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*RAIRO. Analyse numérique*, tome 15, n° 3 (1981), p. 231-235

[http://www.numdam.org/item?id=M2AN\\_1981\\_\\_15\\_3\\_231\\_0](http://www.numdam.org/item?id=M2AN_1981__15_3_231_0)

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## SIMULTANEOUS APPROXIMATION IN NEGATIVE NORMS OF ARBITRARY ORDER (\*)

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Communiqué by J A NITSCHÉ

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Résumé — On améliore un résultat de Bramble et Scott concernant l'approximation optimale dans différentes normes simultanées, dans le cas d'une échelle de Hilbert. On montre la décroissance exponentielle de l'erreur correspondant à la meilleure approximation, dans une norme très faible. Ce résultat entraîne la simultanéité de l'approximation optimale pour des normes négatives d'ordre arbitraire dans l'échelle de Hilbert.

Abstract — We are sharpening a result of Bramble and Scott concerning optimal approximation simultaneously in different norms in the case of a Hilbert scale. We show that in some very weak norm the error of the best approximation decays exponentially. This fact implies optimal approximation simultaneously in negative norms of any order of the Hilbert scale.

### 1. INTRODUCTION

Approximation in negative norms plays an important role in the solution of parabolic partial differential equations. Let us consider a parabolic problem in an abstract form

$$v'(t) = -Av(t), \quad t > 0 \quad (1)$$

with an initial condition

$$v(0) = u, \quad (2)$$

where  $A$  denotes a positive definite selfadjoint operator acting in a Hilbert space  $H$  with norm  $\|\cdot\|$ . To solve this equation by a semidiscrete Galerkin method, one has to choose an approximation  $\varphi$  on the initial element  $u$  in a

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(\*) Reçu le 18 avril 1980

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subspace  $S_h$  from a family of finite dimensional subspaces  $\{ S_h \mid 0 < h \leq 1 \}$  and to solve thereafter an approximate equation

$$\begin{aligned} v'_h(t) &= A_h v_h(t) \\ v_h(0) &= \varphi \end{aligned}$$

in the subspace. The error of the Galerkin approximation splits in two parts

$$v_h(t) - v(t) = (T_h(t) - T(t)) \varphi + T(t) (\varphi - u),$$

where  $T(t)$ ,  $T_h(t)$  denote the semigroups generated by the operators  $A$ ,  $A_h$  respectively. We will consider here only the second term  $T(t) (\varphi - u)$ , which does not depend on the Galerkin method but on the choice of the approximation  $\varphi$  on the initial element  $u$ . This term may be interpreted as the error by solving (1), (2) exactly but with the wrong initial element  $\varphi$ . As it was pointed out by Bramble *et al.* [3], this error depends on the error  $\varphi - u$  in negative norms. Since the most suitable negative norm is not given a priori, it is desirable to get results on approximation simultaneously in different norms.

Bramble and Schatz [2] have shown that a certain order of approximation in one norm of a Banach scale implies a better order in weaker norms. Bramble and Scott [4] improved this result by showing that there exist elements  $\varphi_h$  which achieve simultaneously the optimal order of convergence in a bounded range of the Banach scale.

We are sharpening the result of [4] in two ways. First we show that there are approximations  $\varphi_h$  such that the error  $T(t) (\varphi_h - u)$  in the norm of the Hilbert space  $H$  decays exponentially. This may be interpreted as fast convergence  $\varphi_h \rightarrow u$  in the very weak norm  $\| T(t) \cdot \|$ . As a consequence we get optimal approximation simultaneously in any negative norm.

**2. PRELIMINARIES AND ASSUMPTIONS**

We suppose that  $A$  is a selfadjoint operator with domain  $D(A)$  and lower bound 1, i.e.

$$(Au, u) \geq \| u \|^2 \tag{3}$$

for all  $u \in D(A)$ . Let  $E_\lambda$  denote the spectral resolution of the identity generated by the operator  $A$ . The solution of the equations (1), (2) is given by (Krein [5], pp. 82-84)

$$v(t) = T(t) u = \int_1^\infty e^{-\lambda t} dE_\lambda u, \tag{4}$$

where  $T(t)$ ,  $t \geq 0$  is the semigroup generated by the operator  $A$ .

We define the negative norm of order  $-\alpha \leq 0$  by

$$\|u\|_{-\alpha} = \|A^{-\alpha}u\|, \quad u \in H, \quad (5)$$

where the fractional powers can be defined via the spectral resolution (Krein [5], pp. 127, 128) and with this definition (5) can be written

$$\|u\|_{-\alpha}^2 = \int_1^{\infty} \lambda^{-2\alpha} d(E_{\lambda}u, u). \quad (6)$$

In this way the operator  $A$  generates a Hilbert scale (Lions-Magenes [6], chapter I).

Let  $\{\varphi_h \mid 0 < h \leq 1\}$  be a family of subspaces of  $H$  and suppose the following approximation assumption is valid.

(AA) For each  $u \in D(A)$  there exists a  $\varphi \in S_h$  such that

$$\|\varphi - u\| \leq h \|Au\|. \quad (7)$$

Such assumptions are typical for finite element spaces (Babuska and Aziz [1]). By an appropriate scaling of the parameter  $h$  and the norms (e.g. replacing  $A$  by a fractional power of  $A$ ) and by shift theorems for elliptic operators other cases may be brought in this form.

First we prove a lemma, which is in some sense an analogue of the lemma on p. 941 of [4].

LEMMA 1 : For each  $u \in H$  and  $\varepsilon > 0$  there exists an element  $w \in D(A)$  with

$$\|Aw\| \leq \varepsilon^{-1} \|u\| \quad (8)$$

$$\|u - w\| \leq \|u\| \quad (9)$$

$$\|T(t)(u - w)\| \leq \exp(-t/\varepsilon) \|u\|. \quad (10)$$

*Proof* : The element

$$w = \int_1^{\delta} dE_{\lambda}u$$

with  $\delta = \max(1, \varepsilon^{-1})$  satisfies the requirements as one can see using (4) and the properties of the spectral resolution.

**THEOREM 1 :** *Suppose that the assumption (AA) is fulfilled and let  $t_0 > 0$  fixed. Then for each  $u \in H$  and each  $0 < h \leq 1$  there exists an element  $\varphi \in S_h$  such that for  $0 < t \leq t_0$*

$$\| u - \varphi \| \leq 4 \| u \| \quad (11)$$

$$\| T(t)(u - \varphi) \| \leq 4 \exp(-t/2h) \| u \| . \quad (12)$$

*Proof :* Let

$$E(u) = \inf_{\varphi \in S_h} (\| T(t_0)(u - \varphi) \| + \exp(-t_0/2h) \| u - \varphi \|)$$

and

$$\beta = \sup_{\| u \| \leq 1} E(u) .$$

From the definition of  $\beta$  and the assumption (AA) it follows in a standard way

$$E(w) \leq \beta h \| Aw \| , \quad w \in D(A) . \quad (13)$$

The seminorm property of  $E(u)$  gives with the choice of  $w$  according to lemma 1

$$\begin{aligned} E(u) &\leq E(u - w) + E(w) \\ &\leq \| T(t_0)(u - w) \| + \exp(-t_0/2h) \| u - w \| + \beta h \| Aw \| \\ &\leq (\exp(-t_0/\varepsilon) + \exp(-t_0/2h) + \beta h/\varepsilon) \| u \| . \end{aligned}$$

Putting  $\varepsilon = 2h$  and taking the supremum over all  $\| u \| \leq 1$ , we get

$$\beta = \sup E(u) \leq 4 \exp(-t_0/2h) .$$

This gives the assertion for  $t = t_0$ . By the logarithmic convexity of  $\| T(t) \|$  (Kreĭn [5], p. 128) the estimate (12) follows for  $0 < t \leq t_0$ .

We will now show that the estimates of theorem 1 imply simultaneous approximability in all negative norms.

**THEOREM 2 :** *For each  $0 < h \leq 1$  and each  $u \in H$  there exists an element  $\varphi \in S_h$  such that for all  $\alpha \geq 0$  and  $0 \leq t \leq t_0$*

$$\| \varphi \| \leq 5 \| u \| , \quad (14)$$

$$\| u - \varphi \|_{-\alpha} \leq C(\alpha) h^\alpha \| u \| . \quad (15)$$

*In the case  $u \in D(A)$  one has further*

$$\| u - \varphi \|_{-\alpha} \leq C(\alpha) h^{\alpha+1} \| Au \| . \quad (16)$$

*Proof* : We choose  $\varphi \in S_h$  such that the assertions of theorem 1 are valid for some fixed  $t > 0$ . Then (14) is true. From the obvious inequality

$$\lambda^{-\alpha} \leq \max(\varepsilon^\alpha, \exp(t(\varepsilon^{-1} - \lambda)))$$

for  $\lambda \geq 1$ ,  $\varepsilon > 0$ ,  $t > 0$  we get with the aid of (4) and (6) the estimate

$$\|u\|_{-\alpha} \leq \sqrt{2} \max(\varepsilon^\alpha \|u\|, \exp(t/\varepsilon) \|T(t)u\|) \quad (17)$$

for every  $\varepsilon > 0$ ,  $\alpha \geq 0$ . Choosing  $\varepsilon = 3h$  we get from (11), (12)

$$\|u - \varphi\|_{-\alpha} \leq 4\sqrt{2} \max(3^\alpha h^\alpha, \exp(-t/6h)) \|u\|.$$

Since the exponential term tends faster to zero than  $h^\alpha$  we get (15). The case  $u \in D(A)$  can be handled in a similar way using (13).

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