname{div} q \\ \varphi|_\Gamma &= 0. \end{aligned}\right\} \tag{5.25}$$

Then $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ and setting

$$\tilde{q} = \operatorname{grad} \varphi, \tag{5.26}$$

solves (5.24).

We now define:

$$\Pi_h q = \Pi_h \tilde{q}. \tag{5.27}$$

We then have

$$\int_\Omega (\operatorname{div} q - \operatorname{div} \Pi_h q) v_h \, dx = \int_\Omega (\operatorname{div} \tilde{q} - \operatorname{div} \Pi_h \tilde{q}) v_h \, dx = 0 \tag{5.28}$$

and the uniform continuity of Π_h follows from (5.24) and (5.23).

This proves from Lemma 4.1 that Ker $B_h^* \subset$ Ker B^* and therefore that B_h^* is surjective. Proposition 4.2 then implies UCLP condition.

It is also a trivial task to prove Ker $B_h \subset$ Ker B.

Indeed for $q_h \in V_h$, div $q_h|_K \in P_k(K)$. Let then v_h be the $L^2(\Omega)$ projection of $v \in W$ on W_h. Then:

$$\int_\Omega \operatorname{div} q_h (v - v_h) \, dx = 0, \qquad q_h \in V_h, \tag{5.29}$$

hence the result by Lemma 4.2. ■

Example 5.3: We go back to the problem of Example 5.1 but we now consider bilinear finite elements on a rectangular mesh, the degrees of freedom are the values of u_1 and u_2 at the vertices.

It is then classical to get in this way an approximation of $(H_0^1(\Omega))^2$ (*cf.* [4]).

We then consider W_h formed by piecewise constants on the rectangle. The operator B_h still associates to u_h its average divergence on each rectangle. It is an easy task to verify that in this case, the Kernel of the discrete gradient is generated by two piecewise constant functions,

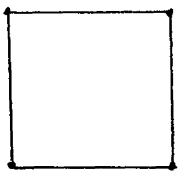

Figure 5.2.