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RAIRO. Analyse numérique, tome 13, n° 1 (1979), p. 31-54

<http://www.numdam.org/item?id=M2AN_1979__13_1_31_0>
**L_\infty-CONVERGENCE OF FINITE ELEMENT GALERKIN APPROXIMATIONS FOR PARABOLIC PROBLEMS** (*

by Joachim A. Nitsche (*)

Abstract. — Using weighted norms L_\infty-error estimates of the Galerkin method for second order parabolic initial-boundary value problems are derived.

0. INTRODUCTION

Let the model problem

\begin{align*}
\dot{u} - \Delta u &= f \quad \text{in } \Omega \times (0, T], \\
u &= 0 \quad \text{on } \partial \Omega \times (0, T], \\
u_t = 0 &= u_0 \quad \text{in } \Omega
\end{align*}

be given. With \( S_h \subseteq \bar{H}_1 \) being a finite dimensional space—we will consider only finite elements—the standard Galerkin approximation \( u_h = u_h(t) \in S_h \) is defined by

\begin{equation}
(\dot{u}_h, \chi) + D(u_h, \chi) = (f, \chi) \quad \text{for } \chi \in S_h \quad \text{and} \quad t \in (0, T]
\end{equation}

with

\begin{equation}
u_h(0) = Q_h u_0.
\end{equation}

Here \((., .)\) is the \( L_2(\Omega)\)-scalar-product and \(D(., .)\) the Dirichlet integral. \(Q_h\) may be any computable projection onto \(S_h\). Substitution of \(f\) by \(\dot{u} - \Delta u\) gives for the error \(e = u - u_h\) the defining relation

\begin{equation}
(\dot{e}, \chi) + D(e, \chi) = 0 \quad \text{for } \chi \in S_h.
\end{equation}

(*) Manuscrit reçu le 6 avril 1978.

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R.A.I.R.O. Analyse numérique/Numerical Analysis, 0399-0516/1979/31/$ 1.00
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Because of the Hilbert-space setting error estimates in Sobolev-norms are available primarily. This part of the convergence analysis is solved now in a satisfactory way. In part (b) of the bibliography a number of papers dealing with this question is listed.

With the help of special techniques in one space dimension there are also results in the maximum-norm. We refer to Archer [1], Cavendish-Hall [4], Douglas-Dupont [6], Douglas-Dupont-Wheeler [7], Thomee [15], Wahlbin [16], and Wheeler [17]. Seemingly $L_\infty$-estimates for general space dimensions are only treated by Bramble-Schatz-Thomee-Wahlbin [3]. The idea is to write (3) in the form

$$e + \tau h \dot{e} = (I - R_h)u.$$  

Here to any $f$ the element $U_h = R_h \Delta^{-1} f = T_h f \in S_h$ is the Ritz approximation on $-\Delta^{-1} f$ defined by

$$D(U_h, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h.$$  

In this way $L_\infty$-estimates for the elliptic problem give rise to corresponding estimates for the parabolic problem. Using Sobolev-type embedding theorems Bramble et al. derive $L_\infty$-estimates in terms of $L_2$-estimates of time derivatives of sufficiently high order depending on the dimension of $\Omega$.

The aim of this paper is to give estimates the type

$$\| e \|_{L_\infty(L_2)} \leq c h^m \left\{ \| u \|_{L_\infty(W^2_2)} + \| \dot{u} \|_{L_\infty(W^2_2)} + \| \ddot{u} \|_{L_\infty(W^2_2)} \right\}. \quad (6)$$  

Here we consider only the case $u_h(0) = R_h u_0$. More general initial conditions and also the discretisation in time will be discussed in a forthcoming paper.

Similar to the elliptic case extensively we use weighted norms, see Natterer [8] and Nitsche [10] and [11]. The corresponding approximation properties of finite elements are derived in sections 2,3. A needed generalization of the boundedness of the $L_2$-projection is given in section 4 and the main error analysis in 5-7.

1. NOTATIONS, FINITE ELEMENTS

In the following $\Omega \subseteq \mathbb{R}^N$ denotes a bounded domain with boundary $\partial \Omega$ sufficiently smooth. For any $\Omega' \subseteq \mathbb{R}^N$ let $W^k_p(\Omega')$ be the Sobolev space of functions having $L_p$-integrable generalized derivatives up to order $k$. In case $p = 2$ we also adopt $H_k(\Omega') = W^k_2(\Omega')$. The norms are indicated by the corresponding subscripts. $H_1(\Omega')$ is the closure in $H_1(\Omega')$ of the functions with compact support.

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For $T > 0$ fixed the spaces $L_p(W^k_0(t)) = L_p(0, T, W^k_0(t))$ consist of functions $u = u(t) \in W^k_0(t)$ such that $\|u(t)\|_{L^p(t)}$ is $L_p$-integrable (respective a.e. bounded for $p = \infty$) in $(0, T)$ with the norms

$$\|u\|_{L_p(t)} = \left(\int_0^T \|u(t)\|_{W^k_0(t)}^p dt\right)^{1/p}. \quad (1.1)$$

In case $\Omega' = \Omega$ we drop $\Omega$, i.e. $H_k = H_k(\Omega)$, etc. If there is no confusion we will also simply write $u$ instead of $u(t)$.

In addition we consider weighted semi-norms. Let $| \cdot |$ denotes the euclidian distance in $R^N$:

$$\mu = |x - x_0|^2 + \rho^2 \quad (1.2)$$

with $x_0 \in \Omega$ and $\rho > 0$. We define

$$\|\nabla^k v\|_{a, \Omega} = \left\{ \sum_{|\alpha| = k} \int_{\Omega'} \mu^{-\beta} |D^\beta v|^2 dx \right\}^{1/2}. \quad (1.3)$$

($\beta, \gamma$) is the corresponding bilinear form. According to above we drop $\Omega'$ in case $\Omega' = \Omega$. Furthermore, the $L_p(0, T)$-norm of $\|u\|_{a} = \|u(t)\|_{a, \Omega}$ is denoted by $\|\cdot\|_{a}$ with subscript $L_p(a)$.

By $\Gamma_h$ a subdivision of $\Omega$ into generalized simplices $\Delta_i$ is meant, i.e. $\Delta_i$ is a simplex if $\Delta_i$ intersects $\partial\Omega$ in at most a finite number of points and otherwise one of the faces may be curved. $\Gamma_h$ is called $\kappa$-regular if to any $\Delta_i \in \Gamma_h$ there are two spheres with radii $\kappa^{-1} h$ and $\kappa h$ such that $\Delta_i$ contains the one and is contained in the other.

The finite element spaces $S_h = S(\Gamma_h)$ have the following structure: Let the integer $m$ be fixed. Any $\chi \in S_h$ is in $C^0(\Omega)$, i.e. continuous in $\Omega$, and the restriction to $\Delta_i \in \Gamma_h$ is a polynomial of degree less than $m$. In the curved elements we use isoparametric modifications as discussed by Ciarlet-Raviart [5], Zlamal [18]. $S_h$ is the intersection of $S_h$ and $H^1$.

By construction we have $S_h \subseteq H^1$ but in general $S_h \subseteq H^k$ for $k \geq 2$. It is useful to introduce the spaces $H' = H'_k(\Gamma_h)$ consisting of functions in $L_2$ the restriction of which to any $\Delta_i \in \Gamma_h$ is in $H_k(\Delta_i)$. Obviously $S_h \subseteq H'_k$ for all $k$. Parallel to (1.3) we use the “broken” semi-norms

$$\|\nabla^k v\|_{a, \Delta} = \left\{ \sum_{\Delta_i \in \Gamma_h} \|\nabla^k v\|_{a, \Delta_i}^2 \right\}^{1/2}. \quad (1.4)$$

In order to avoid difficulties we will use three different letters for the “constants” in the estimates: $k, \gamma$, and $c$ with the following distinctions:

(i) $k_1, k_2, \ldots$ denote numerical constants depending only on $N$ and $m$ (the space-dimension and the degree of the finite elements used);
(ii) the parameter \( p \) in (1.2) is independent of \( x \) but will change with \( h \). Most of the lemmata and theorems are only valid if \( p \) is not too small compared with \( h \). The corresponding conditions are formulated by “for \( \gamma, h \leq p \)” respective “let \( \gamma, h = p \)”. Of course the \( \gamma \)'s depend on \( N, m, \) the domain \( \Omega \) and the regularity factor \( \kappa \) of \( \Gamma_h \);

(iii) numerical constant with the same dependence as the \( \gamma \)'s but entering directly the estimates are denoted by \( c, c_1, c_2, \ldots \) Normally just \( c \) is used, it may differ at different locations. In order not to loose control in section 5 the constants \( c \) are numbered.

The case \( m = 2 \), i.e. linear finite elements, need special treatment. Then logarithmic terms of \( h \) will appear in the error bounds, see [12] for the elliptic problem. In order not to overburden the paper we assume

\[
m \geq 3,
\]

Furthermore we consider only regular subdivisions with some fixed \( \kappa \). Finally we remark the powers of \( \mu \) for the weights \( \mu^a \) are always within the limit \( |a| \leq 2N \).

2. APPROXIMATION PROPERTIES IN WEIGHTED NORMS

Let \( \Delta \in \Gamma_h \) be any simplex as described in section 1. Then \( \mu(1.2) \) does not change too fast if \( p \) is not too small compared with \( h \):

**Lemma 1:** Let \( \gamma_1 h < p \) with \( \gamma_1 = 2 \kappa \). Then

\[
\sup_{x \in \Delta_i} \mu(x) \leq 3 \inf_{x \in \Delta_i} \mu(x).
\]  

(2.1)

**Proof:** Let \( x, \bar{x} \in \Delta_i \) be points with

\[
\mu = \mu(x) = \inf \{ \mu(x) \mid x \in \Delta_i \}, \quad \bar{\mu} = \mu(\bar{x}) = \sup \{ \mu(x) \mid x \in \Delta_i \}.
\]  

(2.2)

Since in \( \Delta_i \):

\[
|\nabla \mu| \leq 2|x - x_0| \leq 2\bar{\mu}^{1/2},
\]

(2.3)

we get

\[
\bar{\mu} = \mu(\bar{x}) = \mu(x) + (\bar{x} - x) \cdot \nabla \mu \leq \mu + |\bar{x} - x| 2 \bar{\mu}^{1/2}.
\]

(2.4)

We have \( |\bar{x} - X| \leq \kappa h \) and therefore

\[
\bar{\mu} \leq \mu + 2 \kappa h \bar{\mu}^{1/2}.
\]  

(2.5)
Schwartz's inequality in the form
\[ 2 \times h \mu^{1/2} \leq \frac{1}{2} \mu + 2 \times h^2 \]
gives
\[ \mu \leq 2 \mu + 4 \times h^2. \] (2.6)
Now — independent of \( x \):
\[ \mu \geq \rho^2 \geq \gamma^2 h^2 \] (2.7)
and therefore the lemma is shown.

The approximability property
\[ \| \nabla^k (v - \chi) \|_{L_2(\Delta)}^2 \leq ch^{2(l-k)} \| \nabla^l v \|_{L_2(\Delta)}^2 \quad (0 \leq k \leq l \leq m) \] (2.8)
with a proper interpolation resp. approximation \( \chi \in S_h \) is well known. Because of lemma 1 we get from this
\[ \| \nabla^k (v - \chi) \|_{a,\Delta}^2 \leq 3^a h^{2(l-k)} \| \nabla^l v \|_{a,\Delta}^2 \] (2.9)
and after summation over \( \Delta_i \in \Gamma_h \).

**Lemma 2:** Let \( \gamma \leq h \leq \rho \). To any \( v \in H_1 \) there is a \( \chi \in S_h \) according to
\[ \| \nabla^k (v - \chi) \|_{a} \leq ch^{l-k} \| \nabla^l v \|_{a} \quad (0 \leq k \leq l \leq m). \] (2.10)

**Remark:** Since (2.8) holds also for \( v \in \bar{H}_1 \) with a \( \chi \in S_h \) lemma 2 remains valid if \( H_1 \), \( S_h \) is replaced by \( H_1 \cap \bar{H}_1 \) and \( S_h \).

The proof of the next lemma follows the same lines and is omitted here.

**Lemma 3:** Let \( \gamma \leq h \leq \rho \). Then Bernstein-type inequalities hold: For any \( \chi \in S_h \):
\[ \| \nabla^l \chi \|_{a} \leq ch^{l-t} \| \nabla^k \chi \|_{a} \quad (0 \leq k \leq l < m). \] (2.11)

Multiplication of a function in \( S_h \) resp. \( \bar{S}_h \) gives no longer a function in these spaces. But still a certain "super-approximability" property of such functions is valid (see Nitsche-Schatz [13]):

**Lemma 4:** A function \( \mu^{-b} \phi \) with \( \phi \in S_h \) (resp. \( \bar{S}_h \)) can be approximated by a \( \chi \in S_h \) (resp. \( \bar{S}_h \)) according to
\[ \| \nabla^k (\mu^{-b} \phi - \chi) \|_{a} \leq c \{ h^{m-k} \| \phi \|_{a+2b+1} + h^{2-k} \| \nabla \phi \|_{a+2b+1} \}. \] (2.12)

Before proving the lemma let us consider e. g. the case \( a = -b \) and \( k = 0 \). Then (2.12) means
\[ \| \mu^{-b} \phi - \chi \|_{-b} \leq c \{ h^{m} \| \phi \|_{b+m} + h^{2} \| \nabla \phi \|_{b+1} \}. \] (2.13)
Now using (2.11) with $l = 1$, $k = 0$ and the obvious inequality
\[ \| \varphi \|_{b+b'} \leq \rho^{-b'} \| \varphi \|_b \] (2.14)
for $b' \geq 0$ we get
\[ \| \mu^{-b} \varphi - \chi \|_{b-b'} \leq c (h/\rho) \| \varphi \|_b. \] (2.15)

By choosing $\gamma$ in $\gamma h \leq \rho$ sufficiently large the bound on the right hand side becomes as small as wanted.

In order to prove lemma 4 we apply lemma 2 with $l = m$ and get
\[ \| \nabla^k (\mu^{-b} \varphi - \chi) \|_a \leq ch^{m-k} \| \nabla^m (\mu^{-b} \varphi) \|_a. \] (2.16)

Since $\varphi \in S_h$ is piecewise a polynomial of degree $< m$ and because of
\[ |D^k \mu^{-b}| \leq c \mu^{-b-|\xi|/2} \] (2.17)
Leibniz’s rule gives
\[ \| \nabla^m (\mu^{-b} \varphi) \|_a \leq c \sum_{n=0}^{m-1} \| \nabla^n \varphi \|_{a+2b+m-n}. \] (2.18)

The term with $n = 0$ in connection with (2.16) leads to the first term of the right hand side in (2.12). Using lemma 3 and (2.14) we get for the rest
\[ \sum_{n=1}^{m-1} \| \nabla^n \varphi \|_{a+2b+m-n} \leq c h^{1-n} \| \nabla \varphi \|_{a+2b+m-n} \]
\[ \leq ch^{2-m} \| \nabla \varphi \|_{a+2b+1} \sum_{n=1}^{m-1} (h/\rho)^{m-1-n}. \] (2.19)

The last sum is bounded because of $h \leq \rho$, thus the lemma is proved.

3. SHIFT THEOREMS, "A PRIORI" ESTIMATES

Solutions of boundary value problems obey certain shift theorems. Assume $u \in \hat{H}_1$, and $k \geq 0$. Then the norm of $u$ in $H_{k+2}$ is equivalent to that of $\Delta u$ in $H_k$:
\[ c^{-1} \| u \|_{H_{k+1}} \leq \| \Delta u \|_{H_k} \leq c \| u \|_{H_{k+2}}. \] (3.1)

A direct consequence is:

Lemma 5: Let $k \geq 2$ be an integer. Then for any $u \in \hat{H}_1 \cap H_k$:
\[ \| \nabla^k u \|_a \leq c \left\{ \sum_{n=0}^{k-2} \| \nabla^n \Delta u \|_{a+k-2-n} + \| \nabla u \|_{a+k-1} + \| u \|_{a+k} \right\}. \] (3.2)
In order to prove the lemma the shift theorem (3.1) has to be applied to \( \mu^{-b/2} u \) with \( b = a \) resp. \( a + 1, a + 2, \ldots \) and \( k \) resp. \( k - 1, k - 2, \ldots \). The details are left.

There are some exceptions if \( a \) is an integer and one of the indices \( a + k - 2 - n \) in the sum of (3.2) is zero. We will only need

**Lemma 5**: Let \( w \in H_1 \cap H_3 \). Then

\[
\| \nabla^3 w \|_{-1} \leq c \{ \| \nabla \Delta w \|_{-1} + \| \Delta w \| \},
\]

\[
\| \nabla^3 w \|_{-2} \leq c \{ \| \nabla \Delta w \|_{-2} + \| \Delta w \|_{-1} + \| \nabla w \| \}. \tag{3.3}
\]

We will only give the proof of (3.3). We have

\[
\| \nabla^3 w \|_{-1}^2 = \rho^2 \| \nabla^3 w \|^2 + \sum_{i=1}^N \int (x_i - x_{0,i})^2 | \nabla^3 w |^2. \tag{3.5}
\]

The shift theorem gives for the first term

\[
\rho \| \nabla^3 w \| \leq c \rho \{ \| \nabla \Delta w \| + \| \Delta w \| \}
\leq c \{ \| \nabla \Delta w \|_{-1} + \| \Delta w \|_{-1} \} \leq c \{ \| \nabla \Delta w \|_{-1} + \| \Delta w \| \}. \tag{3.6}
\]

For the other terms we apply (3.1) with \( k = 1 \) and \( u = (x_i - x_{0,i}) w \). Since \( \nabla^3 u \) differs from \( (x_i - x_{0,i}) \nabla^3 w \) only by derivatives of \( w \) up to order 2 and the same is true for \( \nabla \Delta u \) and \( (x_i - x_{0,i}) \nabla \Delta w \) we get

\[
\int \int (x_i - x_{0,i})^2 | \nabla^3 w |^2 \leq c \int \int \{ (x_i - x_{0,i})^2 | \nabla \Delta w |^2 + | \nabla^2 w |^2 + | \nabla w |^2 \}. \tag{3.7}
\]

The first integrand is bounded by \( \| \nabla \Delta w \|_{-1}^2 \) whereas the rest is bounded by \( \| \Delta w \|^2 \).

In general in (3.2) the terms with \( u \) and \( \nabla u \) are present. But depending on \( a \) and \( k \) they may be interchangeable resp. can be dropped.

**Lemma 6**: Let \( u \in H_1 \cap H_2 \). Then:

(i) for \( b < 0 \) the norms \( \| \nabla u \|_b \) and \( \| u \|_{b+1} \) are comparable modulo \( \| \Delta u \|_{b-1} \), i.e.:

\[
\| \nabla u \|_b \leq k \{ \| u \|_{b+1} + \| \Delta u \|_{b-1} \}, \tag{3.8}
\]

\[
\| u \|_{b+1} \leq k \{ \| \nabla u \|_b + \| \Delta u \|_{b-1} \}.
\]

(ii) for \( 0 < b < (N/2) - 1 \) \((N > 2)\) both terms are bounded by the last, i.e.:

\[
\| u \|_{b+1} + \| \nabla u \|_b \leq k \| \Delta u \|_{b-1}. \tag{3.9}
\]
(iii) the case $b=(N/2)-1$ gives
\[ N(N-2)\beta^2 \| u \|_{b+2}^2 + 2 \| \nabla u \|_b^2 = 2D(u, \mu^{-b}u). \tag{3.10} \]

(iv) for arbitrary $b$ the term with $\nabla u$ is always bounded by the others
\[ \| \nabla u \|_b \leq k(\| u \|_{b+1} + \| u \|_{b-1}). \tag{3.11} \]
The relation
\[ \| \nabla u \|_b^2 = D(u, \mu^{-b}u) - \int (u \nabla u \nabla \mu^{-b}) \tag{3.12} \]
is an identity which may be written also in the form
\[ \| \nabla u \|_b^2 = D(u, \mu^{-b}u) + \frac{1}{2} \int u \Delta \mu^{-b} = (u, -\Delta u)_b + \frac{1}{2} \int u \Delta \mu^{-b} \tag{3.13} \]

Now direct differentiation gives $-r = |x-x_0|:
\[ \Delta \mu^{-b} = -2b \mu^{-b-2} (N \beta^2 + (N-2b-2)r^2). \tag{3.14} \]
We prove only case (i) in detail, the other proofs follow the same lines. Now let $b<0$. Then $\Delta \mu^{-b}$ is positive and $\mu^{b+1} \Delta \mu^{-b}$ is bounded and bounded away from zero (3.13) then gives
\[ \| \nabla u \|_b^2 \leq (u, -\Delta u)_b + k \| u \|_{b+1} \geq (u, -\Delta u)_b + k^{-1} \| u \|_{b+1}^2. \tag{3.15} \]
Now the assertions of the lemma, part (i) follow from this and the obvious generalization of Schwarz’s inequality $-b'$ being arbitrary:
\[ (u, v)_b \leq \| u \|_{b-b'} \| v \|_{b+b'}. \tag{3.16} \]
For the sake of completeness we note also
\[ D(u, v) \leq \| \nabla u \|_{b'} \| \nabla v \|_{b'}. \tag{3.17} \]

In section 5 we will introduce to $\Phi \in \mathcal{S}_h$ an auxiliary function $w$ defined by
\[ -\Delta w = \mu^{-a-1} \Phi \quad \text{in } \Omega, \]
\[ w = 0 \quad \text{on } \partial\Omega. \tag{3.18} \]

Some of the needed estimates are handled here, the rest will be given in the appendix.

Because of $\mathcal{S}_h \subseteq \mathcal{H}_1$ we have the regularity $w \in \mathcal{H}_1 \cap H_3$. We will need a bound for the $(-a)$-seminorm of the third derivatives. With the help of lemma 5 we get
\[ \| \nabla^3 w \|_{-a} \leq c \{ \| \nabla \Delta w \|_{-a} + \| \Delta w \|_{-a+1} + \| \nabla w \|_{-a+2} + \| w \|_{-a+3} \}. \tag{3.19} \]
First we have
\[ \| \Delta w \|_{-\alpha+1} = \| \Phi \|_{\alpha+3}. \] (3.20)

Next we get
\[ \| \nabla (\Delta w) \|_{-\alpha} = \| \nabla (\mu^{-\alpha} \Phi) \|_{-\alpha} \leq c \{ \| \Phi \|_{\alpha+3} + \| \nabla \Phi \|_{\alpha+2} \}. \] (3.21)

Lemma 3 and (2.14) give
\[ \| \nabla \Delta w \|_{-\alpha} + \| \Delta w \|_{-\alpha+1} \leq ch^{-2} (h/\rho) \| \Phi \|_{\alpha+1}. \] (3.22)

In this way we have shown.

**Lemma 7:** Let \( w \) be defined by (3.18) with \( \alpha \) arbitrary. Then
\[ \| \nabla^3 w \|_{-\alpha} \leq c \{ h^{-2} (h/\rho) \| \Phi \|_{\alpha+1} + \| \nabla w \|_{-\alpha+2} + \| w \|_{-\alpha+3} \}. \] (3.23)

In deriving this lemma we have applied lemma 5. According to lemma 5' there is the modification.

**Lemma 7':** Let \( w \) be defined by (3.18). In case of the exceptional values \( \alpha = 1, 2 \) instead of (3.23) the estimates hold true
\[ \begin{align*}
\| \nabla^3 w \|_{-1} &\leq ch^{-2} (h/\rho) \| \Phi \|_{2}, \\
\| \nabla^3 w \|_{-2} &\leq ch^{-2} (h/\rho) \| \Phi \|_{3} + \| \nabla w \|. 
\end{align*} \] (3.24)

4. \( L^2 \)-**PROJECTIONS**

To any \( v \) the approximations \( \chi \in S_h \) guaranted by lemma 2 may be replaced by \( V_h := P_h v \in S_h \) with the \( L^2 \)-projector \( P_h \) defined by
\[ (V_h, \chi) = (v, \chi) \quad \text{for} \quad \chi \in S_h. \] (4.1)

As a first result we mention:

**Theorem 1:** \( P_h \) is bounded with respect to any weighted norm, i.e. for a fixed there is a \( \gamma_2 \geq \gamma_1 \) depending only on \( N, m, \alpha \) and a such that for \( \gamma_2 h \leq \rho \):
\[ \| P_h v \|_a \leq 2 \| v \|_a. \] (4.2)

This was presented at Second Conference on Finite Elements, Rennes 1975, and appeared in the proceedings of that conference, see [10]. But those were distributed only in a limited number. With the above preparations the proof is rather short and will be reproduced here. Let \( \varphi = P_h v \) and \( \chi \in S_h \) be arbitrary. Then with Schwarz's inequality (3.16):
\[ \begin{align*}
\| \varphi \|_a^2 &= (\varphi, \mu^{-\alpha} \varphi) = (\varphi - v, \mu^{-\alpha} (\varphi - \chi)) + (v, \varphi)_a \\
&\leq \| \varphi - v \|_a \| \mu^{-\alpha} \varphi - \chi \|_{-\alpha} + \| v \|_a \| \varphi \|_a.
\end{align*} \] (4.3)
The consequence (2.15) of lemma 4 gives
\[ \| \varphi \|_{a}^{2} \leq c(h/p) \| \varphi \|_{a}^{2} + (1 + c(h/p)) \| v \|_{a} \| \varphi \|_{a}. \] (4.4)

Now we choose \( \gamma_2 = \text{Max} (\gamma_1, 3c) \) and get in case of \( \gamma_2 h \leq \rho \):
\[ \| \varphi \|_{a} \leq \frac{1}{3} \| \varphi \|_{a} + \frac{4}{3} \| v \|_{a}. \] (4.5)

A well-known consequence of theorem 1 is the "almost best" approximability
\[ \| v - P_h v \|_{a} \leq 3 \inf \{ \| v - \chi \|_{a} | \chi \in S_h \}. \] (4.6)

In addition we have the property of simultaneous approximability of \( P_h v \) on \( v \) which we formulate only in the way needed below:

**Corollary 1:** With the assumptions of theorem 1:
\[ \| v - P_h v \|_{a} + h \| \nabla (v - P_h v) \|_{a} \leq c \inf \{ \| v - \chi \|_{a} + h \| \nabla (v - \chi) \|_{a} | \chi \in S_h \}. \] (4.7)

**Proof:** Let again \( \varphi = P_h v \) for abbreviation and let \( \chi \in S_h \) be arbitrary. Then in using lemma 3 applied to \( \varphi - \chi \in S_h \) we get
\[ h \| \nabla (v - \varphi) \|_{a} \leq h \| \nabla (v - \chi) \|_{a} + h \| \nabla (\varphi - \chi) \|_{a} \]
\[ \leq h \| \nabla (v - \chi) \|_{a} + c \| \varphi - \chi \|_{a} \]
\[ \leq h \| \nabla (v - \chi) \|_{a} + c \| v - \varphi \|_{a} + c \| v - \chi \|_{a} \] (4.8)

and therefore with (4.6):
\[ \| v - \varphi \|_{a} + h \| \nabla (v - \varphi) \|_{a} \leq 3 (1 + c) \{ \| v - \chi \|_{a} + h \| \nabla (v - \chi) \|_{a} \}. \] (4.9)

Since \( \chi \in S_h \) is arbitrary (4.9) is also correct with the infimum taken on the right hand side.

**Remark:** All of the above statements hold true if \( S_h \) is replaced by \( i \).

**Remark:** If \( v \in H^1 \) resp. \( v \in H^1 \cap H^1 \) then according to lemma 2 the right hand side of (4.7) is bounded by \( ch^l \| \nabla^j v \|_{a} \). This gives the simultaneous error estimates
\[ \| \nabla^k (v - P_h v) \|_{a} \leq ch^{l-k} \| \nabla^l v \|_{a} \] (k = 0, 1). (4.10)

For completeness we mention the result of Bramble-Scott [2] on simultaneous approximability which could be applied also here. But since the question of interpolation in weighted norms is not well-developed the direct proof is shorter. Another possibility would have been to apply the ideas of [9].

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5. ESTIMATES IN WEIGHTED NORMS FOR FIXED TIME

In order to derive error estimates for the Galerkin method it is convenient to compare the Galerkin solution \( u_h \) with an appropriate approximation \( U_h \) on \( u \) in the subspace \( \hat{S}_h \). We will take the Ritz approximation \( U_h = R_h u \in \hat{S}_h \) defined by see (5):

\[
D(u - U_h, \chi) = 0 \quad \text{for} \quad \chi \in \hat{S}_h. \tag{5.1}
\]

The error

\[
e = u - u_h \tag{5.2}
\]

can be splitted

\[
e = (u - U_h) - (u_h - U_h) = \varepsilon - \Phi \tag{5.3}
\]

with the effect that now \( \Phi \) is an element of \( \hat{S}_h \). The defining relation for \( \Phi \) is see (3):

\[
(\Phi, \chi) + D(\Phi, \chi) = (\varepsilon, \chi) \quad \text{for} \quad \chi \in \hat{S}_h. \tag{5.4}
\]

Since estimates for \( \varepsilon \), i.e. the error of the Ritz method, are available it will be sufficient to bound \( \Phi \) in terms of \( \varepsilon \) resp. \( \dot{\varepsilon} \). The aim of this section is the proof of

**Theorem 2:** Let \( \alpha = N/2 \) with \( N \neq 3 \) and let \( \gamma_3 h \leq p \) with \( \gamma_3 \) properly chosen. Then

\[
\| \Phi \|_{a+1}^2 + \| \nabla \Phi \|_a^2 \leq c_1 \rho^{-2} \| \dot{\varepsilon} - \Phi \|_{a-1}^2, \tag{5.5}
\]

in case \( N = 3 \):

\[
\| \Phi \|_3^2 + \| \nabla \Phi \|_1^2 \leq c_1 \rho^{-1} \| \dot{\varepsilon} - \Phi \|_1^2. \tag{5.6}
\]

Firstly we will give the proof of (5.5) which is divided into three steps. In order to control the constants in this section they are numbered. \( c \) denotes in this section an upper bound of the constants in the previous sections. In step 1 we show the validity of

\[
\| \nabla \Phi \|_a^2 \leq c_2 \{ \| \dot{\varepsilon} - \Phi \|_{a-1}^2 + \| \Phi \|_{a+1}^2 \} \tag{5.7}
\]

for \( \alpha, N \) arbitrary. Using (5.4) and (3.13), (3.14) we get with \( \chi \in \hat{S}_h \) arbitrary

\[
\| \nabla \Phi \|_a^2 \leq D(\Phi, \mu^{-\alpha} \Phi) + c_1 \| \Phi \|_{a+1}^2, \quad \leq D(\Phi, \mu^{-\alpha} \Phi - \chi) - (\dot{\varepsilon} - \Phi, \mu^{-\alpha} \Phi - \chi) + (\dot{\varepsilon} - \Phi, \Phi)_a + c_1 \| \Phi \|_{a+1}^2. \tag{5.8}
\]

Using Schwarz’s inequality (3.16), (3.17) we derive

\[
\| \nabla \Phi \|_a^2 \leq \frac{1}{4} \| \nabla \Phi \|_a^2 + c_3 \{ \| \dot{\varepsilon} - \Phi \|_{a-1}^2 + \| \Phi \|_{a+1}^2 \}
\]

\[
+ \| \nabla (\mu^{-\alpha} \Phi - \chi) \|_a^2 + \| \mu^{-\alpha} \Phi - \chi \|_{a+1}^2. \tag{5.9}
\]

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Now let \( x \) be an appropriate approximation on \( u \). Lemma 4 with \( k = 0, b = a, \) and \( a = - \alpha + 1 \) gives

\[
\| \mu^{-\alpha} \Phi - \chi \|_{-\alpha + 1} \leq c_4 \left\{ h^m \| \Phi \|_{\alpha + m + 1} + h^2 \| \nabla \Phi \|_{\alpha + 2} \right\}
\]  
(5.10)

and because of (2.14) and \( h/\rho < 1 \):

\[
\| \mu^{-\alpha} \Phi - \chi \|_{-\alpha + 1} \leq c_4 (h/\rho) \left\{ \| \Phi \|_{\alpha + 1} + \| \nabla \Phi \|_\alpha \right\}.
\]  
(5.11)

In the same way we come to

\[
\| \nabla (\mu^{-\alpha} \Phi - \chi) \|_{-\alpha + 1} \leq c_5 (h/\rho) \left\{ \| \Phi \|_{\alpha + 1} + \| \nabla \Phi \|_\alpha \right\}.
\]  
(5.12)

With the last two bounds (5.9) gives

\[
\| \nabla \Phi \|_\alpha^2 \leq \left\{ \frac{1}{4} + 2(c_4^2 + c_5^2) (h/\rho)^2 \right\} \| \nabla \Phi \|_\alpha^2 \\
+ c_6 \left\{ \| \hat{\epsilon} - \Phi \|_{\alpha - 1}^2 + \| \Phi \|_{\alpha + 1}^2 \right\}.
\]  
(5.13)

Now we choose \( \gamma_3 = \max(\gamma_2, 4(c_4 + c_5)) \). Then obviously the coefficient of \( \| \nabla \Phi \|_\alpha^2 \) on the right hand side is less than 1/2 and so (5.7) is shown.

In order to get an estimate for \( \| \Phi \|_{\alpha + 1} \) we introduce an auxiliary function \( w \) defined by

\[
- \Delta w = \mu^{-\alpha - 1} \Phi \quad \text{in } \Omega,
\]
\[
w = 0 \quad \text{on } \partial \Omega.
\]  
(5.14)

Then with any \( \chi \in \hat{S}_h \) we have

\[
\| \Phi \|_{\alpha + 1}^2 = D(\Phi, w) = D(\Phi, w - \chi) - (\hat{\epsilon} - \Phi, w - \chi) + (\hat{\epsilon} - \Phi, w).
\]  
(5.15)

In step 2 of the proof of (5.5) we will show

\[
\| \Phi \|_{\alpha + 1}^2 \leq c_7 (h/\rho) \left\{ \| \Phi \|_{\alpha + 1}^2 + \| \nabla \Phi \|_\alpha^2 \right\} \\
+ \delta \| w \|_{\alpha + 1}^2 + c_8 (1 + \delta^{-1}) \| \hat{\epsilon} - \Phi \|_{\alpha - 1}^2
\]  
(5.16)

with \( \delta > 0 \) arbitrary. The two terms with \( \delta \) come from

\[
(\hat{\epsilon} - \Phi, w) \leq \| \hat{\epsilon} - \Phi \|_{\alpha - 1} \| w \|_{\alpha + 1} \leq \delta \| w \|_{\alpha + 1}^2 + \frac{1}{4 \delta} \| \hat{\epsilon} - \Phi \|_{\alpha - 1}^2.
\]  
(5.17)

With \( \chi \) chosen properly next we have

\[
D(\Phi, w - \chi) \leq \| \nabla \Phi \|_\alpha \| \nabla (w - \chi) \|_{-\alpha} \leq \| \nabla \Phi \|_\alpha c h^2 \| \nabla^3 w \|_{-\alpha}.
\]  
(5.18)

Firstly let us consider the case \( N > 4 \). Then we have to apply lemma 7. Since
then \(-\alpha + 2 = -N/2 + 2\) is negative part (i) of lemma 6 can be used. In this way we get
\[
D(\Phi, w - \chi) \leq c_9 \| \nabla \Phi \|_\alpha \{ (h/\rho) \| \Phi \|_{\alpha + 1} + h^2 \| \nabla w \|_{-\alpha + 2} \}. \tag{5.19}
\]
An essential aid is the next lemma the proof of which is given in the appendix:

**Lemma 8**: Let \(N \geq 4\) and \(\alpha = N/2\). For \(w\) defined by (5.14) the a priori estimate
\[
\| \nabla w \|_{-\alpha + 2} \leq c_{10} \rho^{-4} \| \Phi \|_{\alpha + 1}^2 \tag{5.20}
\]
is valid.

Obviously the right hand side of (5.19) is bounded by that of (5.16). For \(N = 4\) we have by lemma 7'—note \(\alpha = 2\) in this case:
\[
h^2 \| \nabla^3 w \|_{-2} \leq c (h/\rho) \| \Phi \|_{\alpha + 1} + ch^2 \| \nabla w \|. \tag{5.21}
\]
Applying lemma 8 also here shows that the term \(D(\Phi, w - \chi)\) is bounded by the right hand side of (5.16). Finally for \(N = 2\) lemma 7' gives directly
\[
D(\Phi, w - \chi) \leq \| \nabla \Phi \|_\alpha c (h/\rho) \| \Phi \|_{\alpha + 1} \leq \frac{1}{2} c (h/\rho) \{ \| \nabla \Phi \|_{\alpha}^2 + \| \Phi \|_{\alpha + 1}^2 \}. \tag{5.22}
\]
It remains to bound the middle term in (5.15).

We have
\[
(\hat{\epsilon} - \Phi, w - \chi) \leq \| \hat{\epsilon} - \Phi \|_{\alpha - 1} \| w - \chi \|_{-\alpha + 1} \tag{5.23}
\]
and
\[
\| w - \chi \|_{-\alpha + 1} \leq ch^3 \| \nabla^3 w \|_{-\alpha + 1} \leq ch^2 \| \nabla^3 w \|_{-\alpha}. \tag{5.24}
\]
With the help of the bounds given above for \(\| \nabla^3 w \|_{-\alpha}\) we see that this term is bounded in the same way by the right hand side of (5.16).

In step 3 of the proof of (5.5) we apply a lemma which also is proved in the appendix.

**Lemma 9**: Let \(N \geq 2\), \(\alpha = N/2\). Then for any \(w \in H_1 \cap H_2\):\[
\| w \|_{-\alpha + 1}^2 \leq c_{11} \rho^{-2} \| \Delta w \|_{-\alpha - 1}. \tag{5.25}
\]
For \(w\) defined by (5.14) this gives
\[
\| w \|_{-\alpha + 1}^2 \leq c_{11} \rho^{-2} \| \Phi \|_{\alpha + 1}^2. \tag{5.25}
\]
Therefore we may rewrite (5.16):
\[
\| \Phi \|_{\alpha + 1}^2 \leq \{ c_7 (h/\rho) + c_{11} \delta \rho^{-2} \} \{ \| \Phi \|_{\alpha + 1}^2 + \| \nabla \Phi \|_{\alpha}^2 \} + c_8 (1 + \delta^{-1}) \| \hat{\epsilon} - \Phi \|_{\alpha - 1}^2 \tag{5.27}
\]
and compare this with (5.7). If
\[
\left\{ c_7 \left( \frac{h}{\rho} \right) + c_{11} \delta \rho^{-2} \right\} \left\{ 1 + c_2 \right\} < 1 \tag{5.28}
\]
then \( \| \Phi \|_{a+1} \) and \( \| \nabla \Phi \|_a \) are bounded by \( \| \dot{\varepsilon} - \Phi \|_{a-1} \). We may choose
\[
\delta = \rho^2 \left\{ 4 c_{11} \left( 1 + c_2^2 \right) \right\}^{-1} \tag{5.29}
\]
and \( \gamma_4 h \leq \rho \) with
\[
\gamma_4 = \max(\gamma_3, 4 c_7 (1 + c_2)) \tag{5.30}
\]
to guarantee this. In this way (5.5) is proved.

Now we turn over to (5.6) of theorem 2. We have already
\[
\| \nabla \Phi \|^2 \leq c_2 \left\{ \| \dot{\varepsilon} - \Phi \|^2 + \| \Phi \|^2 \right\} \tag{5.31}
\]
since the power \( \alpha \) was not restricted in step 1. Similar to above we define \( w \) by
\[
-\Delta w = \mu^{-2} \Phi \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega, \tag{5.32}
\]
and get now—using lemma 7':
\[
\| \Phi \|^2 = D(\Phi, w - \chi) - (\dot{\varepsilon} - \Phi, w - \chi) + (\dot{\varepsilon} - \Phi, w) \\
\leq c h^2 \| \nabla \Phi \|_1 \| \nabla^3 w \|_{-1} + \| \dot{\varepsilon} - \Phi \left\{ \| w \| + h^3 \| \nabla^3 w \| \right\} \\
\leq c_{12} \left( \frac{h}{\rho} \right) \| \nabla \Phi \|_1 \| \Phi \|_2 + c_{13} \| \dot{\varepsilon} - \Phi \| \left\{ \| w \| +(h/\rho) \| \Phi \|_2 \right\}. \tag{5.33}
\]

In the analogue way (5.6) is then proved with the only difference that instead of (5.25) now the following lemma—see appendix—has to be applied.

**Lemma 9'**: Let \( N = 3 \). Then for any \( w \in \bar{H}_1 \cap H_2 \):
\[
\| w \|^2 \leq c \rho^{-1} \| \Delta w \|_2. \tag{5.34}
\]

### 6. ERROR ESTIMATES IN WEIGHTED NORMS

Theorem 2 gives in case \( N = 2,3 \):
\[
\| \Phi \|_2 \leq c \rho^{-4+N} \left\{ \| \Phi \|^2 + \| \dot{\varepsilon} \|^2 \right\}. \tag{6.1}
\]

Since by differentiation of (5.5):
\[
(\Phi, \chi) + D(\Phi, \chi) = (\dot{\varepsilon}, \chi) \quad \text{for } \chi \in \bar{S}_h \tag{6.2}
\]

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we get putting \( \chi = \Phi \) and integrating
\[
\| \Phi (t) \|^2 \leq \| \Phi (0) \|^2 + 2 \int_0^t \| \dot{\varepsilon} \| \| \Phi \| \, dt \tag{6.3}
\]
and therefore by Gronwall’s lemma
\[
\| \Phi (t) \|^2 \leq c \left\{ \| \Phi (0) \|^2 + \int_0^t \| \dot{\varepsilon} \|^2 \, dt \right\}. \tag{6.4}
\]
Since our initial condition — see (5.3) and the remarks in the introduction regarding the choice of the initial value of \( u_h \) — is
\[
\Phi (0) = 0 \tag{6.5}
\]
\( \chi = \Phi (0) \) in (5.4) gives
\[
\| \Phi (0) \|^2 = (\dot{\varepsilon} (0), \Phi (0)) \leq \| \dot{\varepsilon} (0) \|^2. \tag{6.6}
\]
Therefore we can rewrite (6.4) in the form
\[
\| \Phi \|_{L^\infty (U_\beta)} \leq c \left\{ \| \dot{\varepsilon} \|_{L^\infty (U_\beta)} + \| \dot{\varepsilon} \|^2 \right\}. \tag{6.7}
\]
In connection with (6.1) we have shown — note that \( L^\infty (a) \) is the \( L^\infty (0, T) \) norm of \( \| \cdot \|_a \):

**Theorem 3':** Let \( N = 2, 3 \). Then
\[
\| \Phi \|_{L^\infty (U_\beta)}^2 \leq c \rho^{-4 + N} \left\{ \| \dot{\varepsilon} \|^2_{L^\infty (U_\beta)} + \| \dot{\varepsilon} \|^2_{L^2 (U_\beta)} \right\}. \tag{6.8}
\]
In the case \( N \geq 4 \) the \((\alpha - 1)\)-norm of \( \Phi \) in (5.5) still is a weighted norm which has to be discussed further. The structure of the defining relation of \( \Phi \) and \( \Phi \) is the same. Therefore we will work with \( \Phi \) firstly and show

**Theorem 4:** Let \( N \geq 4 \) and \( \beta = (N/2) - 1 \). Then
\[
\| \Phi (t) \|_\beta^2 \leq \| \Phi (0) \|_\beta^2 + c \int_0^t \| \dot{\varepsilon} \|_\beta^2 \, dt. \tag{6.9}
\]
Now we will apply this with \( \Phi, \dot{\varepsilon} \) replaced by \( \Phi, \dot{\varepsilon} \). Further we have — the proof is given below.

**Lemma 10:** Let \( \Phi (0) = 0 \) and \( a, N \) arbitrary. Then
\[
\| \Phi (0) \|_a^2 \leq c \| \dot{\varepsilon} (0) \|^2_a. \tag{6.10}
\]
With the help of (6.9), (6.10) theorem 2 leads to the counterpart of theorem 3'.
THEOREM 3: Let $N \geq 4$, $\alpha = N/2$ and $\beta = \alpha - 1$. Then
\[ \| \Phi \|_{L_2(\alpha+1)}^2 \leq c \rho^{-2} \left\{ \| \hat{E} \|_{L_2(0)}^2 + \| \hat{E} \|_{L_2(0)}^2 \right\}. \] (6.11)

Proof of lemma 10: We take
\[ \chi = P_h(\mu^{-a} \Phi(0)) \] (6.12)
with $P_h$ being the $L_2$-projector in (5.4):
\[ \| \Phi(0) \|_{a}^2 = (\Phi(0), \chi) = -D(\Phi(0), \chi) + (\hat{E}(0), \chi) \leq \| \hat{E}(0) \|_a \| \chi \|_{-a}. \] (6.13)
Because of theorem 1 we get
\[ \| \chi \|_{-a} \leq c \| \mu^{-a} \Phi \|_{-a} = c \| \Phi(0) \|_a. \] (6.14)

Proof of theorem 4: We start with the identity $-\chi \in \mathcal{S}_h$ is arbitrary
\[ (\Phi, \Phi)_\beta + D(\Phi, \mu^{-\beta} \Phi) = (\Phi, \mu^{-\beta} \Phi - \chi) + D(\Phi, \mu^{-\beta} \Phi - \chi) \]
\[ = (\hat{E}, \mu^{-\beta} \Phi - \chi) + (\hat{E}, \Phi)_\beta. \] (6.15)
The choice $\chi = P_h(\mu^{-\beta} \Phi)$ causes that the first term on the right hand side disappears. Further in our case of $\beta (3:10)$ gives
\[ D(\Phi, \mu^{-\beta} \Phi) = \| \nabla \Phi \|_{\beta}^2 + N(N-2) \frac{1}{2} \rho^2 \| \Phi \|_{\beta+2}^2. \] (6.16)
Therefore with the special $\chi$:
\[
(\Phi, \Phi)_\beta + \| \nabla \Phi \|_{\beta}^2 + k \rho^2 \| \Phi \|_{\beta+2}^2
= D(\Phi, \mu^{-\beta} \Phi - \chi) - (\hat{E}, \mu^{-\beta} \Phi - \chi) + (\hat{E}, \Phi)_\beta. \] (6.17)
Now lemma 4 with $b = -a = \beta$ and $k = 0$ resp. $k = 1$ in connection with theorem 1 gives
\[ \| \nabla^k (\mu^{-\beta} \Phi - \chi) \|_{-\beta} \leq c h^{-k} \left\{ h^m \| \Phi \|_{\beta+m} + h^2 \| \nabla \Phi \|_{\beta+1} \right\} \]
\[ \leq c h^{1-k} (h/\rho) \left\{ \rho \| \Phi \|_{\beta+2} + \| \nabla \Phi \|_{\beta} \right\}. \] (6.18)
In this way we get for the first two terms on the right hand side of (6.17):
\[ D(\Phi, \mu^{-\beta} \Phi - \chi) + (\hat{E}, \mu^{-\beta} \Phi - \chi) \]
\[ \leq c (h/\rho) \left\{ \| \nabla \Phi \|_{\beta} + h \| \hat{E} \|_{\beta} \right\} \left\{ \| \nabla \Phi \|_{\beta} + \rho \| \Phi \|_{\beta+2} \right\}. \] (6.19)
In the way analogue to the proof of theorem 2 — see especially (5.27) — we get with $\gamma_5 h \leq \rho$ and $\gamma_5 \geq \gamma_4$ chosen properly
\[ (\Phi, \Phi)_\beta + \| \nabla \Phi \|_{\beta}^2 + \rho^2 \| \Phi \|_{\beta+2}^2 \leq c \left\{ \| \hat{E} \|_{\beta}^2 + \| \Phi \|_{\beta}^2 \right\}. \] (6.20)
respective
\[ \frac{d}{dt} \| \Phi(t) \|_2^2 = 2(\Phi, \dot{\Phi})_\beta \leq c \left\{ \| \dot{\Phi} \|^2_\beta + \| \Phi \|^2_\beta \right\}. \] 

(6.21)

Then Gronwall's lemma gives (6.9).

7. POINTWISE ERROR ESTIMATES

Up to now we had conditions on \( \rho \) of the type \( \gamma_1 h \leq \rho \). Now we fix \( \rho = \gamma_5 h \).

Let \( t \in [0, T] \) be fixed. There is an \( \hat{x} = \hat{x}_t \in \Omega \) such that
\[ \Phi(\hat{x}, t) = \pm \| \Phi(t) \|_{L_u}. \] 

(7.1)

We identify \( x_0 \) entering \( \mu (1.2) \) with this \( \hat{x} \). Further let \( \Lambda \in \Gamma_h \) be the simplex (or one of the simplices) with \( \hat{x} \in \Lambda \).

The function \( \Phi \) restricted to \( \Lambda \) is a polynomial of degree less than \( m \), i.e., an element of a finite dimensional space. Therefore any two norms are equivalent. Because of the \( \kappa \)-regularity of \( \Delta \) there is a \( k = k(N, m, \kappa) \) such that
\[ \| \Phi \|_{L_u(\Lambda)}^2 \leq k \left\{ h^{-N} \int_{\Lambda} \Phi^2 \, dx \right\}. \] 

(7.2)

Since \( x_0 \in \Lambda \) we have in \( \Delta \):
\[ \gamma_5^2 h^2 \leq \mu \leq (\gamma_5^2 + \kappa^2) h^2 \] 

(7.3)

and therefore with \( \alpha = N/2 \):
\[ h^{-N} \int_{\Lambda} \int_{\Lambda} \Phi^2 \, dx \leq c \, p^2 \int_{\Lambda} \int_{\Lambda} \mu^{-\alpha-1} \Phi^2 \, dx \leq c \, \rho^2 \| \Phi \|_{\alpha+1}^2 \] 

(7.4)

resp. combining (7.1), (7.2), (7.4):
\[ \| \Phi(t) \|_{L_u} \leq c \, \rho \| \Phi(t) \|_{\alpha+1} \] 

(7.5)

With the help of theorem 3 we deduce for \( N \geq 4 \) with \( \beta = N/2 - 1 \):
\[ \| \Phi \|_{L_u, L_{u\beta}} \leq c \left\{ \| \dot{\Phi} \|_{L_u(\beta)} + \| \ddot{\Phi} \|_{L_2(\beta)} \right\}. \] 

(7.6)

In case \( N \leq 3 \) the same arguments give — see (7.4):
\[ h^{-N} \int_{\Lambda} \int_{\Lambda} \Phi^2 \, dx \leq c \, \rho^{N-4} \int_{\Lambda} \int_{\Lambda} \mu^{-2} \Phi^2 \, dx \leq c \, \rho^{N-4} \| \Phi \|_2^2. \] 

(7.7)
Because of theorem 3' (7.6) is valid for $N \leq 3$ with $\beta = 0$.

At the end the weighted norms may be replaced by $L_p$-norms. The factor $\mu^{-\beta}$ is $L_q$-integrable for $q < N/(N-2)$. Since then $q'$ defined by $q^{-1} + q'^{-1} = 1$ is greater than $N/2$ for any $p > N$:

$$\| u \|_{L_p} \leq c_p \| u \|_{L_q}. \quad (7.6)$$

In this way we get

**Theorem 5:** Let $p = 2$ for $N \leq 3$ and $p > N$ for $N \geq 4$. Then

$$\| \Phi \|_{L_2(L_p)} \leq c \{ \| \hat{\varepsilon} \|_{L_2(L_p)} + \| \tilde{\varepsilon} \|_{L_2(L_p)} \}. \quad (7.9)$$

Scott [14] and Nitsche [10] gave the error estimates for the Ritz-method

$$\| \varepsilon \|_{L_2} = \| u - R_h u \|_{L_2} \leq c h^k \| u \|_{W^k_2}. \quad (7.10)$$

for $k \leq m$. Because of $e = u - u_h = \varepsilon - \Phi$ see (5.3) — we have the final result:

**Theorem 6:** Assume the regularity of the solution $u$ of the initial-boundary value problem (1):

(i) $u \in L_\infty(0, T, W^k_\infty(\Omega))$;

(ii) $\dot{u} \in L_\infty(0, T, W^k_\infty(\Omega))$;

(iii) $\ddot{u} \in L_2(0, T, W^k_\infty(\Omega))$.

Then the error $e = u - u_h$ between the exact solution $u$ and the Galerkin approximation $u_h$ defined by (2) is of order $h^k$ with $k \leq m$ — the order of the finite elements used.

**Remark:** For $N \leq 3$ the regularity assumptions on $\dot{u}, \ddot{u}$ can be lowered:

$$\dot{u} \in L_\infty(0, T, W^2_2(\Omega)), \quad \ddot{u} \in L_2(0, T, W^2_2(\Omega))$$

is sufficient.

**Remark:** Having theorem 5 in mind one would expect assumptions of the type:

(ii') $\dot{u} \in L_\infty(0, T, W^k_p(\Omega))$;

(iii') $\ddot{u} \in L_2(0, T, W^k_p(\Omega))$,

instead of (ii), (iii) of theorem 6. As was pointed out by Scott the estimates (7.10) togethert with the $L_2$-bounds

$$\| \varepsilon \|_{L_2} \leq c h^k \| u \|_{W^2_2}. \quad (7.11)$$

do not imply

$$\| \varepsilon \|_{L_2} \leq c h^k \| u \|_{W^2_2}. \quad (7.12)$$

This is the reason for the formulation with $L_\infty$-norms in theorem 6.
The convergence rate up to $h^n$ is optimal with respect to the power of $h$. But in order to get this bounds for the second time derivative are needed. We can get from (6.9) a reduced convergence result but without needing $\bar{e}$. With $\Phi(0)=0$ we have

$$\|\Phi\|_{L_\infty(\Omega)} \leq c\|\bar{e}\|_{L_\infty(\Omega)}. \quad (7.13)$$

For $\beta = N/2 - 1$ now $c\|\Phi\|_\beta$ is an upper bound of $h\|\Phi\|_{L_\infty}$ if $x_0$ (1.2) is chosen properly. This gives

**Theorem 7:** Let $N \geq 3$ and $p > N$. Then

$$\|\Phi\|_{L_\infty(L_\infty)} \leq ch^{-1}\|\bar{e}\|_{L_\infty(L_{p,\Omega})}. \quad (7.14)$$

The counterpart of theorem 6 is then

**Theorem 8:** The error of the Galerkin approximation is of order $h^{k-1}$ ($k \leq m$) provided the regularity assumptions

(i) $u \in L_\infty(0, T, W^{k-1}_\infty(\Omega));$

(ii) $u \in L_2(0, T, W^k(\Omega)).$

hold.

8. APPENDIX: PROOF OF LEMMATA 8, 9

For bounded domains $\Omega' \subseteq R^N$ let

$$\lambda(\Omega') = \sup \left\{ \left\| \nabla w \right\|_{L^2(\Omega')}^{2-\alpha+2}, \left\| \Delta w \right\|_{L^2(\Omega')}^{2-\alpha-1} \right\} \quad (8.1)$$

and

$$\Lambda(\Omega') = \sup \left\{ \left\| w \right\|_{L^2(\Omega')}^{2-\alpha+3}, \left\| \Delta w \right\|_{L^2(\Omega')}^{2-\alpha-1} \right\} \quad (8.2)$$

Because of the definition of $w$ (5.14) lemma 8 is proved if we can show $\lambda(\Omega) \leq c\rho^{-4}$. Firstly we consider the case $N > 4$. Then $-\alpha + 2$ is negative and lemma 6, (i) gives

$$\lambda(\Omega') \leq k\{ \lambda(\Omega') + \rho^{-4} \}, \quad \Lambda(\Omega') \leq k\{ \lambda(\Omega') + \rho^{-4} \} \quad (8.3)$$

with $k$ independent of $\Omega'$. Obviously $\Lambda$ is monotone in $\Omega'$, i.e. $\Lambda(\Omega') \leq \Lambda(\Omega)$ for $\Omega' \subseteq \Omega'$. Next let $K = K_R(x_0)$ be a sphere of radius $R = \text{diam}(\Omega)$ with center $x_0$. Then $\Omega \subseteq K$ and hence $\Lambda(\Omega) \leq \Lambda(K)$. The supremum $\Lambda(K)$ is attained for a positive function $w_K$ with $-\Delta w_K > 0$ because of the maximum principle, and $w_K$
solves the eigenvalue problem
\[ \Delta (\mu^{a+1} \Delta w) = \Lambda^{-1} \mu^{a-3} w \quad \text{in } K, \]
\[ w = \Delta w = 0 \quad \text{on } \partial K. \]  
(8.4)

Without loss of generality we can assume \( w_K = w_K(r) \) with \( r = |x - x_0| \) since \( \mu \) depends only on \( r \), for otherwise the spherical average of \( w_K \) solves the same eigenvalue problem and is also positive. Therefore we can restrict the space of admissible functions without changing \( \Lambda \):
\[ \Lambda(K) = \sup \left\{ \left\| \frac{w}{\Delta w} \right\|_{\frac{a+3}{a-1}, K} \right\|_{V_K} \} \]  
(8.5)

with \( V_K = \dot{H}_1(K) \cap H_2(K) \cap \{ w \mid w = w(r) \} \). Now with lemma 6, (i) we get
\[ \lambda(\Omega) \leq k \left\{ \rho^{-4} + \Lambda(K) \right\} \leq k \left\{ \rho^{-4} + \sup \left\{ \left\| \nabla w \right\|_{\frac{a+2}{a-1}, K} \right\|_{V_K} \right\} \right\}. \]  
(8.6)

Functions \( w \in V_K \) have the representation \( (w' = dw/dr) \):
\[ w' = r^{1-N} \int_0^r s^{N-1} \Delta w \, ds. \]  
(8.7)

Schwarz’s inequality gives
\[ \left| w' \right|^2 \leq \rho^{2-2N} f(r) \int_0^r s^{N-1} \mu^{a+1} \left| \Delta w \right|^2 \, ds \]  
(8.8)

with
\[ f(r) = \int_0^r s^{N-1} \mu^{a-1} \, ds \leq c \left\{ \begin{array}{ll} \rho^{-N-2} \rho^N & \text{for } r \leq \rho, \\ \rho^{-2} & \text{for } r \geq \rho, \end{array} \right\}. \]  
(8.9)

because of \( a = N/2 \).

Therefore
\[ \left\| \nabla w \right\|_{\frac{a+2}{a+2}, K} = k \int_0^R r^{N-1} \mu^{a-2} \left| w' \right|^2 \, dr \]
\[ \leq k \int_0^R r^{1-N} \mu^{a-2} f(r) \, dr \int_0^r s^{N-1} \mu^{a+1} \left| \Delta w \right|^2 \, ds \]
\[ = k \int_0^R s^{N-1} \mu^{a+1} \left| \Delta w \right|^2 \, ds \int_s^R r^{1-N} \mu^{a-2} f(r) \, dr \]
\[ \leq k \left\| \Delta w \right\|_{\frac{a+2}{a-1}, K} \int_0^R r^{1-N} \mu^{a-2} f(r) \, dr. \]  
(8.10)
The last integral is bounded by \( c \rho^{-4} \). This completes the proof in case \( N > 4 \).

For \( N = 4 \) without using lemma 6 we directly consider the supremum of \( \| \nabla w \|^2 / \| \Delta w \|^2 \) and get the same result with the same arguments.

**Proof of lemma 9:** The proof follows the above lines. In the definition of \( \lambda \), \( \Lambda \) we replace the indices of \( \| \nabla w \| \) resp. \( \| w \| \) by \( -\alpha \) resp. \( -\alpha + 1 \). Then \( -\alpha = N/2 \) is negative. Up to formula (8.9) nothing is changed. But then

\[
\| \nabla w \|^{2, -\alpha}_{\mathbb{K}} = \int_0^R r^{N-1} \mu^2 \| w' \|^2 \, dr \leq \| \Delta w \|^{2, -\alpha - 1}_{\mathbb{K}} \int_0^R r^{1-N} \mu^2 f(r) \, dr \quad (8.11)
\]

and the last integral is bounded by \( c (1 + R^2 \rho^{-2}) \leq c' \rho^{-2} \).

The proof of lemma 9' is analogue to the preceding one and is omitted here.

There is an interesting remark to be added. In (8.1) resp. (8.2) the \((-\alpha + 2)\)-norm of the first derivatives resp. the \((-\alpha + 3)\)-norm of the function itself is compared with the \((-\alpha - 1)\)-norm of the second derivatives. Roughly speaking each differentiation in weighted norms may be considered as reducing the weight-power by one. Then \( \| \nabla w \|_{-\alpha+2} \) and \( \| w \|_{-\alpha+3} \) would be something like \( \| \Delta w \|_{-\alpha+1} \). Since this is compared with \( \| \Delta w \|_{-\alpha-1} \) the behavior \( \lambda, \Lambda \approx \rho^{-4} \) is "understandable". Of course this "rule" is only valid for special \( \alpha \) and has to be checked in each case. Just lemma 9 is an example that it may be violated.

**REFERENCES**

**Part a: literature cited**


Part b: additional literature


R.A.I.R.O. Analyse numérique/Numerical Analysis

vol. 13, n° 1, 1979