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# INTERIOR ERROR ESTIMATES FOR SEMI-DISCRETE GALERKIN APPROXIMATIONS FOR PARABOLIC EQUATIONS (*) (**)

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Abstract — *The initial boundary value problem for the heat equation in a domain $\Omega$ and the corresponding standard Galerkin method is considered A certain regularity of the initial data in some subdomain $\Omega_1$ leads to the same regularity of the solution in $\Omega_1$ and for all times It is shown that the error between the exact solution and the Galerkin approximation is also of (almost) optimal order in the interior of $\Omega_1$ Of course certain properties of the underlying approximation spaces are needed, they are typical for finite elements*

Resume — *On considere le probleme aux limites avec conditions initiales pour l'equation de la chaleur dans un domaine $\Omega$, ainsi que l approximation habituelle de Galerkin correspondante Une certaine regularite des donnees initiales dans un sous-domaine $\Omega_1$ conduit a la même regularite de la solution dans $\Omega_1$ pour tous les temps On montre que l'erreur entre la solution exacte et l'approximation de Galerkin est aussi d ordre (presque) optimal dans l'interieur de $\Omega_1$ Naturellement, certaines proprietes des espaces d approximations sont utilisees, qui sont caracteristiques des espaces d elements finis*

**1.** In order to avoid technical details we restrict ourselves to the model problem

$$\begin{aligned} \dot{u} &= \Delta u \quad \text{in} \quad \Omega \times (0, T], \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0, T], \\ u_{t=0} &= v \quad \text{in} \quad \Omega \end{aligned} \tag{1}$$

The boundary of $\Omega \subseteq \mathbb{R}^N$ is assumed to be sufficiently smooth With the help of a finite element space $S_h \subseteq \overset{\circ}{H}_1(\Omega)$ the Galerkin approximation $u_h = u_h(t) \in S_h$ is defined by

$$\begin{aligned} (\dot{u}_h, \chi) + D(u_h, \chi) &= 0 \quad \text{for} \quad \chi \in S_h \wedge t > 0, \\ u_{h|t=0} &= P_h v \end{aligned} \tag{2}$$

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Here $(.,.)$ resp. $D(.,.)$ is the $L_2$-inner product resp. the Dirichlet integral and (for simplicity) $P_h$ is the $L_2$-projector onto $S_h$.

For the corresponding elliptic problem

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega\,, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{3}$$

interior estimates of the error $e = u - u_h$ of the Ritz approximation $u_h \in S_h$ defined by

$$D(u_h, \chi) = (f, \chi) \quad \text{for} \quad \chi \in S_h \tag{4}$$

were derived in [1], [5], [6], [7]. They are of the following type (*) :

Assume $f \in L_2(\Omega)$ and in addition $f \in H_{k-2}(\Omega_1)$ for some domain $\Omega_1 \subseteq \Omega$ and $k > 2$. Further let $\Omega_2$ be contained properly in $\Omega_1$. If $S_h$ is of degree $r$ with $r \geqslant k$ then the error $e$ is of order $k$ in $\Omega_2$, i.e.

$$\| e \|_{\Omega_2} \leqslant ch^k \{ \| f \|_{k-2.\Omega_1} + \| f \| \} . \tag{5}$$

The norms of $f$ could be replaced by the appropriate norms of $u$ because of the shift theorem.

Now let us assume the regularity

$$v \in L_2(\Omega) \cap H_k(\Omega_1) \tag{6}$$

and let $\Omega_2 \subset\subset \Omega_1$. Then the solution of (1) has the regularity

$$u \in L_\infty(L_2(\Omega)) \cap L_2(H_1(\Omega)) \cap L_\infty(H_k(\Omega_1)) \tag{7}$$

with the abbreviation $L_p(Z) = L_p(0, T; Z)$ for some $T > 0$ fixed. Corresponding to (5) we would expect in the parabolic case an estimate of the type

$$\| e \|_{L_\infty(L_2(\Omega_2))} \leqslant ch^k \{ \| v \|_{k.\Omega_1} + \| v \| \} . \tag{8}$$

In the next two sections we prove this error bound for $k < r$ (being the degree of $S_h$). In case $k = r$ the factor $ch^k$ is to be replaced by $c_\varepsilon \, h^{r-\varepsilon}$ with $\varepsilon > 0$ arbitrary small.

This problem was already treated in Thomee [9]. There the local error in the $L_\infty(L_2(\Omega_2))$ norm is bounded by the $L_2(L_2(\Omega_1))$ norm besides of a remainder. Although this result does not give the final answer it turns out to be the main step. With respect to the notations as already mentioned as well as to the

(*) We use the notations of [7] resp. [9].

assumptions on $S_h$ we refer to [9]. Since we do not extend our result to difference quotients the uniformity of the subdivisions in defining $S_h$ is not necessary of course.

**2.** We start with an interior estimate for $u_h$, see lemma 3.3 in [9] :

LEMMA 1 (Thomée) : *Let $u_h$ be the solution of (2). Further assume*

$$\Omega_2 \subset\subset \Omega_1 \subseteq \Omega$$

*and let $q > 0$ be fixed. Then*

$$\| u_h(t) \|_{\Omega_2}^2 \leqslant c \left\{ \| P_h v \|_{\Omega_1}^2 + \int_0^t [\| u_h \|_{\Omega_1}^2 + h^q \| \dot{u}_h \|_{\Omega_1}^2] \, d\tau \right\}. \tag{9}$$

Because of

$$\| \dot{u}_h \|_{\Omega_1}^2 \leqslant \| \dot{u}_h \|^2 = -\frac{1}{2} \partial_t \| u_h' \|^2 \tag{10}$$

we get

$$\int_0^t \| \dot{u}_h \|_{\Omega_1}^2 \, d\tau \leqslant \frac{1}{2} \| u_h'(0) \|^2 . \tag{11}$$

Further we have

$$\begin{aligned} \| u_h'(0) \| &\leqslant ch^{-1} \| u_h(0) \| \\ &\leqslant ch^{-1} \| v \| . \end{aligned} \tag{12}$$

In this way the last term in (9) may be replaced by $h^{q-2} \| v \|^2$.

Next let us assume there is an additional domain $\Omega_0$ according to

$$\Omega_1 \subset\subset \Omega_0 \subseteq \Omega .$$

Using the result of [6] or of Douglas *et al.* [3] we may estimate

$$\| P_h v \|_{\Omega_1} \leqslant c \{ \| v \|_{\Omega_0} + h^r \| v \| \} . \tag{13}$$

By replacing $\Omega_0$ by $\Omega_1$ we get with $q = 2r + 2$.

LEMMA 2 : *Let $\Omega_2 \subset\subset \Omega_1 \subseteq \Omega$. Then*

$$\| u_h(t) \|_{\Omega_2}^2 \leqslant c \left\{ \| v \|_{\Omega_1}^2 + h^{2r} \| v \|^2 + \int_0^t \| u_h \|_{\Omega_1}^2 \, d\tau \right\}. \tag{14}$$

Now we use an induction argument. Let $\Omega_1$, $\Omega_2$ with $\Omega_2 \subset\subset \Omega_1$ and $p \in \mathbb{N}$ be given. We can choose domains $\Omega^\nu$ according to

$$\Omega_2 = \Omega^p \subset\subset \Omega^{p-1} \subset\subset \dots \subset\subset \Omega^0 = \Omega_1 . \tag{15}$$

Repeated application of (14) and the interchange of the order of integration leads to

$$\| u_h(t) \|_{\Omega_2}^2 \leqslant c \left\{ \| v \|_{\Omega_1}^2 + h^{2r} \| v \|^2 + \int_0^t (t - \tau)^{p-1} \| u_h \|_{\Omega_1}^2 \, d\tau \right\} . \tag{16}$$

Since anyway

$$\| u_h \|_{\Omega_1} \leqslant \| u_h \| \leqslant \| v \| \tag{17}$$

we get :

LEMMA 3 : *Assume $\Omega_2 \subset\subset \Omega_1 \subseteq \Omega$ and let $p > 0$ be arbitrary. Then*

$$\| u_h(t) \|_{\Omega_2} \leqslant c \{ \| v \|_{\Omega_1} + (t^p + h^r) \| v \| \} . \tag{18}$$

This is the counterpart of the a priori estimate

$$\| u(t) \|_{\Omega_2} \leqslant c \{ \| v \|_{\Omega_1} + t^p \| v \| \} \tag{19}$$

for the solution of (1) which is easily derived (for instance using the exponential decay of the fundamental solution).

**3.** Now we are ready to prove (8). Let $\Omega_2 \subset\subset \Omega_1$ be fixed. We choose an $\Omega'$ according to

$$\Omega_2 \subset\subset \Omega'_0 \subset\subset \Omega_1 . \tag{20}$$

Next let $\omega$ be a cut-off function with respect to $\Omega'$, $\Omega_1$, i.e. $\omega \in C^\infty(\Omega)$ with $0 \leqslant \omega \leqslant 1$ and

$$\omega = \begin{cases} 1 & \text{in } \Omega' \\ 0 & \text{in } \Omega - \Omega_1 \end{cases} . \tag{21}$$

We will use the splitting

$$v = v^1 + v^2 := \omega v + (1 - \omega) v \tag{22}$$

and denote by $u^i$, $u_h^i$ the solutions of (1), (2) with the initial data $v^i$.

We have

$$u = u^1 + u^2 , \quad u_h = u_h^1 + u_h^2 . \tag{23}$$

The regularity assumption (6) leads to $v^1 \in H_k(\Omega)$ and

$$\| v^1 \|_k \leqslant c \| v \|_{k.\Omega_1} \tag{24}$$

Therefore (see e.g. Bramble *et al.* [2])

$$\| u^1 - u_h^1 \| \leqslant ch^k \| v \|_{k.\Omega_1} \tag{25}$$

is guaranteed. On the other hand $v^2$ is in $L_2(\Omega)$ with $\| v^2 \| \leqslant \| v \|$ and $v^2$ vanishes in $\Omega'$. The estimates (18), (19) for $u^2$ and $u_h^2$ with $\Omega_1$ replaced by $\Omega'$ give

$$\| u^2 - u_h^2 \|_{\Omega_2} \leqslant c(t^p + h^r) \| v \| . \tag{26}$$

In this way

$$\| e(t) \|_{\Omega_2} = \| u - u_h \|_{\Omega_2} \leqslant c \{ h^k \| v \|_{k.\Omega_1} + (t^p + h^r) \| v \| \} \tag{27}$$

is shown. In order to eliminate the time dependence we make use of the time dependent error estimate

$$\| e(t) \| \leqslant ch^r t^{-r/2} \| v \| \tag{28}$$

due to Helfrich [4], Thomee [8] which in connection with (27) gives

$$\| e(t) \|_{\Omega_2} \leqslant c \{ h^k \| v \|_{k.\Omega_1} + [h^r + \mathrm{Min}\,(t^p, h^r t^{-r/2})] \| v \| \} . \tag{29}$$

The minimum is maximal for $t = c_p h^\alpha$ with $\alpha = 2 r/(r + 2 p)$ leading to

$$\mathrm{Min}\,(t^p, h^r t^{-r/2}) = ch^\beta \tag{30}$$

with

$$\beta = \frac{2 pr}{2 p + r} = r - \frac{r^2}{2 p + r} . \tag{31}$$

For any $k < r$ we may choose $p$ such that $\beta \geqslant k$.

Then (29) gives (8). In case of $k = r$ and $\varepsilon > 0$ fixed we can choose $p$ such that $\beta \geqslant r - \varepsilon$. Since the choice of $p$ influences the number of iterations the constant in (8) then depends on $p$ resp. $\varepsilon$.

*Remark* : In the previous estimates we did not control the constants. Therefore we dıd not take ınto account a contrıbutıon of $(p - 1)^{-1}$ ın the denomınator of the ıntegral ın (16) By a careful analysıs the « garbage » term $h^{-\varepsilon}$ ın (8) ın case of $k = r$ could be replaced by a logarıthmıc one

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