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ON THE PARTITIONED MATRIX  $\begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$   
AND ITS ASSOCIATED SYSTEM  $AX = T, A^* Y + QX = Z$  (\*)

by Vladimiro VALERIO <sup>(1)</sup> (\*\*)

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Abstract — Inverses of the partitioned matrix  $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$ , where  $Q$  is possibly nonnegative definite, and solutions of its associated system  $AX = T, A^* Y + QX = Z$  are the object of this note. Some results in an earlier paper are extended. Finally, conditions for inverting the square regular matrix  $N$ , when  $Q$  is also singular, and a different construction of the inverse  $N^{-1}$  are given using a particular  $g$ -inverse of  $Q$ .

Résumé — L'objet de cet article est l'étude des inverses de matrices partitionnées sous la forme  $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$ , où  $Q$  peut être semi-définie positive, ainsi que l'étude des solutions du système associé  $AX = T, A^* Y + QX = Z$ . On généralise les résultats d'un article antérieur. Enfin, utilisant un  $g$ -inverse particulier de  $Q$ , on donne des conditions pour inverser la matrice carrée inversible  $N$  quand  $Q$  est singulière, ainsi qu'une construction différente de l'inverse  $N^{-1}$ .

LIST OF SYMBOLS

- $\alpha$  lower case greek alfa
- $\beta$  lower case greek beta
- $*$  star
- $\Rightarrow$  arrow
- $\oplus$  circle with plus inside

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## 1. INTRODUCTION

An increasing number of papers has been appeared in the last ten years on the generalized inverses of a partitioned matrix. One of the approaches depends on the Schur-complement  $M/A = D - CA^{-1}B$  defined for a square regular matrix  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , where  $A$  is also regular. Its generalization to rectangular and singular matrices under any partition has also been studied in [6, 7, 11, 14] and [15]. Partitioned matrices are given in [3] and [10] which give conditions on the rank and the range of the partition in order to define their generalized inverses; [8] considers the Moore-Penrose inverse of  $M$ . Some particular aspects, useful for correcting least squares estimates, are found in [9, 10, 12, 16] and [18], where the matrix is in the form  $(A : a)$  and  $a$  is a vector. In [5] we have partitioned matrices like  $A = [U, V]$  in which conditions for the existence of the Moore-Penrose inverse are given. A more detailed discussion on the latter is in [2].

In the present note we consider the partitioned matrix  $N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}$  where  $Q$  is  $n \times n$ , if it is not otherwise stated, and the associated system  $AX = T$ ,  $A^*Y + QX = Z$ . A matrix partitioned like  $N$  could be found in [19] and [20].

The above system arises in many problems of applied Mechanics, where  $Q$  is also symmetric and  $pd$ , and in calculating space structures (trusses) or continuous structures finding a discrete structure which matches the continuous one. We refer to an earlier paper [21] and give additional results. Theorem 1 gives a particular set of solution to the considered system if we observe that  $X$  and  $Y$  are possibly two different kind of unknowns [22]. Finally, conditions for inverting the square regular matrix  $N$  when  $Q$  is singular and a different construction of the regular inverse  $N^{-1}$  are given using a particular  $g$ -inverse of  $Q$ .

## 2. DEFINITIONS AND NOTATIONS

We denote by  $C^{m,n}$  the vector space of all  $m \times n$  matrices defined over the complex number field. For a given matrix  $A$   $r(A)$  is its rank,  $R(A)$  is the range or the space spanned by the columns of  $A$ ,  $A^*$  is the conjugate transpose of  $A$ .  $A^-$  is any  $g$ -inverse of  $A$  satisfying  $AA^-A = A$  and  $A_r$  is a reflexive  $g$ -inverse satisfying also  $A^-AA^- = A^-$ . In general we use the notations of [19].

Let  $A \in C^{m,n}$  and  $X \in C^{n,p}$ , we consider the system

$$\begin{pmatrix} AX = T \\ A^*Y + QX = Z \end{pmatrix} \quad (1)$$

We have  $Q \in C^{n,m}$ ,  $Y \in C^{m,p}$ ,  $T \in C^{m,p}$  and  $Z \in C^{n,p}$ . System (1) can be constrained in the form  $NU = W$ , where  $N \in C^{n+m,n+m}$ ,  $U \in C^{n+m,p}$  and  $W \in C^{n+m,p}$ . In particular

$$N = \begin{pmatrix} O & A \\ A^* & Q \end{pmatrix}, \quad U = \begin{pmatrix} Y \\ X \end{pmatrix}, \quad W = \begin{pmatrix} T \\ Z \end{pmatrix}.$$

3. MAIN RESULTS

We use the following lemmas.

LEMMA 1 : *A necessary and sufficient condition that  $AX = T$  is consistent is that  $AA^-T = T$ .*

LEMMA 2 : *Let  $G = \begin{bmatrix} -H^- & H^-AK^- \\ K^-A^*H^- & K^- - K^-A^*H^-AK^- \end{bmatrix}$  be a parti-*

*tioned matrix in which  $K = Q + A^*A$  and  $H = AK^-A^*$ . Then :*

- (α) *G is a g-inverse of N ;*
- (β) *if  $R(A^*) \subset R(Q)$ , G is a g-inverse of N replacing the expression of K by Q.*

A proof of lemma 1 and lemma 2 is in [19]. But for lemma 2(β) we can give the following alternative proof. The generalized Schur-complement <sup>(1)</sup> of Q reduces to  $N/Q = AQ^-A^*$ , thus according to [14] and [15], G is a g-inverse of N iff the rank is additive on the Schur-complement ; that's true if

$$R(A^*) \subset R(Q)$$

in view of [14, corollary 19.1].

THEOREM 1 : *If system (1) is consistent  $R(Z - QA^-T) \subset R(A^*)$  is n.s. for  $\forall X/AX = T \Leftrightarrow X \in U$ .*

*Proof :* If  $AX = T$  and  $X \in U$ , there exists a solution of  $A^*Y + QA^-T = Z$  for any Z and  $QA^-T$ . Thus in view of lemma 1 :  $R(Z - QA^-T) \subset R(A^*)$ , and vice versa. ■

By straightforward multiplication we obtain :

COROLLARY 1 : *If  $K^-$  and  $H^-$  (respectively  $Q^-$  and  $H^-$ ) in the expression for G in lemma 2(α) (lemma 2(β)) are replaced by  $K_r^-$  and  $H_r^-$  ( $Q_r^-$  and  $H_r^-$ ), G is a reflexive g-inverse of N no further conditions being required.*

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<sup>(1)</sup> For the Schur-complement and other references see [11].

LEMMA 3 : *The set of all solutions of system (1) is given by*

$$Y = H^{-1} AK^{-1}Z - H^{-1}T,$$

$$X = K^{-1}A^*H^{-1}T + (I - K^{-1}A^*H^{-1}A)K^{-1}Z;$$

where  $H$  and  $K$  are defined as in lemma 2.

As far as the uniqueness of solution of system (1) is concerned we state the following.

LEMMA 4 : *System (1) has a unique solution only if  $r(A) = m$  and  $r(Q) \geq n - m$ .*

THEOREM 2 : (a) *A necessary and sufficient condition that system (1) has a unique solution is that : (i)  $r(A) = m$  and  $R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = C^n$ , or what is the same (ii)  $r(A) = m$ ,  $r(Q) \geq n - m$  and  $A$  and  $Q$  are virtually disjoint, or (iii)  $K = (Q + A^*A)$  has full rank.*

(b)  *$r(A) = m$  and  $r(Q) = n$  are n.s. that system (1) has a unique solution iff  $R(A^*) \subset R(Q)$ .*

*Proof of (a) :* The matrix  $N$  is not singular, so its rows are linearly independent hence  $r(A) = m$  and  $R(A) \oplus R(Q^*) = C^n$ . The same for its columns, thus  $R(A^*) \oplus R(Q) = C^n$ . This condition is obviously equivalent to (ii). (iii) follows from lemma 3, and if (iii) holds then (i) holds.

*Proof of (b) :* The matrix  $G$  as defined in lemma 2(b) is the regular inverse of  $N$  with  $R(A^*) \subset R(Q)$ , hence  $H^{-1}$  and  $Q^{-1}$  exist, so that  $r(A) = m$  and  $r(Q) = n$ . For the only if part we consider that if  $r(A) = m$  and  $r(Q) = n$  then  $R(A^*) \subset R(Q)$  since  $m \leq n$  and both  $A$  and  $Q$  have full rank. ■

An alternative proof of theorem 2(b) is in [7, theorem 1].

We point out that theorem 2(a) provides a general statement for the uniqueness of solution of system (1). A particular case of (a), when  $r(Q) = n - m$  is stated in [19, p. 19] when the matrix is  $\begin{pmatrix} A & U \\ V^* & O \end{pmatrix}$ , and  $U$  and  $V$  have the same dimension. Theorem 2 emphasizes that the inverse of a matrix partitioned like in  $N$  <sup>(2)</sup> can be constructed even if  $Q$  is not of full rank (for  $Q$  with full rank see [13, p. 107]), but only  $r(Q) \geq n - m$ . Theorem 2 holds for any  $Q$ .

On the other hand, it is natural to expect some  $g$ -inverse of  $Q$  gets involved in computing the regular inverse of  $N$  whenever  $Q$  is singular just as the regular inverse plays when  $Q$  is not singular. The following lemma clears up this

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(<sup>2</sup>) This result can be extended to the general form  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ .

apparent contradiction by showing how a particular  $g$ -inverse of  $Q$  arises from the formula of lemma 2 under the conditions of theorem 2(a).

LEMMA 5 : Let  $A \in C^{m,n}$  and  $Q \in C^{n,n}$ , if  $r(A) = m$ ,  $(Q + A^* A)^{-1}$  exists and is one choice of  $Q^-$  with maximum rank iff  $A$  and  $Q$  are virtually disjoint,  $R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = C^n$ .

We do not prove this lemma since it follows easily from [19, theorem 2.7.1],

LEMMA 6(a) : Under the conditions of theorem 2(a)

$$G = \begin{bmatrix} O & A_{QO}^{*-} \\ A_{QO}^- & \tilde{Q}^- - A_{QO}^- A \tilde{Q}^- \end{bmatrix}$$

is the regular inverse of  $N$ , where  $A_{QO}^- = \tilde{Q}^- A^* H^-$  is a  $g$ -inverse of  $A$ ,  $H = A \tilde{Q}^- A^*$  and  $\tilde{Q}^-$  is a selected  $g$ -inverse of  $Q$  with maximum rank as defined in lemma 5.

The solution of system (1) is

$$Y = A_{QO}^{*-} Z, \\ X = A_{QO}^- T + (I - A_{QO}^- A) \tilde{Q}^- Z.$$

(b) If theorem 2(b) holds then

$$G = \begin{bmatrix} -H^{-1} & A_{QO}^{*-1} \\ A_{QO}^{-1} & Q^{-1} - A_{QO}^{-1} A Q^{-1} \end{bmatrix}$$

is the regular inverse of  $N$ , where  $A_{QO}^{-1} = Q^{-1} A^* H^{-1}$  is the  $g$ -inverse of  $A$  as defined by [4] and  $H$  is defined in lemma 2(b). The solution of system (1) is

$$Y = A_{QO}^{*-1} Z - H^{-1} T, \\ X = A_{QO}^{-1} T + (I - A_{QO}^{-1} A) Q^{-1} Z.$$

Examples

$$N = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \\ Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad r(A) = 2, \quad r(Q) = 1.$$

It easy to verify that  $R(A^*) \not\subset R(Q)$  and

$$R(A^*) \oplus R(Q) = R(A) \oplus R(Q^*) = R^3,$$

thus  $A$  and  $Q$  are disjoint. The conditions of theorem 2(a) are fulfilled and  $G$  as defined in lemma 6(a) is the regular inverse of  $N$ . Thus  $\tilde{Q}^- = (Q + A^* A)^{-1}$ ,  $H = A\tilde{Q}^- A^*$ ,  $A_{Q0}^- = \tilde{Q}^- A^* H^{-1}$  and by easy computation

$$N^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 \end{bmatrix}.$$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}; \quad A = (1 \quad 0); \quad Q = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix};$$

$$r(A) = 1, \quad r(Q) = 2.$$

In this case  $R(A^*) \subset R(Q)$  and theorem 2(b) holds. Then by lemma 6(b)  $H = A Q^{-1} A^*$  and

$$N^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

**4. OTHER INVERSES OF  $N$**

As stated in lemma 4 system (1) does not have a unique solution whenever  $A \in C^{m,n}$  and  $m > n$ . However we can find other particular solutions when system (1) is possibly inconsistent. A set of equivalent conditions is stated in [18] in order to obtain a  $g$ -inverse minimum norm, least squares or both them for the system  $AX = T$ . We denote these by  $A_m^-, A_1^-, A^+$ : the last one is the Moore-Penrose inverse of  $A$ . Thus we have the following :

**THEOREM 3 :** *Let  $G$  be a partitioned matrix as defined in lemma 2(b),*

(a)  *$G$  is a minimum norm inverse of  $N$  if  $(I - H^- H) A = 0$ ,  $Q^-$  is replaced by  $Q_m^-$  and  $R(A^*) \subset R(Q^*)$ .*

(b)  *$G$  is a least squares inverse of  $N$  if  $Q^-$  is replaced by  $Q_1^-$  and*

$$A^*(I - HH^-) = 0.$$

(c)  $G$  is the Moore-Penrose inverse of  $N$  if  $Q^-$  and  $H^-$  are replaced by  $Q^+$  and  $H^+$  and  $R(A^*) \subset R(Q^*)$ ,  $R(AQ^+) \subset R(H)$  and  $R((Q^+ A^*)^*) \subset R(H^*)$

*Remark* If  $Q$  is Hermitian, then  $G$  is the Moore-Penrose inverse of  $N$  if  $Q^-$  and  $H^-$  are replaced by  $Q^+$  and  $H^+$  and  $R(AQ^+) \subset R(H)$  only

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