RAIRO. ANALYSE NUMÉRIQUE

F. BREZZI

Finite element approximations of the von Kármán equations

RAIRO. Analyse numérique, tome 12, nº 4 (1978), p. 303-312

<http://www.numdam.org/item?id=M2AN_1978__12_4_303_0>

R.A.I.R.O. Analyse numérique/Numerical Analysis
(vol. 12, n° 4, 1978, p. 303 à 312)

FINITE ELEMENT APPROXIMATIONS OF THE VON KÁRMÁN EQUATIONS (*)

by F. BREZZI ([1])

Communiqué par P. G. CIARLET

Abstract. — *We analyse a general technique in order to prove the convergence and optimal error bounds for suitable finite element approximations of the von Kármán plate bending equations.*

0. INTRODUCTION

Consider a thin isotropic elastic plate which occupies a given region Ω in the x, y plane. Suppose that the plate is clamped along the entire boundary $\partial\Omega$ and that a uniformly distributed load $p(x, y)$ is acting on the plate. Under suitable assumptions one can prove that the transverse displacement $w_2(x, y)$ in the z-axis direction is a solution of the von Kármán equations [1] :

$$\left.\begin{aligned} \Delta^2 w_1(x, y) &= -1/2\,[w_2, w_2] \quad \text{in } \Omega,\\ \Delta^2 w_2(x, y) &= [w_1, w_2]+p \quad \text{in } \Omega,\end{aligned}\right\} \tag{0.0}$$

where: $w_1(x, y)$ is another unknown function depending on the "in-plane" displacements of the plate, the expression $[u, v]$ is defined by

$$[u, v]=u_{xx}v_{yy}+u_{yy}v_{xx}-2\,u_{xy}v_{xy}, \tag{0.1}$$

and a certain number of physical constants have been put equal to 1 for simplicity. Since the plate is clamped on $\partial\Omega$, w_2 will also satisfy the boundary conditions $w_2=\partial w_2/\partial n=0$ on $\partial\Omega$; $w_1(x, y)$ will also satisfy some suitable boundary conditions which for simplicity we assume to be $w_1=\partial w_1/\partial n=0$.

(*) Reçu octobre 1977.
([1]) Università di Pavia and Laboratorio di Analisi Numerica del C.N.R., 27100 Pavia (Italie).

Problem (0, 0) can therefore be written, in a more precise way, as follows:

$$\left.\begin{array}{ll} \textit{find } w=(w_1, w_2)\in(H_0^2(\Omega))^2 \textit{ such that:} & \\ \Delta^2 w_1 = -1/2\,[w_2, w_2] & \text{in } \Omega, \\ \Delta^2 w_2 = [w_1, w_2]+p & \text{in } \Omega. \end{array}\right\} \qquad (0.2)$$

Equations (0 . 2) have long since been carefully analysed, from the theoretical point of view (*see* e. g. [2] and the references contained therein). It can be shown that for each given $p\in H^{-2}(\Omega)$ there exists at least one solution (in general not unique) of problem (0 . 2).

In the present paper we analyse a general technique in order to prove the convergence and the error bounds of suitable finite element approximations of equations (0 . 2). In order to simplify the exposition only the so called "displacement" f. e. approximations will be presented in detail. However the same technique can be applied to prove error bounds for different and, perhaps, more interesting approaches, such as mixed, hybrid, etc. The convergence of a particular type of mixed approximations has been proved in [3], while a hybrid approach has been used in [4].

The tools that will be used are of the classical type: we refer for instance to [5], and to the references included therein, for similar approaches, though in different contexts.

The main results, for simplicity, are summarised in theorem 3, at the end of the paper.

1. SOME FURTHER REMARKS ON THE CONTINUOUS PROBLEM

We introduce the following notations:

$$V=(H_0^2(\Omega))^2, \qquad H=(L^\infty(\Omega))^2 \quad (^2), \qquad (1.0)$$

$$a(u, v)=\int_\Omega (\Delta u_1\, \Delta v_1+\Delta u_2\, \Delta v_2)\, dx, \qquad (1.1)$$

$$b(w, u, v)=1/2\int_\Omega \{-v_2([w_1, u_2]+[w_2, u_1])+v_1[w_2, u_2]\}\, dx, \qquad (1.2)$$

$$(p, v)=\int_\Omega p v_2\, dx, \qquad (1.3)$$

and we observe that problem (0 . 2) becomes:

$$\left.\begin{array}{l} \textit{find } w\in V \textit{ such that:} \\ a(w, v)+b(w, w, v)=(p, v), \qquad \forall v\in V. \end{array}\right\} \qquad (1.4)$$

(^{2}) The norms on V and H will be denoted by $\|.\|$ and $|.|$ respectively.

We also note the following properties that will be used later on:

$$\forall u, v \in V, \quad a(u, v) \leqq A \|u\| . \|v\|, \tag{1.5}$$

$$\forall v \in V, \quad a(v, v) \geqq \alpha \|v\|^2, \tag{1.6}$$

$$\forall w, u, v \in V, \quad b(w, u, v) \leqq B \|w\| . \|u\| . |v| \leqq \bar{B} \|w\| . \|u\| . \|v\|, \tag{1.7}$$

$$\forall w, u, v \in V, \quad b(w, u, v) = b(u, w, v). \tag{1.8}$$

Suppose now that w is an *isolated solution* of (1.4); by definition this will imply that the bilinear form

$$(u, v) \to a(u, v) + 2\, b(w, u, v) \tag{1.9}$$

is non singular on $V \times V$. More precisely: *for any given $f \in V'$, there exists a unique $u \in V$ which satisfies*

$$a(u, v) + 2\, b(w, u, v) = (f, v), \qquad \forall v \in V. \tag{1.10}$$

Moreover one has

$$\|u\| \leqq L \|f\|_{V'} \tag{1.11}$$

with L independent of f.

We will need the following lemma:

LEMMA 1: *Suppose that (1.9) is non singular. Then there exists a positive constant δ such that for any given $\tilde{w}$ with*

$$\|\tilde{w} - w\| \leqq \delta \tag{1.12}$$

the bilinear form

$$(u, v) \to a(u, v) + 2\, b(\tilde{w}, u, v) \tag{1.13}$$

is non singular.

Proof: The lemma is almost obvious; nevertheless we shall state the proof for sake of completeness. It is known (*cf.* e. g. [6, 7]) that a bilinear form

$$(u, v) \to l(u, v) \tag{1.14}$$

is non singular iff there exist two positive constants c_1 and c_2 such that

$$\operatorname*{Sup}_{\|v\|=1} l(u, v) \geqq c_1 \|u\|; \qquad \sup_{\|u\|=1} l(u, v) \geqq c_2 \|v\|. \tag{1.15}$$

Let now (c_1, c_2) be the two constants associated with the bilinear form (1.9), and let z be an element of V. We have

$$\begin{aligned}
&\sup_{\|v\|=1} a(u, v)+2b(w+z, u, v)\\
&\quad \geqq \sup_{\|v\|=1} a(u, v)+2b(w, u, v)-2\operatorname*{Sup}_{\|v\|=1} b(z, u, v)\\
&\quad \geqq c_1\|u\| -2\bar{B}\|z\|.\|u\| \geqq (c_1/2)\|u\| \qquad (1.16)
\end{aligned}$$

provided that $2\bar{B}\|z\| \leqq c_1/2$.

In a similar way we also get

$$\sup_{\|u\|=1} a(u, v)+2b(w+z, u, v) \geqq (c_2/2)\|v\| \qquad (1.17)$$

if $2\bar{B}\|z\| \leqq c_2/2$. Therefore the lemma holds with

$$\delta=(4\bar{B})^{-1}\min(c_1, c_2). \qquad (1.18)$$

2. THE DISCRETISED PROBLEM: EXISTENCE OF THE SOLUTION AND ERROR BOUNDS

Suppose now that we are given, for any $h\in]0, h_0]$, a finite dimensional subspace V_h of $V=(H_0^2(\Omega))^2$ with the following properties:

$$\left\{\begin{array}{ll} \displaystyle\operatorname*{Inf}_{v_h\in V_h}\|u-v_h\|_{(H^s(\Omega))} \leqq ch^{r-s}\|u\|_{(H^r(\Omega))^2}, & s=1, 2, \\ \textit{for all } u\in(H^r(\Omega))^2\cap(H_0^2(\Omega))^2, & s\leqq r\leqq k+1, \end{array}\right\} \qquad (2.0)$$

where the constant c is independent of u and h, and k is a given parameter associated with the family $\{V_h\}$ (in the applications, k=degree of the piecewise polynomial approximation). Families of spaces V_h satisfying (2.0) are well known; *cf.* e. g. [8, 9, 10].

We consider the following approximate problem:

$$\left.\begin{array}{l} \textit{Find } w_h\in V_h \textit{ such that:}\\ a(w_h, v_h)+b(w_h, w_h, v_h)=(p, v_h), \qquad \forall v_h\in V_h \end{array}\right\} \qquad (2.1)$$

The existence of a solution of (2.1) and the corresponding error bound will be obtained by means of a "modified Newton" method. We need, therefore, some more information on the behaviour of the linearized operator. For that purpose suppose first that $\tilde{w}_h\in V_h$ is defined by

$$\|w-\tilde{w}_h\| \leqq \|w-v_h\|, \qquad \forall v_h\in V_h. \qquad (2.2)$$

Since w is isolated, we have from (2.0), (2.2) and lemma 1 that, *for h sufficiently small the bilinear form*

$$(u, v) \to a(u, v) + 2b(\tilde{w}_h, u, v) \tag{2.3}$$

is non singular on $V \times V$. This will not imply, *a priori*, that (2.3) is non singular on $V_h \times V_h$; nevertheless this will be true in our case, at least for h sufficiently small, as shown in the next lemma.

LEMMA 2: *If the bilinear form* (2.3) *is non singular on $V \times V$ then, for h small enough, it is non singular on $V_h \times V_h$.*

Proof; Since V_h is finite dimensional, we need only to prove that

$$\sup_{\substack{\|v_h\|=1 \\ v_h \in V_h}} a(u_h, v_h) + 2b(\tilde{w}_h, u_h, v_h) \geqq K\|u_h\|, \qquad \forall u_h \in V_h. \tag{2.4}$$

For any given $u_h \in V_h \subseteq V$, we have, from the non singularity of (2.3) on $V \times V$, that

$$\sup_{\|v\|=1} a(u_h, v) + 2b(\tilde{w}_h, u_h, v) \geqq (c_1/2)\|u_h\|. \tag{2.5}$$

Therefore there exists a $\bar{v} \in V$ such that $\|\bar{v}\| = 1$ and

$$a(u_h, \bar{v}) + 2b(\tilde{w}_h, u_h, \bar{v}) \geqq (c_1/4)\|u_h\|. \tag{2.6}$$

Let now $\bar{v}_h \in V_h$ be the solution of the problem

$$a(z_h, \bar{v}_h) = a(z_h, \bar{v}), \qquad \forall z_h \in V_h. \tag{2.7}$$

It is easy to see that $\bar{v}_h$ exists, is unique, and satisfies

$$\|\bar{v}_h\| \leqq (A/\alpha)\|v\| = A/\alpha. \tag{2.8}$$

Suppose now that the domain Ω has the following property: for any given $f \in L^2(\Omega)$ there exists a $\varphi \in H^{2+\varepsilon}(\Omega) \cap H_0^2(\Omega)$ $(0 < \varepsilon < \varepsilon_0(\Omega))$ with $\Delta^2 \varphi = f$ and

$$\|\varphi\|_{H^{2+\varepsilon}(\Omega)} \leqq \bar{c}\|\Delta^2 \varphi\|_{L^2(\Omega)} \tag{2.9}$$

(This is a very weak regularity assumption on Ω; for instance any polygon will satisfy it). If (2.9) is satisfied one can apply the duality technique of Aubin-Nitsche (*see* e. g. [9, 10]) in order to get an error bound for $\bar{v} - \bar{v}_h$ in $L^2(\Omega)$. Then, by means of the Sobolev injection of $H^{1+\sigma}(\sigma > 0)$ in L^∞ and interpolating between L^2 and H^2 (*cf.* [14, chap. 1]), one gets

$$|\bar{v}_h-\bar{v}|\leqq\bar{\bar{c}}h^{\eta}\|\bar{v}\|=\bar{\bar{c}}h^{\eta}$$

for some positive η (for instance $\eta=\varepsilon_0/4$).

Therefore we have

$$\begin{aligned}a(u_h,\bar{v}_h)+2b(\tilde{w}_h,u_h,\bar{v}_h)&=a(u_h,\bar{v})+2b(\tilde{w}_h,u_h,\bar{v})+2b(\tilde{w}_h,u_h,\bar{v}_h-\bar{v})\\&\geqq(c_1/4)\|u_h\|-2B\|\tilde{w}_h\|.\|u_h\|\bar{\bar{c}}h^{\eta}\end{aligned}\tag{2.10}$$

and for h sufficiently small we obtain (2.4) with $K=\alpha c_1/8A$.

REMARK: Results of this kind are quite classical. See for instance [11 or 12] for similar cases.

Using the statement of lemma 2 we are now able to define the following map φ: $V_h\to V_h$; for any given $u_h\in V_h$, $\varphi_h=\varphi(u_h)$ is defined as the unique solution of

$$\left.\begin{aligned}&a(\varphi_h,v_h)+2b(\tilde{w}_h,\varphi_h,v_h)=2b(\tilde{w}_h,u_h,v_h)-b(u_h,u_h,v_h)+(p,v_h),\\&\forall v_h\in V_h.\end{aligned}\right\}\tag{2.11}$$

It is easy to see that *any solution of* (2.1) *is a fixed point of* φ *and vice-versa.* In order to prove the existence of a solution of (2.1) we can therefore show that φ has a fixed point.

THEOREM 1: *There exist two positive constants* $R_1(h)$ *and* R_2 *such that, for any* $u_h\in V_h$,

$$R_1(h)\leqq\|u_h-\tilde{w}_h\|\leqq R_2\quad\Rightarrow\quad\|\varphi(u_h)-\tilde{w}_h\|\leqq\|u_h-\tilde{w}_h\|.\tag{2.12}$$

Proof: We have from (2.4) that there exists a $\tilde{v}_h\in V_h$ such that $\|\tilde{v}_h\|=1$ and

$$\|\varphi(u_h)-\tilde{w}_h\|\leqq(2/K)\,(a(\varphi(u_h)-\tilde{w}_h,\tilde{v}_h))+2b(\tilde{w}_h,\varphi(u_h)-\tilde{w}_h,\tilde{v}_h).\tag{2.13}$$

On the other hand, using (2.11) we have :

$$\begin{aligned}&a(\varphi(u_h)-\tilde{w}_h,\tilde{v}_h)+2b(\tilde{w}_h,\varphi(u_h)-\tilde{w}_h,\tilde{v}_h)\\&\quad=2b(\tilde{w}_h,u_h,\tilde{v}_h)-b(u_h,u_h,\tilde{v}_h)+(p,\tilde{v}_h)\\&\qquad-a(\tilde{w}_h,\tilde{v}_h)-2b(\tilde{w}_h,\tilde{w}_h,\tilde{v}_h)=[-a(\tilde{w}_h,\tilde{v}_h)-b(\tilde{w}_h,\tilde{w}_h,\tilde{v}_h)+(p,\tilde{v}_h)]\\&\qquad-[b(u_h,u_h,\tilde{v}_h)-2b(\tilde{w}_h,u_h,\tilde{v}_h)+b(\tilde{w}_h,\tilde{w}_h,v_h)]\\&\quad=[a(w,\tilde{v}_h)-a(\tilde{w}_h,\tilde{v}_h)+b(w,w,\tilde{v}_h)-b(\tilde{w}_h,\tilde{w}_h,\tilde{v}_h)]-b(u_h-\tilde{w}_h,u_h-\tilde{w}_h,\tilde{v}_h)\\&\quad\leqq\|v_h\|\{(A+2\bar{B}\|w\|)\|w-\tilde{w}_h\|+\bar{B}\|w-\tilde{w}_h\|^2+\bar{B}\|u_h-\tilde{w}_h\|^2\}\end{aligned}\tag{2.14}$$

where, in the last inequality the formula

$$x^2-y^2=(x-y)^2-2y(x-y)$$

has been used. We set now

$$\mu(h)=(2/K)\{(A+2\bar{B}\|w\|)\|w-\tilde{w}_h\|+B\|w-\tilde{w}_h\|^2\}, \tag{2.15}$$

$$\lambda=2\bar{B}/K \tag{2.16}$$

and from (2.13), (2.14) we get

$$\|\varphi(u_h)-\tilde{w}_h\|\leqq\mu(h)+\lambda\|u_h-\tilde{w}_h\|^2$$

which implies (2.12) with

$$R_1(h)=(2\lambda)^{-1}(1-\sqrt{1-4\lambda\mu}), \tag{2.17}$$

$$R_2=(2\lambda)^{-1}\leqq(2\lambda)^{-1}(1+\sqrt{1-4\lambda\mu}). \tag{2.18}$$

COROLLARY 1: *For any given R with*

$$R_1(h)\leqq R\leqq R_2, \tag{2.19}$$

there exists a solution w_h of (2.1) *such that:*

$$\|w_h-\tilde{w}_h\|\leqq R. \tag{2.20}$$

Proof: We have from theorem 1 that for any R satisfying (2.19), the continuous mapping φ maps the closed sphere with center $\tilde{w}_h$ and radius R into itself.

Therefore φ has at least a fixed point, w_h, in the closed sphere, which is a solution of (2.1) and verifies (2.20).

COROLLARY 2: *There exists at least a solution w_h of* (2.1) *which satisfies*

$$\|w-w_h\|\leqq ch^{r-2}\|w\|_{(H^r(\Omega))^2}, \qquad 2\leqq r\leqq k+1. \tag{2.21}$$

Proof: Putting $R=R_1(h)$ in the statement of corollary 1 we have

$$\|w-w_h\|\leqq\|w-\tilde{w}_h\|+R_1(h). \tag{2.22}$$

On the other hand, from (2.15) we have

$$\mu(h)\leqq\nu_1\|w-\tilde{w}_h\| \tag{2.23}$$

with ν_1 constant independent of h; therefore from (2.17) we have, always for h small enough,

$$R_1(h)\leqq\nu_2\|w-\tilde{w}_h\| \tag{2.24}$$

with ν_2 constant independent of h. And finally (2.22), (2.24) and (2.0) give us (2.21).

3. A CONSTRUCTIVE PROCEDURE FOR SOLVING THE DISCRETISED PROBLEM

The results of the previous paragraph, although optimal from the theoretical point of view, are of little practical interest. In fact the solution w_h of the discretised problem (2.1) has been characterised as a fixed point of a map, φ, which in order to be computed requires the explicit knowledge of $\tilde{w}_h$, that is, in some sense, of the solution itself. It is therefore evident that, in a practical case, a different procedure for the computation of w_h has to be sought.

We will show in the following that, if the initial guess w_h^0 is sufficiently close to w_h (and so to w), the *Newton iterates*, defined by

$$a(w_h^{n+1}, v_h)+2b(w_h^n, w_h^{n+1}, v_h)=b(w_h^n, w_h^n, v_h)+(p, v_h), \qquad \forall v_h \in V_h \tag{3.0}$$

converge quadratically to w_h.

From lemma 1 and 2 we have that *there exist two positive constants* $\overline{\delta}$, $\overline{K}$ *independent of* h, *such that for each* z_h *such that* $\| z_h-\tilde{w}_h \| \leqq \overline{\delta}$ *and for each* $u_h \in V_h$ *we have:*

$$\sup_{\substack{\| v_h \| =1 \\ v_h \in V_h}} a(u_h, v_h)+2b(z_h, u_h, v_h) \geqq \overline{K} \| u_h \|. \tag{3.1}$$

We shall also suppose from now on that

$$R_1(h) \leqq \overline{\delta}/2, \tag{3.2}$$

which will obviously hold for h small enough.

THEOREM 2: *Let* ρ *be defined as*

$$\rho=\min(\overline{K}/\overline{B}, \overline{\delta}/2) \tag{3.3}$$

and let w_h *be a solution of* (2.1) *which satisfies:*

$$\| w_h-\tilde{w}_h \| \leqq \overline{\delta}/2; \tag{3.4}$$

then if the initial guess w_h^0 *verifies*

$$\| w_h-w_h^0 \| \leqq \rho, \tag{3.5}$$

the Newton iterates (3.0) *are well defined and converge quadratically to* w_h.

Proof: We note first that (3.3)...(3.5) imply

$$\| \tilde{w}_h-w_h^0 \| \leqq \overline{\delta}, \tag{3.6}$$

and therefore (3.0) makes sense for $n=0$. We shall prove that if w_h^n satisfies

$$\| w_h - w_h^n \| \leqq \rho, \tag{3.7}$$

then w_h^{n+1} also satisfies

$$\| w_h - w_h^{n+1} \| \leqq \rho, \tag{3.8}$$

and moreover

$$\| w_h - w_h^{n+1} \| \leqq (\overline{B}/\overline{K}) \| w_h - w_h^n \|^2. \tag{3.9}$$

Therefore, always together with (3.3) and (3.5), we will have by induction that, for each n,

$$\| w_h - w_h^n \| \leqq \delta, \tag{3.10}$$

and so the iterates (3.0) are well defined. Let us now prove that (3.7) implies (3.8) and (3.9). From (3.1) we have first that there exists a $\bar{v}_h \in V_h$ such that $\| \bar{v}_h \| = 1$ and

$$a(w_h^{n+1} - w_h, \bar{v}_h) + 2\, b(w_h^n, w_h^{n+1} - w_h, \bar{v}_h) \geqq \overline{K} \| w_h^{n+1} - w_h \|. \tag{3.11}$$

On the other hand from (3.0) we have

$$\begin{aligned} & a(w_h^{n+1} - w_h, \bar{v}_h) + 2\, b(w_h^n, w_h^{n+1} - w_h, \bar{v}_h) \\ & \quad = b(w_h^n, w_h^n, \bar{v}_h) + (p, \bar{v}_h) - a(w_h, \bar{v}_h) - 2\, b(w_h^n, w_h, \bar{v}_h) \\ & \quad = b(w_h^n, w_h^n, \bar{v}_h) - 2\, b(w_h^n, w_h, \bar{v}_h) + b(w_h, w_h, \bar{v}_h) \\ & \quad = b(w_h^n - w_h, w_h^n - w_h, \bar{v}_h) \leqq \overline{B} \| w_h - w_h^n \|^2, \end{aligned} \tag{3.12}$$

which together with (3.11) gives us (3.9). From (3.9) and (3.7) we obtain now

$$\| w_h - w_h^{n+1} \| < (\overline{B}/\overline{K}) \rho^2$$

which implies (3.8) since $\rho \leqq \overline{K}/\overline{B}$.

We also note that the definition of ρ in (3.3) does not depend on h or on w_h. In particular we can conclude that the solution w_h is unique in a sphere of center $\tilde{w}_h$ and radius ρ, since, of course, the choice $w_h^0 = \tilde{w}_h$ cannot give rise to a sequence which converges to two different limits at the same time.

All the previous results can be summarized in the following theorem.

THEOREM 3: *If w is an isolated solution of* (1.4) *and if* $\{ V_h \}_h$ *is a family of subspaces of V satisfying* (2.0), *then there exist an $h_1 > 0$ and an $\rho > 0$ such that for each $h \in]0, h_1]$ the problem* (2.2) *has a unique solution w_h in the sphere of centre*

$\tilde{w}_h$ *(projection of* w *on* V_h*) and radius* ρ*. Such a solution satisfies:*

$$\| w_h - w \| \leqq c h^{r-2} \| w \|_{(H^r(\Omega))^2}, \qquad 2 \leqq r \leqq k+1.$$

Moreover if the initial guess w_h^0 *satisfies*

$$\| w_h - w_h^0 \| \leqq \rho$$

then the Newton iterates (3 . 0) *are well defined and converge quadratically to* w_h.

REMARK: The method discussed above has the "advantage" of converging towards both kinds of physical solutions: stable and unstable. This is sometimes of some interest; on other occasions, a method that converges only to the stable solutions may be preferable for pratical purposes. In that case some modification of the Newton procedure, as in [13], is suggested.

REFERENCES

1. T. VON KÁRMÁN, *Festigkeitsprobleme in Maschinbau,* Encycl. der Math. Wissenschaften, vol. 4, 1910, pp. 348-352.
2. M. S. BERGER, *On von Kármán Equations and the Buckling of a thin Elastic Plate,* Comm. Pure Appl. Math., vol. 20, 1967, pp. 687-718.
3. T. MIYOSHI, *A Mixed Finite Element Method for the Solution of the von Kármán Equations,* Num. Math., vol. 26, 1976, pp. 255-269.
4. A. GOBETTI and L. D. MARINI, to appear.
5. H. B. KELLER, *Approximation Methods for Nonlinear Problems with Applications to two-Point Boundary Value Problems,* Math. Comp., vol. 29, 1975, pp. 464-474.
6. J. NEČAS, *Les Méthodes directes en théorie des équations elliptiques,* Masson, Paris, 1967.
7. I. BABUŠKA, *Error Bound for the Finite Element Method,* Num. Math., vol. 16, 1971, pp. 322-333.
8. P. G. CIARLET and P. A. RAVIART, *General Lagrange and Hermite Interpolation in* $\mathbf{R}^n$ *with Applications to the Finite Element Methods,* Arch. Rat. Mech. Anal., vol. 46, 1972, pp. 177-199.
9. G. STRANG and G. FIX, *An Analysis of the Finite Element Method,* Prentice Hall, Englewood Cliffs, 1973.
10. P. G. CIARLET, *The Finite Element Method for Elliptic Problems,* North Holland, Amsterdam, 1977.
11. S. G. MIKHLIN, *The Numerical Performance of Variational Methods,* Wolters Noordholf, Groningen, 1971.
12. A. L. SCHATZ, *An Observation Concerning Ritz-Galerkin Methods with Indefinite Bilinear Forms,* Math. of Comp., vol. 28, 1974, pp. 959-962.
13. D. J. ALLMAN, *Some Fundamental Aspects of the Finite Element Analysis of Nonlinear Elastic Plate Bending,* Finite Elements in Nonlinear Solid and Structural Mechanics, Geilo (Norway), August 29-Sept 1, 1977, pp. C. 06.
14. J. L. LIONS and E. MAGENES, *Problèmes aux limites non homogènes et applications,* vol. 1, Dunod, Paris, 1968.