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ON THE FINITE ELEMENT APPROXIMATION
OF SOLUTIONS FOR RADIATION PROBLEM (*)

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Abstract. — Let \( \Omega \subseteq \mathbb{R}^n \), \( n = 3 \) or \( n = 2 \), be an exterior domain and let \( f \in L_2(\Omega) \) be a finitely supported function. We study a finite element approximation scheme for the solution \( u \) of the problem

\[
\Delta u + k^2 u = f, \quad u|_{\partial\Omega} = 0, \quad \frac{\partial}{\partial r} u - iku \in L_2(\Omega) \quad \text{with} \quad k > 0.
\]

Résumé. — Soit \( \Omega \subseteq \mathbb{R}^n \), \( n = 3 \) ou \( n = 2 \), un domaine extérieur, et soit \( f \in L_2(\Omega) \) une fonction de support fini. On étudie une approximation par éléments finis de la solution \( u \) du problème

\[
\Delta u + k^2 u = f, \quad u|_{\partial\Omega} = 0, \quad \frac{\partial}{\partial r} u - iku \in L_2(\Omega) \quad \text{avec} \quad k > 0.
\]

Let \( \Omega \) denote an exterior domain in \( \mathbb{R}^n \), \( n = 3 \) or \( n = 2 \). Given a finitely supported function \( f \in L_2(\Omega) \) and a number \( k > 0 \) the radiation problem

\[
\begin{aligned}
\Delta u_0 + k^2 u_0 &= f, \quad \varphi u_0 \in \tilde{H}_1(\Omega), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n) \\
\frac{\partial u_0}{\partial r} - iku_0 &\in L_2(\Omega)
\end{aligned}
\]

has an unique solution \( u_0 \) [11], [18]. Because of the condition \( \frac{\partial u_0}{\partial r} - iku_0 \in L_2(\Omega) \) we shall, by convention, say that \( u_0 \) is outgoing. Actually the condition for \( f \) can be weakened; it suffices to assume \((1 + |x|) f \in L_2(\Omega)\), [11]. Also the operator \( \Delta \) can be replaced by a more general second order operator with variable

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coefficients. With regard to the higher order operators we refer to [24]. We consider a finite element scheme for the approximation of the solution $u_0$. The approximation of the solutions of elliptic equations in unbounded domains in general meets difficulties, which do not occur in the case of bounded domains. If the finite element method is considered there are not many articles which deal with such problems. In the paper [7] a whole space problem is reduced to an infinite number of algebraic equations. In [2] the equation $\Delta u - u = f$ is considered in the whole space and the method, as pointed out, is evidently also applicable in exterior domains. However there is a significant difference between an exterior domain and a bounded domain. The solution $u_0$ of (1) can no longer be obtained by an inversion of a compact operator arising in damped problem. Accordingly this way, in contrast to the case of bounded domains [22], is lost in the approximation of $u_0$. We use an approximation of $u_0$, which in a natural way comes from the theory of existence of radiation solutions. Specifically, the well known limiting absorption principle says that the solution $u_0$ can be obtained as the limit of the solutions $w_\varepsilon$ of the problems

$$
\left\{ \begin{array}{l}
[\Delta + (k + i\varepsilon)^2] u_\varepsilon = f, \quad \varepsilon > 0 \\
u_\varepsilon \in H_1^0(\Omega)
\end{array} \right.
$$

(2)

as $\varepsilon \to 0$. This is the basic idea in proving the existence of the solution for the radiation problems. It has been used in a great number of articles from [6] to [24]. The finite element approximation $u_h$, which we are going to use is defined as follows. Take an increasing sequence of the numbers $R = R(h) \to \infty$ as well as a decreasing sequence $\varepsilon = \varepsilon(h) \to 0$ with $h \to 0$. For every $h$ we use a suitable finite dimensional (complex) trial subspace $S_h \subset H_1^0(\Omega(R))$, $\Omega(R) = \{ x \in \Omega \mid |x_i| < R, 1 \leq i \leq n \}$. The approximation $u_h \in S_h$ is defined by

$$
(\nabla u_h \mid \nabla \varphi) - (k + i\varepsilon)^2 (u_h \mid \varphi) = -(f \mid \varphi) \quad \forall \varphi \in S_h.
$$

(3)

Depending on the choice of the subspaces $S_h$ and the sequences $\varepsilon(h), R(h)$ various approximation results for the differences $\| u_0 - u_h \|_{0,K}, \| \nabla (u_0 - u_h) \|_{0,K}$ over bound sets $K \subset \Omega$ are obtained. For example, if $n = 2$ and if the boundary $\Gamma = \partial \Omega$ is smooth or polygonal, then a choice leads to the error estimate

$$
\| u - u_h \|_{0,K} \leq c(K) h^{2/3} \| f \| \quad (4i)
$$

and an other choice to

$$
\| u - u_h \|_{1,K} \leq c(K) h^{1/3} \| f \| . \quad (4ii)
$$

It is perhaps worth of noticing that our convergence results are obtained only, when the rates of the convergences $R(h) \to \infty, \varepsilon(h) \to 0$ are suitable related.
For the approximation of solutions for radiation problems using integral equations we refer to [3], [9], [16], [17] in the case of smooth boundaries and to [21] in the case of a non-smooth boundary. Another approximation can be found in [25]. There exist also articles, which use an approach based on Neumann-expansions [14].

Let us fix some notations. Besides of the Euclidean norm $|x|$ the maximum norm $\|x\| = \max \{ |x_i| \mid 1 \leq i \leq n \}$ is needed. Define $Q(R) = \{ x \mid \|x\| < R \}$, $\Omega(R) = \Omega \cap Q(R)$, $\Gamma(R) = \partial Q(R)$. Since $f$ is assumed to be finitely supported, its support lies in $\Omega(R_0)$ for a fixed number $R_0 > 0$. We take $R_0$ so large that $d = d(\Gamma, \Gamma(R_0)) > 1$ is satisfied, $\Gamma = \partial \Omega$. The only requirement for the subspaces $S_h$, $0 < h \leq h_0$, enters in the following assumption. Let $v(h) \in H_1(\Omega(R(h)))$ be the solution of the Dirichlet problem

$$\Delta v(h) - v(h) = g, \quad g \in L_2(\Omega(R(h))) \tag{5}$$

and let $v(h) \in S_h$ be the approximation of $v(h)$:

$$(\nabla v(h) \mid \nabla \varphi) + (v(h) \mid \nabla \varphi) = -(g \mid \varphi), \quad \forall \varphi \in S_h. \tag{6}$$

**Assumption 1**: There exist a constant $c$ and numbers $k(l), \ l = 0, 1, 0 < k(1) \leq k(0) < \infty$, such that

$$\| v(h) - v(h)_h \|_{L_2(\Omega(R(h))} \leq c h^{k(l)} \| g \|_{L_0, \Omega(R(h))} \tag{7}$$

for every $g \in L_2(\Omega(R(h)))$, $0 < h \leq h_0$.

We now give some examples where this condition is satisfied. It is of course essential in (7) that the constant $c$ is independent of the radius $R(h)$. Roughly speaking the constant $c$ comes from the regularity theorems for the problem (5). In giving examples of (7) the next lemma is useful. In the following $c$ denotes a generic constant, independent of the functions occurring and of the parameters $h, R, \varepsilon$.

**Lemma 1**: If $v \in H_1(\Omega(R))$ is the solution of $\Delta v - v = g, \ g \in L_2(\Omega(R))$, then $v \in H_2(\Omega(R))$ and

$$\| v \|_{2, \Omega(R)} \leq c \| g \|_{0, \Omega(R)} \tag{8}$$

for every $R > 0$.

**Proof**: Because $Q(R)$ is convex the result $v \in H_2(\Omega(R))$ follows from [12]. The equation $\Delta v - v = g, \ v \in H_1(\Omega(R))$ implies

$$\| v \|_{1, \Omega(R)} \leq \| g \|_{0, \Omega(R)} \tag{9}$$

vol. 17, no 2, 1983
Further according to [12]

\[ \| \mathbf{w} \|_{2,Q(1)} \leq c_0 \| \Delta \mathbf{w} \|_{0,Q(1)}, \]  

(10)

when \( \mathbf{w} \in \hat{H}_1(Q(1)) \), \( \Delta \mathbf{w} \in L_2(Q(1)) \). Applying (10) to \( \mathbf{w}(x) = v(Rx) \) the inequality:

\[ \| \mathbf{v} \|_{2,Q(R)}^2 = \sum_{|\alpha|=2} \| \partial^\alpha \mathbf{v} \|_{0,Q(R)}^2 = R^{-4} \| \mathbf{w} \|_{2,Q(1)}^2 \]

\[ \leq c_0^2 R^{-4} \| \Delta \mathbf{w} \|_{0,Q(1)}^2 = c_0^2 \| \Delta \mathbf{v} \|_{0,Q(R)}^2 \]

\[ \leq 2 c_0^2 (\| \mathbf{v} \|_{0,Q(R)}^2 + \| \mathbf{g} \|_{0,Q(R)}^2) \leq 4 c_0^2 \| \mathbf{g} \|_{0,Q(R)}^2 \]  

(11)

is obtained. Thus (9), (11) imply (8). \( \square \).

**Lemma 2**: Let the boundary \( \Gamma \) be smooth; \( \Gamma \in C^2 \). If \( \mathbf{v} \in \hat{H}_1(\Omega(R)) \) is the solution of \( \Delta \mathbf{v} = \mathbf{v}, \mathbf{g} \in L_2(\Omega(R)) \), then \( \mathbf{v} \in H_2(\Omega(R)) \) and

\[ \| \mathbf{v} \|_{2,\Omega(R)} \leq c \| \mathbf{g} \|_{0,\Omega(R)} \]  

(12)

for every \( R \geq R_0 \).

The proof of Lemma 2 is obvious. It uses Lemma 1 and a regularity result for bounded domains with \( C^2 \)-boundaries ([8]: Theorem 8.13).

Using the above lemma we can give an explicit example of (7) in the case of a smooth boundary:

**Example 1**: Let \( n = 2 \), \( \Gamma \in C^2 \). Let \( \mathcal{G}_h \) be a family of regular triangulations of the domain \( \Omega(R(h)) \). (For this notation see e.g. [4], [23].) Near the boundary curved elements are used [26], [27]. If \( S_h \) denotes the trial subspace of continuous piecewise linear functions (except over the curved triangles) which vanish on the nodes of the triangulation lying on the boundary of \( \Omega(R) \), then the error estimate

\[ \| \mathbf{v}(h) - \mathbf{v}(h)_h \|_{1,\Omega(R(h))} \leq c h^{2-1} \| \mathbf{g} \|_{0,\Omega(R(h))} \]

(13)

is valid. For the proof of (13) see [26: Theorem 3]. That \( c \) is independent of \( R \) is a consequence of Lemma 2.

**Example 2**: Let \( n = 2 \), \( \Gamma \) polygonal. The accuracy (13) is obtained, if one uses the trial subspaces as in example 1 (without the curved elements) such that appropriate singular elements in the neighbourhood of the vertices of \( \Gamma \) are added to \( S_h \). See [23], [15].

We will now discuss the error \( u_0 - u_h \). The rate of the convergence \( u_\epsilon \to u_0 \) must first be studied (although the notations \( u_\epsilon, u_h \) are formally the same, there

R.A.I.R.O. Analyse numérique/Numerical Analysis
should be no possibility to a confusion). In the articles which use the limiting absorption principle it has been proved that \( u_\varepsilon \to u_0 \) in \( H_1(\Omega(R)) \) for every \( R \). However, all the existing proofs are, as far as we know, theoretical; results for the rate of convergence do not seem to exist. In the following the idea of Phillips in [20] is crucial. According to [20] the solution \( u_0 \) as well as \( u_\varepsilon \) can be represented as a perturbation of a corresponding whole space solution. On the other hand, the rate of the convergence \( u_\varepsilon \to u_0 \) can, in the whole space case, easily be seen from the behavior of the fundamental solution. It was assumed in [20] that the boundary \( \Gamma \) was smooth; \( \Gamma \in C^2 \). However, such strong requirements cannot not be used if domains with polygonal boundaries are to be considered. Therefore we treat a slightly modified form of the discussion in [20] in some detail. We assume only that the domain \( \Omega \) has the segment property [1]. For \( \zeta = k + i\varepsilon \), \( 0 \leq \varepsilon \leq \varepsilon_0 \), we consider the fundamental solution

\[
S_\zeta(x, y) = S_\zeta(|x - y|) = a(|x - y|^{-1})^2 H_{n-2}^1(\zeta |x - y|)
\]

of the equation

\[
(\Delta + \zeta^2) u = 0.
\]

The constant \( a \) in (14) is independent of \( \zeta \); in fact \( a = -i 4^{-1}(2\pi)^{\frac{n}{2} - n/2} \). The function \( H_{n-2}^1 \) is the Hankel function of first kind and of order \( \nu \) [19: p. 66]. The principal properties of these functions is discussed in [19]. In particular when \( n = 3 \) the formula (14) becomes

\[
S_\zeta(x, y) = -\frac{1}{4\pi} e^{i|\zeta - y|},
\]

and for \( n = 2 \) we get

\[
S_\zeta(x, y) = -\frac{i}{4} H_0^1(\zeta |x - y|).
\]

For the dimensions \( n = 2, 3 \) the fundamental solutions (14) have a square integrable singularity at \( y = x \). In the case \( n = 2 \) the singularity takes the form

\[
S_\zeta(x, y) = \frac{1}{2\pi} (\log \zeta |x - y|) J_0(\zeta |x - y|) + F_0(\zeta |x - y|)
\]

with the Bessel function \( J_0(z) \) and with an entire function \( F_0(z) \). Our choice represents the outgoing case for \( \varepsilon = 0 \). For \( \varepsilon > 0 \) the function \( S_\zeta(|x - y|) \) converges exponentially to zero as \( |x - y| \to \infty \); for the asymptotic pro-
Properties of the Hankel function with large arguments see [19 : p. 139], [5 : p. 524-526]. If \( g \in L_2(\mathbb{R}^n) \) has a compact support \( \text{supp} \ g \subset \Omega(R_0) \), then the equation

\[
  u_\zeta^0(x) = (R_\zeta^0 \ g)(x) = \int_{\mathbb{R}^n} S_\zeta(| x - y |) \ g(y) \ dy
\]

(18)
defines the (unique) whole-space solution \( u_\zeta^0 \) of the equation

\[
  (\Delta + \zeta^2) u_\zeta^0 = g
\]

(19)
such that \( u_\zeta^0 \in \tilde{H}_1(\mathbb{R}^n), \ \varepsilon > 0 \), and such that \( u_\zeta^0 \) is outgoing for \( \varepsilon = 0 \).

From (16), (17) follows that, if \( \varepsilon_1 > 0 \) is fixed, then

\[
  \left| S_\zeta(x, y) \right| \leq c(R) \ | x - y |^{-(n-1)/2}
\]

for all \( \zeta = k + i\varepsilon, \ 0 < \varepsilon \leq \varepsilon_1 \), and for all \( x, y \in Q(R) \ | x \neq y, \ R > 0 \). Thus, we get by (18) for \( R > R_0, \ x \in \Omega(R_0) \)

\[
  \left| u_\zeta^0(x) \right|^2 \leq c_1(R) \left( \int_{\Omega(R_0)} \ | x - y |^{-(n-1)} \ dy \right) \ | g |^2
\]

\[
  \leq c_2(R) \ | g |^2
\]

with \( \| g \| := \| g \|_{0,\Omega(R_0)} \) where the integral is estimated by means of [10], p. 161 Satz. Accordingly, we have

\[
  \| u_\zeta^0 \|_{0,\Omega(R)} \leq c(R) \ | g |.
\]

(20)

Since it holds

\[
  \Delta u_\zeta^0 = g - \zeta^2 u_\zeta^0,
\]

we have by the interior regularity result [1 : Theorem 6.3] that \( u_\zeta^0 \in H^{1,\text{loc}}(\mathbb{R}^n) \) and that

\[
  \| u_\zeta^0 \|_{2,\Omega(R)} \leq c(R) \left( \| g - \zeta^2 u_\zeta^0 \|_{0,\Omega(R+1)} + \| u_\zeta^0 \|_{0,\Omega(R+1)} \right)
\]

which yields by (20)

\[
  \| u_\zeta^0 \|_{2,\Omega(R)} \leq c(R) \ | g |,
\]

(21)

for \( 0 < \varepsilon \leq \varepsilon_1 \).

Let \( v \) be the solution of

\[
  \begin{cases}
  \Delta v - iv = 0, \\
  v|_\Gamma = \xi, \quad \xi \in H^{1,\text{loc}}(\mathbb{R}^n) \\
  v|_{\Gamma(R_0)} = 0
  \end{cases}
\]

(22)
in the following sense: Take $\delta = d/3$ and define $U(\delta) = \{ x \in \mathbb{R}^n \mid d(x, \Omega^c) < \delta \}$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a fixed smoothing function such that $\psi(x) = 1$ for $x \in U(\delta)$ and $\psi(x) = 0$, $x \in U(2 \delta)^c$. Let $w \in H_1(\Omega(R_0))$ be the solution of
\[
(\nabla w \mid \nabla \varphi) + i(w \mid \varphi) = ((\Delta - i) \psi \xi \mid \varphi), \quad \forall \varphi \in H_1(\Omega(R_0)),
\]
and define $v = w + \psi \xi \in H_1(\Omega(R_0))$. The mapping $Q : H_2^{\text{loc}}(\mathbb{R}^n) \to H_1(\Omega(R_0)) \cap H_2^{\text{loc}}(\Omega(R_0))$, $Q \xi = v$, is linear and satisfies
\[
\| Q \xi \|_{1,\Omega(R_0)} + \| Q \xi \|_{2,\Omega(R_0)} \setminus \{ u \mid u \neq v \} \leq c \| \xi \|_{2,\Omega(R_0)}.
\]
Define $v_\zeta = Q R_0^0 g$. According to (21), (24) the estimate
\[
\| v_\zeta \|_{1,\Omega(R_0)} + \| \nabla \psi \cdot \nabla v_\zeta \|_{1,\Omega(R_0)} \leq c \| g \|
\]
is valid. The formula
\[
T_\zeta g = 2 \nabla \psi \cdot \nabla v_\zeta + v_\zeta \Delta \psi + (\zeta^2 + i) \psi v_\zeta
\]
defines a linear operator $T_\zeta : L_2(\Omega(R_0)) \to L_2(\Omega(R_0))$. Because of (25) and the segment property the operator $T_\zeta$ is even compact. Suppose that $1 - T_\zeta$ has the inverse $(1 - T_\zeta)^{-1}$. If $g := (1 - T_\zeta)^{-1} f$ and if $u'_\zeta := (1 - \psi Q) R_0^0 g$, then one can verify that $u'_\zeta$ is a solution of (2), $\epsilon > 0$, and $u'_\zeta$ is a solution of (1) for $\epsilon = 0$. The uniqueness of solution to (1) and (2) indicates that the solution $u_\zeta$ has the representation
\[
u_\zeta = (1 - \psi Q) R_0^0 (1 - T_\zeta)^{-1} f.
\]
Take $\epsilon = 0$. The existence $(1 - T_\zeta)^{-1}$ is seen as in [19] and we omit it. In the proof of the following theorem we will see that $T_{k+ie} : T_k, \epsilon \to 0$. Therefore, the formula (27) also holds for $\zeta = k + ie, 0 \leq \epsilon \leq \epsilon_1$.

**Theorem 4:** For every $R \to 0$ there exists a number $c(R) \to 0$ such that
\[
\| u_0 - u_\epsilon \|_{1,\Omega(R)} \leq c(R) \| f \|
\]
\[
0 \leq \epsilon \leq \epsilon_1.
\]

**Proof:** For $| x |, | y | \leq R$ we have
\[
| S_{k+ie}(| x - y |) - S_k(| x - y |) | \leq \begin{cases} c(R) \ln | x - y | + 1, & n = 2, \\ c(R) | x - y |^{-1}, & n = 3. \end{cases}
\]
\[
\text{(29)}
\]

vol. 17, no 2, 1983
The representation (18) leads to the estimate
\[ \| u_0^0 - u_\varepsilon^0 \|_{0,Q(R)} \leq c(R) \| g \|. \] (30)

Since
\[ \Delta(u_0^0 - u_\varepsilon^0) = k^2(u_\varepsilon^0 - u_0^0) + \varepsilon(2ik - \varepsilon)u_\varepsilon^0, \]
the interior regularity [1: Theorem 6.3] implies that
\[ \| u_0^0 - u_\varepsilon^0 \|_{2,Q(R)} \leq c(R) \left( \| u_\varepsilon^0 - u_0^0 \|_{0,Q(2R)} + \varepsilon \| u_\varepsilon^0 \|_{0,Q(2R)} \right) \]
\[ \leq \varepsilon c(R) \| g \|. \] (31)

From
\[ (T_{k+\varepsilon} - T_k)g = 2\nabla\psi \cdot \nabla(v_{k+\varepsilon} - v_k) + \Delta\psi \cdot (v_{k+\varepsilon} - v_k) + \]
\[ + ((k + i\varepsilon)^2 + i)\psi v_{k+\varepsilon} - (k^2 + i)\psi v_k \]
we get using (25), (31)
\[ \| (T_{k+\varepsilon} - T_k)g \| \leq \varepsilon c \| g \| \] (32)
and in particular \( T_{k+\varepsilon} \rightarrow T_k \) with respect of the operator norm as \( \varepsilon \rightarrow 0 \).

Now, the formula
\[ u_\varepsilon - u_0 = (1 - \psi Q) \left[ R_{k+\varepsilon}^0 (1 - T_{k+\varepsilon})^{-1} - R_k^0 (1 - T_k)^{-1} \right] f \] (33)
is true for \( 0 \leq \varepsilon \leq \varepsilon_1 \). Because of the continuity of the inverse the inequality
\[ \| (1 - T_{k+\varepsilon})^{-1} - (1 - T_k)^{-1} \| \leq c \varepsilon \] (34)
is obtained. The rest of the proof follows in a straightforward manner from (33) using (34), (31) and (25).

Our next step is to discuss the difference of \( u_\varepsilon \) and \( u_\varepsilon^R \) where \( u_\varepsilon^R \) is the solution of the Dirichlet problem
\[ \begin{aligned} \Delta u_\varepsilon^R + (k + i\varepsilon)^2 u_\varepsilon^R &= f, \\ u_\varepsilon^R \in H_1(\Omega(R)). \end{aligned} \] (35)

For this purpose the following bound is needed:

**Lemma 3**: The solution \( u_\varepsilon \) obeys the estimate
\[ |u_\varepsilon(x)| + ||\nabla u_\varepsilon(x)|| \leq c |x|^{\frac{n-1}{2}} e^{-\frac{1}{2}\varepsilon|x|} \| f \|. \] (36)

R.A.I.R.O. Analyse numérique/Numerical Analysis
for \( |x| \geq 2 R_1, R_1 = \sqrt{n} R_0, 0 \leq \varepsilon < \varepsilon_1. \)

**Proof**: If \( y \in \Omega(R_0), \) then \( |y| \leq \sqrt{n} \| y \| \leq \sqrt{n} R_0 = R_1. \) For \( |x| \geq 2 R_1 \) we obtain \( |x - y| \geq \frac{1}{2} |x|. \) Since

\[
 u_c(x) = (R^D_{k+ic}(1 - T_{k+ic})^{-1} f)(x)
\]

for \( x \notin \Omega(R_0), \) the estimate

\[
 |u_c(x)| + |\nabla u_c(x)| \leq \int_{\Omega(R_0)} (|S_{k+ic}(|x - y|)| + |\nabla_y S_{k+ic}(|x - y|)|, (1 - T_{k+ic})^{-1} f(y)| dy \quad (37)
\]

is valid. Recalling the asymptotic formula of the Hankel functions for large arguments \([19: p. 139], \) (see also \([5: p. 524-526]\)) as well as the recurrence relations for the derivatives \([19: p. 67], \) we find that for \( |x| \geq 2 R_1 \)

\[
 |S_{k+ic}(|x - y|)| + |\nabla_y S_{k+ic}(|x - y|)| \leq c |x|^{-\frac{n-1}{2}} e^{-\frac{1}{2} \varepsilon |x|}. \quad (38)
\]

The formulae (37), (38), (34) lead to (36). \( \square \)

The following lemma holds for all open sets \( \Omega, \) bounded or not. The proof is simple, and will be omitted.

**Lemma 4**: Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( \Omega \neq \emptyset. \) Assume that \( k > 0, \varepsilon > 0. \) The equation

\[
 \begin{cases}
 \Delta u_c + (k + ic)^2 u_c = f, & f \in L_2(\Omega), \\
 u_c \in \hat{H}^1(\Omega)
\end{cases}
\]

has the unique solution \( u_c, \) and the estimate

\[
 \|u_c\|_{1,\Omega} \leq \varepsilon^{-1} c(k) \|f\|_{0,\Omega} \quad (39)
\]

is valid.

*We are now ready to establish*:

**Theorem 2**: The difference \( u_c - u^R_c \) obeys the estimate

\[
 \|u_c - u^R_c\|_{1,\Omega(\frac{R}{2})} \leq c \varepsilon^{-1} R^{\frac{n-3}{2}} e^{-\frac{1}{4} \varepsilon R} \|f\|, 
\]

\( 0 < \varepsilon \leq \varepsilon_1, R \geq 4 R_1. \)
Proof : Fix a smoothing function \( \varphi \in C^\infty(\mathbb{R}^n) \) such that \( \varphi(x) = 1, 3/4 \leq ||x||, \varphi(x) = 0, ||x|| \leq 1/2 \). Take \( \varphi_R(x) = \varphi(R^{-1}x), \psi_R(x) = 1 - \varphi_R(x) \). The function \( \psi_R \) is identically one in \( Q(R_0) \) and vanishes in a neighbourhood of the boundary \( \Gamma(R) \). Since \( w := \psi_R u_e - u_e^R \in \tilde{H}^1(\Omega(R)) \), we conclude from (39)

\[
\|w\|_{1,\Omega(R)} \leq c e^{-1} \|u_e\|_{1,\Omega(R)} + 2 \nabla\psi_R \cdot \nabla u_e \|_{0,\Omega(R)} = c e^{-1} \|u_e\|_{1,\Omega(R)} + 2 \nabla\psi_R \cdot \nabla u_e \|_{Q(\frac{1}{2}R, \frac{3}{4}R)} \ .
\]  

(41)

We have \( |\partial^a \varphi_R(x)| \leq M(||x||) R^{-|a|} \) for every multi-indices \( a \). Since

\[
| x | \geq ||x|| \geq \frac{1}{2} R \geq 2 R_1 \quad \text{in} \quad Q\left(\frac{1}{2} R, \frac{3}{4} R\right)
\]

the inequality

\[
\|w\|_{1,\Omega(R)} \leq c e^{-1} R^{\frac{n-3}{2}} e^{-\frac{1}{4}e^R} \|f\|
\]  

(42)

is obtained from (41). Because \( \psi_R(x) \equiv 1, x \in \Omega(\frac{1}{2} R) \), the estimate (40) follows. \[ \square \]

As the final step we have

**Theorem 3** : Let \( \varepsilon = \varepsilon(h) = h^\delta, 0 < \delta < k(l) \). For sufficiently small \( h \) the estimate

\[
\|u_e^R - (u_e^R)_h\|_1 \leq c h^{k(l) - 2\delta} \|f\| ,
\]  

(43)

\( R \geq R_0 \), is valid.

**Proof** : Denote by \( K(R) \) the solution operator \( K(R) = (\Delta_0 - 1)^{-1} : L^2(\Omega(R)) \rightarrow \tilde{H}^1(\Omega(R)) \), where \( \Delta_0 \) means the Laplacian with the homogeneous Dirichlet boundary condition. The equation (35) is then equivalent to

\[
(I + [1 + (k + i\varepsilon)^2] K(R)) u_e^R = K(R) f, \quad u_e^R \in L^2(\Omega(R)) \ .
\]  

(44)

In the same way, if \( K_h(R) \) is the solution operator \( K_h(R) : L^2(\Omega(R)) \rightarrow S_h \)

defined by \( K_h(R) f = v_h, \)

\[
(\nabla v_h \cdot \nabla \varphi) + (v_h \cdot \varphi) = -(f \cdot \varphi), \quad \forall \varphi \in S_h ,
\]

then the equation

\[
(\nabla(u_e^R)_h \cdot \nabla \varphi) - (k + i\varepsilon)^2 ((u_e^R)_h \cdot \varphi) = -(f \cdot \varphi), \quad \forall \varphi \in S_h ,
\]

R.A.I.R.O. Analyse numérique/Numerical Analysis
is equivalent to

\[(I + [1 + (k + i\varepsilon)^2] K_h(R)) (u^0_h) = K_h(R) f, \quad (u^0_h) \in L_2(\Omega(R)).\]  

(45)

For brevity we write

\[U = I + [1 + (k + i\varepsilon)^2] K(R),\]
\[U_h = I + [1 + (k + i\varepsilon)^2] K_h(R).\]

The operator \(K\) is compact with respect both of the norms \(\| . \|_{L_2(\Omega(R))}, \quad l = 0, 1\) (for \(l = 1\) see [22]). Let \(\| . \|_h, \quad l = 0, 1\) be the operator norm in \(L_2(\Omega(R))\) for \(l = 0\) and in \(H^1(\Omega(R))\) for \(l = 1\). Since \(U\) is one-to-one, the inverse exists. According to the Assumption 1

\[\| U - U_h \|_l \leq ch^{h(l)}, \quad l = 0, 1. \]  

(46)

Therefore, the inverse \(U_h^{-1}\) exists if \(h\) is sufficiently small. Moreover, we get from a Neumann-expansion

\[\| U^{-1} - U_h^{-1} \|_l \leq \| U^{-1}(U - U_h) \|_l (1 - \| U^{-1}(U - U_h) \|_l)^{-1} \| U^{-1} \|_l \]

(47)

if

\[\| U^{-1}(U - U_h) \|_l < 1. \]  

(48)

To obtain the inequality (48) an estimate for the norm \(\| U^{-1} \|_j\) is needed. Let us first consider the case \(l = 0\). Define \(\mu = 1 + (k + i\varepsilon)^2\). Because \(K(R)\) is selfadjoint in \(L_2(\Omega(R))\) the inequality

\[\| U^{-1} \|_0 = \| \mu^{-1} (\mu^{-1} + K(R))^{-1} \|_0 \leq \| \mu^{-1} \| \| \operatorname{Im} \mu^{-1} \|^{-1} \leq ce^{-1} = ch^{-\delta}\]

(49)

is true [13 : p. 272]. Since \(\delta < k(l) \leq k(0)\) the inequality (48) \((l = 0)\) is satisfied if \(h\) is small enough. From (47) we then have

\[\| U^{-1} - U_h^{-1} \|_0 \leq ch^{h(0) - 2\delta}. \]  

(50)

The norm \(\| U^{-1} \|_1\) can be estimated as follows. Let \(u, v \in H^1(\Omega(R))\) and let \(v = U^{-1} u\). Then

\[(\Delta_0 - 1)(u - v) = (1 + (k + i\varepsilon)^2) v.\]
According to (49) this implies that
\[
\| u - v \|_{1,\Omega(R)} \leq (1 + |k + |e| |^2) \| v \|_{0,\Omega(R)} \\
\leq ch^{-\delta} \| u \|_{0,\Omega(R)}.
\]
Hence
\[
| v \|_{1,\Omega(R)} \leq | u \|_{1,\Omega(R)} + | u - v \|_{1,\Omega(R)} \leq ch^{-\delta} | u \|_{1,\Omega(R)}.
\]
Therefore the bound (49) is valid for the norm | U^{-1} |, too. By analogy with (50) the inequality
\[
| U^{-1} - U_h^{-1} | \leq ch^{k(l)-2\delta}
\]
is obtained if \( \delta < k(1) \). Finally,
\[
| u^R - (u^R)_h | \leq | U^{-1}(K(R) f - U_h^{-1}(K_h(R) f)) | \\
\leq | (U^{-1}((K(R) - K_h(R)) f)) | + | (U^{-1} - U_h^{-1}) K_h(R) f | \\
\leq ch^{k(l)-2\delta} | f |.
\]
We now choose \( R(h) = h^{-(\delta + a)}, a > 0 \). Write \( u_h = (u^R)_h \). If \( R_2 \geq R_1 \) is fixed, we have \( \Omega(R_2) \subset \Omega \left( \frac{1}{4} R(h) \right) \) for sufficiently small \( h \). Then Theorems 1-3 imply that
\[
| u_0 - u_h |_{1,\Omega(R_2)} \leq c(R_2) \left[ h^\delta + h^{-\delta} h^{a-h^{-a}} + h^{k(l)-2\delta} \right] | f |.
\]
if \( 0 < \delta < k(l) \). If \( h \) is small enough ; \( 0 < h \leq h_1 \) (where \( h_1 \) depends on the choice of \( \delta, a \), then the middle term has the upper bound \( h^{k(l)} \). Therefore
\[
| u_0 - u_h |_{1,\Omega(R_2)} \leq c(R_2) (h^\delta + h^{k(l)-2\delta}) | f |.
\]
The best error bound
\[
| u_0 - u_h |_{1,\Omega(R_2)} \leq c(R_2) h^{\frac{1}{2} k(l)} | f |
\]
is achieved by choosing \( \delta = k(l)/3 \).
THEOREM 4: Let the Assumption 1 be satisfied. If \( u_h = (u^{R(h)})_h \), where 
\[
\varepsilon(h) = h^{\sigma/3},
\]
\[
R(h) = h^{-\left(\frac{\sigma}{3} + \varepsilon\right)}, \quad \alpha > 0,
\]
then the error estimate \((54)\) is valid.

We note once more that it is a different choice of \( \varepsilon(h), R(h) \), which gives the best bound for the error with respect of the \( \| \cdot \|_{0,K} \) and \( \| \cdot \|_{1,K} \) norms. For example in the Examples 1 and 2 the choice \( \delta = 2/3 \) gives the best error bound
\[
\| u_0 - u_h \|_{0,K} \leq c(K) h^{2/3} \| f \|
\]
for the \( \| \cdot \|_{0,K} \) norm but no convergence with respect of the \( \| \cdot \|_{1,K} \) norm. On the other hand \( \delta = 1/3 \) gives the best bound
\[
\| u_0 - u_h \|_{1,K} \leq c(K) h^{1/3} \| f \|
\]
with respect of the \( \| \cdot \|_{1,K} \) norm, but no better estimate with respect of the \( \| \cdot \|_{0,K} \) norm.

BIBLIOGRAPHY


R.A.I.R.O. Analyse numérique/Numerical Analysis