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ON THE FINITE ELEMENT APPROXIMATION OF SOLUTIONS FOR RADIATION PROBLEM (*)

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Abstract. — *Let $\Omega \subset \mathbb{R}^n$, $n = 3$ or 2, be an exterior domain and let $f \in L_2(\Omega)$ be a finitely supported function. We study a finite element approximation scheme for the solution u of the problem*

$$\Delta u + k^2 u = f, \quad u|_{\partial\Omega} = 0, \quad \frac{\partial}{\partial r} u - iku \in L_2(\Omega) \quad \text{with} \quad k > 0 .$$

Résumé. — *Soit $\Omega \subset \mathbb{R}^n$, $n = 3$ ou 2, un domaine extérieur, et soit $f \in L_2(\Omega)$ une fonction de support fini. On étudie une approximation par éléments finis de la solution u du problème*

$$\Delta u + k^2 u = f, \quad u|_{\partial\Omega} = 0, \quad \frac{\partial}{\partial r} u - iku \in L_2(\Omega) \quad \text{avec} \quad k > 0 .$$

Let Ω denote an exterior domain in $\mathbb{R}^n$, $n = 3$ or $n = 2$. Given a finitely supported function $f \in L_2(\Omega)$ and a number $k > 0$ the radiation problem

$$\left.\begin{array}{l} \Delta u_0 + k^2 u_0 = f, \quad \varphi u_0 \in \overset{\circ}{H}_1(\Omega), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n) \\ \dfrac{\partial u_0}{\partial r} - iku_0 \in L_2(\Omega) \end{array}\right\} \tag{1}$$

has an unique solution u_0 [11], [18]. Because of the condition $\frac{\partial u_0}{\partial r} - iku_0 \in L_2(\Omega)$ we shall, by convention, say that u_0 is outgoing. Actually the condition for f can be weakened; it suffices to assume $(1 + |x|) f \in L_2(\Omega)$, [11]. Also the operator Δ can be replaced by a more general second order operator with variable

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coefficients. With regard to the higher order operators we refer to [24]. We consider a finite element scheme for the approximation of the solution u_0. The approximation of the solutions of elliptic equations in unbounded domains in general meets difficulties, which do not occur in the case of bounded domains. If the finite element method is considered there are not many articles which deal with such problems. In the paper [7] a whole space problem is reduced to an infinite number of algebraic equations. In [2] the equation $\Delta u - u = f$ is considered in the whole space and the method, as pointed out, is evidently also applicable in exterior domains. However there is a significant difference between an exterior domain and a bounded domain. The solution u_0 of (1) can no longer be obtained by an inversion of a compact operator arising in damped problem. Accordingly this way, in contrast to the case of bounded domains [22], is lost in the approximation of u_0. We use an approximation of u_0, which in a natural way comes from the theory of existence of radiation solutions. Specifically, the well known limiting absorption principle says that the solution u_0 can be obtained as the limit of the solutions u_ε of the problems

$$\left.\begin{array}{l}[\Delta + (k + i\varepsilon)^2]\, u_\varepsilon = f, \quad \varepsilon > 0 \\ u_\varepsilon \in \overset{\circ}{H}_1(\Omega)\end{array}\right\} \tag{2}$$

as $\varepsilon \to 0$. This is the basic idea in proving the existence of the solution for the radiation problems. It has been used in a great number of articles from [6] to [24]. The finite element approximation u_h, which we are going to use is defined as follows. Take an increasing sequence of the numbers $R = R(h) \to \infty$ as well as a decreasing sequence $\varepsilon = \varepsilon(h) \to 0$ with $h \to 0$. For every h we use a suitable finite dimensional (complex) trial subspace $S_h \subset H_1^0(\Omega(R))$, $\Omega(R) = \{ x \in \Omega \mid |x_i| < R, 1 \leqslant i \leqslant n \}$. The approximation $u_h \in S_h$ is defined by

$$(\nabla u_h \mid \nabla \varphi) - (k + i\varepsilon)^2 (u_h \mid \varphi) = -(f \mid \varphi) \quad \forall \varphi \in S_h . \tag{3}$$

Depending on the choice of the subspaces S_h and the sequences $\varepsilon(h)$, $R(h)$ various approximation results for the differences $\| u_0 - u_h \|_{0,K}$, $\| \nabla(u_0 - u_h) \|_{0,K}$ over bound sets $K \subset \Omega$ are obtained. For example, if $n = 2$ and if the boundary $\Gamma = \partial\Omega$ is smooth or polygonal, then a choice leads to the error estimate

$$\| u - u_h \|_{0,K} \leqslant c(K)\, h^{2/3} \| f \| \tag{4i}$$

and an other choice to

$$\| u - u_h \|_{1,K} \leqslant c(K)\, h^{1/3} \| f \| . \tag{4ii}$$

It is perhaps worth of noticing that our convergence results are obtained only, when the rates of the convergences $R(h) \to \infty$, $\varepsilon(h) \to 0$ are suitable related.

For the approximation of solutions for radiation problems using integral equations we refer to [3], [9], [16], [17] in the case of smooth boundaries and to [21] in the case of a non-smooth boundary. Another approximation can be found in [25]. There exist also articles, which use an approach based on Neumann-expansions[14].

Let us fix some notations. Besides of the Euclidean norm $|x|$ the maximum norm $\|x\| = \max\{|x_i| \mid 1 \leqslant i \leqslant n\}$ is needed. Define $Q(R) = \{x \mid \|x\| < R\}$, $\Omega(R) = \Omega \cap Q(R)$, $\Gamma(R) = \partial Q(R)$. Since f is assumed to be finitely supported, its support lies in $\Omega(R_0)$ for a fixed number $R_0 > 0$. We take R_0 so large that $d = d(\Gamma, \Gamma(R_0)) > 1$ is satisfied, $\Gamma = \partial\Omega$. The only requirement for the subspaces S_h, $0 < h \leqslant h_0$, enters in the following assumption. Let $v(h) \in H_1(\Omega(R(h)))$ be the solution of the Dirichlet problem

$$\Delta v(h) - v(h) = g\,, \quad g \in L_2(\Omega(R(h))) \tag{5}$$

and let $v(h)_h \in S_h$ be the approximation of $v(h)$:

$$(\nabla v(h)_h \mid \nabla\varphi) + (v(h)_h \mid \nabla\varphi) = -(g \mid \varphi)\,, \quad \forall\varphi \in S_h\,. \tag{6}$$

Assumption 1 : There exist a constant c and numbers $k(l)$, $l = 0, 1$, $0 < k(1) \leqslant k(0) < \infty$, such that

$$\| v(h) - v(h)_h \|_{l,\Omega(R(h))} \leqslant c h^{k(l)} \| g \|_{0,\Omega(R(h))} \tag{7}$$

for every $g \in L_2(\Omega(R(h)))$, $0 < h \leqslant h_0$.

We now give some examples where this condition is satisfied. It is of course essential in (7) that the constant c is independent of the radius $R(h)$. Roughly speaking the constant c comes from the regularity theorems for the problem (5). In giving examples of (7) the next lemma is useful. In the following c denotes a generic constant, independent of the functions occuring and of the parameters h, R, ε.

LEMMA 1 : *If $v \in \overset{\circ}{H}_1(Q(R))$ is the solution of $\Delta v - v = g$, $g \in L_2(Q(R))$, then $v \in H_2(Q(R))$ and*

$$\| v \|_{2,Q(R)} \leqslant c \| g \|_{0,Q(R)} \tag{8}$$

for every $R > 0$.

Proff : Because $Q(R)$ is convex the result $v \in \mathrm{H}_2(Q(R))$ follows from [12]. The equation $\Delta v - v = g$, $v \in H_1(Q(R))$ implies

$$\| v \|_{1,Q(R)} \leqslant \| g \|_{0,Q(R)}\,. \tag{9}$$

Further according to [12]

$$\| w \|_{2,Q(1)} \leqslant c_0 \| \Delta w \|_{0,Q(1)}, \tag{10}$$

when $w \in \overset{\circ}{H}_1(Q(1))$, $\Delta w \in L_2(Q(1))$. Applying (10) to $w(x) = v(Rx)$ the inequality :

$$\begin{aligned} | v |^2_{2,Q(R)} = \sum_{|\alpha|=2} \| \partial^\alpha v \|^2_{0,Q(R)} = R^{n-4} | w |^2_{2,Q(1)} \leqslant \\ \leqslant c_0^2 R^{n-4} \| \Delta w \|^2_{0,Q(1)} = c_0^2 \| \Delta v \|^2_{0,Q(R)} \\ \leqslant 2 c_0^2 (\| v \|^2_{0,Q(R)} + \| g \|^2_{0,Q(R)}) \leqslant 4 c_0^2 \| g \|^2_{0,Q(R)} \end{aligned} \tag{11}$$

is obtained. Thus (9), (11) imply (8). □.

LEMMA 2 : *Let the boundary* Γ *be smooth*; $\Gamma \in C^2$. *If* $v \in \overset{\circ}{H}_1(\Omega(R))$ *is the solution of* $\Delta v - v = g$, $g \in L_2(\Omega(R))$, *then* $v \in H_2(\Omega(R))$ *and*

$$\| v \|_{2,\Omega(R)} \leqslant c \| g \|_{0,\Omega(R)} \tag{12}$$

for every $R \geqslant R_0$.

The proof of Lemma 2 is obvious. It uses Lemma 1 and a regularity result for bounded domains with C^2-boundaries ([8] : Theorem 8.13).

Using the above lemma we can give an explicit example of (7) in the case of a smooth boundary :

Example 1 : Let $n = 2$, $\Gamma \in C^2$. Let $\mathcal{T}_h$ be a family of regular triangulations of the domain $\Omega(R(h))$. (For this notation see e.g. [4], [23].) Near the boundary curved elements are used [26], [27]. If S_h denotes the trial subspace of continuous piecewise linear functions (except over the curved triangles) which vanish on the nodes of the triangulation lying on the boundary of $\Omega(R)$, then the error estimate

$$\| v(h) - v(h)_h \|_{l,\Omega(R(h))} \leqslant ch^{2-l} \| g \|_{0,\Omega(R(h))} \tag{13}$$

is valid. For the proof of (13) see [26 : Theorem 3]. That c is independent of R is a consequence of Lemma 2.

Example 2 : Let $n = 2$, Γ polygonal. The accuracy (13) is obtained, if one uses the trial subspaces as in example 1 (without the curved elements) such that appropriate singular elements in the neighbourhood of the vertices of Γ are added to S_h. See [23], [15].

We will now discuss the error $u_0 - u_h$. The rate of the convergence $u_\varepsilon \to u_0$ must first be studied (although the notations u_ε, u_h are formally the same, there should be no possibility to a confusion). In the articles which use the limiting absorption principle it has been proved that $u_\varepsilon \to u_0$ in $H_1(\Omega(R))$ for every R. However, all the existing proofs are, as far as we know, theoretical; results for the rate of convergence do not seem to exist. In the following the idea of Phillips in [20] is crucial. According to [20] the solution u_0 as well as u_ε can be represented as a perturbation of a corresponding whole space solution. On the other hand, the rate of the convergence $u_\varepsilon \to u_0$ can, in the whole space case, easily be seen from the behavior of the fundamental solution. It was assumed in [20] that the boundary Γ was smooth; $\Gamma \in C^2$. However, such strong requirements can not be used if domains with polygonal boundaries are to be considered. Therefore we treat a slightly modified form of the discussion in [20] in some detail. We assume only that the domain Ω has the segment property [1]. For $\zeta = k + i\varepsilon$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$, we consider the fundamental solution

$$S_\zeta(x, y) = S_\zeta(|x - y|) = a(\zeta\,|x - y|^{-1})^{\frac{n-2}{2}}\, H^1_{\frac{n-2}{2}}(\zeta\,|x - y|) \quad (14)$$

of the equation

$$(\Delta + \zeta^2)\, u = 0\,. \quad (15)$$

The constant a in (14) is independent of ζ; in fact $a = -\, i\, 4^{-1}(2\,\pi)^{(2-n)/2}$. The function H^1_ν is the Hankel function of first kind and of order ν [19 : p. 66]. The principal properties of these functions is discussed in [19]. In particular when $n = 3$ the formula (14) becomes

$$S_\zeta(x, y) = -\frac{1}{4\pi}\,\frac{e^{i\zeta|x-y|}}{|x - y|}, \quad (16)$$

and for $n = 2$ we get

$$S_\zeta(x, y) = -\frac{i}{4}\, H^1_0(\zeta\,|x - y|)\,.$$

For the dimensions $n = 2, 3$ the fundamental solutions (14) have a square integrable singularity at $y = x$. In the case $n = 2$ the singularity takes the form

$$S_\zeta(x, y) = \frac{1}{2\pi}(\log \zeta\,|x - y|).J_0(\zeta\,|x - y|) + F_0(\zeta\,|x - y|) \quad (17)$$

with the Bessel function $J_0(z)$ and with an entire function $F_0(z)$. Our choice represents the outgoing case for $\varepsilon = 0$. For $\varepsilon > 0$ the function $S_\zeta(|x - y|)$ converges exponentially to zero as $|x - y| \to \infty$; for the asymptotic properties of the Hankel function with large arguments see [19 : p. 139], [5 : p. 524-526]. If $g \in L_2(\mathbb{R}^n)$ has a compact support supp $g \subset \Omega(R_0)$, then the equation

$$u_\varepsilon^0(x) = (R_\zeta^0 g)(x) = \int_{\mathbb{R}^n} S_\zeta(|x - y|)\, g(y)\, dy \tag{18}$$

defines the (unique) whole-space solution u_ε^0 of the equation

$$(\Delta + \zeta^2)\, u_\varepsilon^0 = g \tag{19}$$

such that $u_\varepsilon^0 \in \overset{\circ}{H}_1(\mathbb{R}^n)$, $\varepsilon > 0$, and such that u_ε^0 is outgoing for $\varepsilon = 0$.

From (16), (17) follows that, if $\varepsilon_1 > 0$ is fixed, then

$$|S_\zeta(x, y)| \leqslant c(R)\, |x - y|^{-(n-1)/2}$$

for all $\zeta = k + i\varepsilon$, $0 < \varepsilon \leqslant \varepsilon_1$, and for all $x, y \in Q(R)\; x \neq y$, $R > 0$. Thus, we get by (18) for $R > R_0$, $x \in \Omega(R_0)$

$$\begin{aligned} |u_\varepsilon^0(x)|^2 &\leqslant c_1(R) \left(\int_{\Omega(R_0)} |x - y|^{-(n-1)}\, dy \right) \|g\|^2 \\ &\leqslant c_2(R)\, \|g\|^2 \end{aligned}$$

with $\|g\| := \|g\|_{0,\Omega(R_0)}$, where the integral is estimated by means of [10], p. 161 Satz. Accordingly, we have

$$\|u_\varepsilon^0\|_{0,Q(R)} \leqslant c(R)\, \|g\| \,. \tag{20}$$

Since it holds

$$\Delta u_\varepsilon^0 = g - \zeta^2 u_\varepsilon^0 \,,$$

we have by the interior regularity result [1 : Theorem 6.3] that $u_\varepsilon^0 \in H_2^{\mathrm{loc}}(\mathbb{R}^n)$ and that

$$\|u_\varepsilon^0\|_{2,Q(R)} \leqslant c(R)\,(\|g - \zeta^2 u_\varepsilon^0\|_{0,Q(R+1)} + \|u_\varepsilon^0\|_{0,Q(R+1)})$$

which yields by (20)

$$\|u_\varepsilon^0\|_{2,Q(R)} \leqslant c(R)\, \|g\| \,, \tag{21}$$

for $0 < \varepsilon \leqslant \varepsilon_1$

Let v be the solution of

$$\left.\begin{aligned} &\Delta v - iv = 0\,, \\ &v|_\Gamma = \xi\,, \qquad \xi \in H_2^{\mathrm{loc}}(\mathbb{R}^n) \\ &v|_{\Gamma(R_0)} = 0 \end{aligned}\right\} \tag{22}$$

in the following sense : Take $\delta = d/3$ and define $U(\delta) = \{ x \in \mathbb{R}^n \mid d(x, \Omega^c) < \delta \}$. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a fixed smoothing function such that $\psi(x) = 1$ for $x \in U(\delta)$ and $\psi(x) = 0$, $x \in U(2\,\delta)^c$. Let $w \in \overset{\circ}{H}_1(\Omega(R_0))$ be the solution of

$$(\nabla w \mid \nabla \varphi) + i(w \mid \varphi) = ((\Delta - i)\,\psi\xi \mid \varphi)\,, \quad \forall \varphi \in \overset{\circ}{H}_1(\Omega(R_0))\,, \tag{23}$$

and define $v = w + \psi\xi \in H_1(\Omega(R_0))$. The mapping $Q : H_2^{\mathrm{loc}}(\mathbb{R}^n) \to H_1(\Omega(R_0)) \cap H_2^{\mathrm{loc}}(\Omega(R_0))$, $Q\xi = v$, is linear and satisfies

$$\| Q\xi \|_{1,\Omega(R_0)} + \| Q\xi \|_{2,\Omega(R_0) \setminus (U(\delta) \cup U(2\delta)^c)} \leqslant c \, \| \xi \|_{2,\Omega(R_0)} \,. \tag{24}$$

Define $v_\zeta = QR_\zeta^0\, g$. According to (21), (24) the estimate

$$\| v_\zeta \|_{1,\Omega(R_0)} + \| \nabla\psi . \nabla v_\zeta \|_{1,\Omega(R_0)} \leqslant c \, \| g \| \tag{25}$$

is valid. The formula

$$T_\zeta\, g = 2\, \nabla\psi . \nabla v_\zeta + v_\zeta\, \Delta\psi + (\zeta^2 + i)\, \psi v_\zeta \tag{26}$$

defines a linear operator $T_\zeta : L_2(\Omega(R_0)) \mapsto L_2(\Omega(R_0))$. Because of (25) and the segment property the operator T_ζ is even compact. Suppose that $1 - T_\zeta$ has the inverse $(1 - T_\zeta)^{-1}$. If $g := (1 - \mathrm{T}_\zeta)^{-1}\ f$ and if $u_\varepsilon' := (1 - \psi Q)\, R_\zeta^0\, g$, then one can verify that u_ε' is a solution of (2), $\varepsilon > 0$, and u_ε' is a solution of (1) for $\varepsilon = 0$. The uniqueness of solution to (1) and (2) indicates that the solution u_ε has the representation

$$u_\varepsilon = (1 - \psi Q)\, R_\zeta^0 (1 - T_\zeta)^{-1}\, f\,. \tag{27}$$

Take $\varepsilon = 0$. The existence $(1 - T_\zeta)^{-1}$ is seen as in [19] and we omit it. In the proof of the following theorem we will see that $T_{k+i\varepsilon} \mapsto T_k$, $\varepsilon \to 0$. Therefore, the formula (27) also holds for $\zeta = k + i\varepsilon$, $0 \leqslant \varepsilon \leqslant \varepsilon_1$.

THEOREM 1 : *For every $R > 0$ there exists a number $c(R) > 0$ such that*

$$\| u_0 - u_\varepsilon \|_{1,\Omega(R)} \leqslant \varepsilon c(R) \, \| f \| \,, \tag{28}$$

$0 \leqslant \varepsilon \leqslant \varepsilon_1$.

Proof : For $|x|, |y| \leqslant R$ we have

$$| S_{k+i\varepsilon}(| x - y |) - S_k(| x - y |)| \leqslant \begin{cases} \varepsilon c(R)\,(|\ln | x - y | | + 1)\,, & n = 2\,, \\ \varepsilon c(R)\, | x - y |^{-1}\,, & n = 3\,. \end{cases} \tag{29}$$

The representation (18) leads to the estimate

$$\| u_0^0 - u_\varepsilon^0 \|_{0,Q(R)} \leqslant \varepsilon c(R) \| g \| . \tag{30}$$

Since

$$\Delta(u_0^0 - u_\varepsilon^0) = k^2(u_\varepsilon^0 - u_0^0) + \varepsilon(2\,ik - \varepsilon)\, u_\varepsilon^0 ,$$

the interior regularity [1 : Theorem 6.3] implies that

$$\begin{aligned}\| u_0^0 - u_\varepsilon^0 \|_{2,Q(R)} &\leqslant c(R)\,(\| u_\varepsilon^0 - u_0^0 \|_{0,Q(2R)} + \varepsilon \| u_\varepsilon^0 \|_{0,Q(2R)}) \\ &\leqslant \varepsilon c(R) \| g \| . \end{aligned} \tag{31}$$

From

$$\begin{aligned}(T_{k+i\varepsilon} - T_k)\, g = 2\,\nabla\psi.\nabla(v_{k+i\varepsilon} - v_k) + \Delta\psi.(v_{k+i\varepsilon} - v_k) + \\ + ((k + i\varepsilon)^2 + i)\,\psi v_{k+i\varepsilon} - (k^2 + i)\,\psi v_k\end{aligned}$$

we get using (25), (31)

$$\| (T_{k+i\varepsilon} - T_k)\, g \| \leqslant \varepsilon c \| g \| \tag{32}$$

and in particular $T_{k+i\varepsilon} \mapsto T_k$ with respect of the operator norm as $\varepsilon \to 0$. Now, the formula

$$u_\varepsilon - u_0 = (1 - \psi Q)\,[R_{k+i\varepsilon}^0(1 - T_{k+i\varepsilon})^{-1} - R_k^0(1 - T_k)^{-1}]\, f \tag{33}$$

is true for $0 \leqslant \varepsilon \leqslant \varepsilon_1$. Because of the continuity of the inverse the inequality

$$\| (1 - T_{k+i\varepsilon})^{-1} - (1 - T_k)^{-1} \| \leqslant c\varepsilon \tag{34}$$

is obtained. The rest of the proof follows in a straightforward manner from (33) using (34), (31) and (25). □

Our next step is to discuss the difference of u_ε and u_ε^R where u_ε^R is the solution of the Dirichlet problem

$$\left.\begin{aligned}\Delta u_\varepsilon^R + (k + i\varepsilon)^2\, u_\varepsilon^R = f ,\\ u_\varepsilon^R \in H_1(\Omega(R)) .\end{aligned}\right\} \tag{35}$$

For this purpose the following bound is needed :

LEMMA 3 : *The solution u_ε obeys the estimate*

$$| u_\varepsilon(x) | + | \nabla u_\varepsilon(x) | \leqslant c\, | x |^{\frac{n-1}{2}}\, e^{-\frac{1}{2}\varepsilon |x|} \| f \| \tag{36}$$

for $|x| \geqslant 2R_1$, $R_1 = \sqrt{n}\, R_0$, $0 \leqslant \varepsilon < \varepsilon_1$.

Proof : If $y \in \Omega(R_0)$, then $|y| \leqslant \sqrt{n}\, \|y\| \leqslant \sqrt{n}\, R_0 = R_1$. For $|x| \geqslant 2R_1$ we obtain $|x - y| \geqslant \frac{1}{2}|x|$. Since

$$u_\varepsilon(x) = (R^0_{k+i\varepsilon}(1 - T_{k+i\varepsilon})^{-1} f)(x)$$

for $x \notin \Omega(R_0)$, the estimate

$$|u_\varepsilon(x)| + |\nabla u_\varepsilon(x)| \leqslant \int_{\Omega(R_0)} (|S_{k+i\varepsilon}(|x-y|)| + \\ + |\nabla_x S_{k+i\varepsilon}(|x-y|)|) . |(1 - T_{k+i\varepsilon})^{-1} f(y)|\, dy \quad (37)$$

is valid. Recalling the asymptotic formula of the Hankel functions for large arguments [19 : p. 139], (see also [5 : p. 524-526]) as well as the recurrence relations for the derivatives [19 : p. 67], we find that for $|x| \geqslant 2R_1$

$$|S_{k+i\varepsilon}(|x-y|)| + |\nabla_x S_{k+i\varepsilon}(|x-y|)| \leqslant c\,|x|^{-\frac{n-1}{2}}\, e^{-\frac{1}{2}\varepsilon|x|}. \quad (38)$$

The formulaes (37), (38), (34) lead to (36). □

The following lemma holds for all open sets Ω, bounded or not. The proof is simple, and will be omitted.

LEMMA 4 : *Let* $\Omega \subset \mathbb{R}^n$ *be an open set*, $\Omega \neq \phi$. *Assume that* $k > 0$, $\varepsilon > 0$. *The equation*

$$\begin{cases} \Delta u_\varepsilon + (k + i\varepsilon)^2 u_\varepsilon = f, & f \in L_2(\Omega), \\ u_\varepsilon \in \overset{\circ}{H}_1(\Omega) \end{cases}$$

has the unique solution u_ε, *and the estimate*

$$\|u_\varepsilon\|_{1,\Omega} \leqslant \varepsilon^{-1} c(k)\, \|f\|_{0,\Omega} \quad (39)$$

is valid.

We are now ready to establish :

THEOREM 2 : *The difference* $u_\varepsilon - u_\varepsilon^R$ *obeys the estimate*

$$\|u_\varepsilon - u_\varepsilon^R\|_{1,\Omega(\frac{1}{2}R)} \leqslant c\varepsilon^{-1} R^{\frac{n-3}{2}} e^{-\frac{1}{4}\varepsilon R} \|f\|, \quad (40)$$

$0 < \varepsilon \leqslant \varepsilon_1$, $R \geqslant 4R_1$.

Proof: Fix a smoothing function $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1, 3/4 \leqslant \| x \|$, $\varphi(x) = 0$, $\| x \| \leqslant 1/2$. Take $\varphi_R(x) = \varphi(R^{-1} x)$, $\psi_R(x) = 1 - \varphi_R(x)$. The function ψ_R is identically one in $Q(R_0)$ and vanishes in a neighbourhood of the boundary $\Gamma(R)$. Since $w := \psi_R u_\varepsilon - u_\varepsilon^R \in \overset{\circ}{H}_1(\Omega(R))$, we conclude from (39)

$$\begin{aligned} \| w \|_{1,\Omega(R)} &\leqslant c\varepsilon^{-1} \| u_\varepsilon \Delta\psi_R + 2 \nabla\psi_R . \nabla u_\varepsilon \|_{0,\Omega(R)} \\ &= c\varepsilon^{-1} \| u_\varepsilon \Delta\varphi_R + 2 \nabla\varphi_R . \nabla u_\varepsilon \|_{Q\left(\frac{1}{2}R,\frac{3}{4}R\right)} . \end{aligned} \tag{41}$$

We have $|\partial^\alpha \varphi_R(x)| \leqslant M(|\alpha|) R^{-|\alpha|}$ for every multi-indices α. Since

$$| x | \geqslant \| x \| \geqslant \frac{1}{2} R \geqslant 2 R_1 \quad \text{in} \quad Q\left(\frac{1}{2} R, \frac{3}{4} R\right)$$

the inequality

$$\| w \|_{1,\Omega(R)} \leqslant c\varepsilon^{-1} R^{\frac{n-3}{2}} e^{-\frac{1}{4}\varepsilon R} \| f \| \tag{42}$$

is obtained from (41). Because $\psi_R(x) \equiv 1$, $x \in \Omega\left(\frac{1}{2} R\right)$, the estimate (40) follows. □

As the final step we have

THEOREM 3 : *Let* $\varepsilon = \varepsilon(h) = h^\delta$, $0 < \delta < k(l)$. *For sufficiently small* h *the estimate*

$$\| u_\varepsilon^R - (u_\varepsilon^R)_h \|_l \leqslant c h^{k(l) - 2\delta} \| f \| , \tag{43}$$

$R \geqslant R_0$, *is valid.*

Proof : Denote by $K(R)$ the solution operator $K(R) = (\Delta_0 - 1)^{-1}$: $L_2(\Omega(R)) \to \overset{\circ}{H}_1(\Omega(R))$, where Δ_0 means the Laplacian with the homogeneous Dirichlet boundary condition. The equation (35) is then equivalent to

$$(I + [1 + (k + i\varepsilon)^2] K(R)) u_\varepsilon^R = K(R) f , \quad u_\varepsilon^R \in L_2(\Omega(R)) . \tag{44}$$

In the same way, if $K_h(R)$ is the solution operator $K_h(R) : L_2(\Omega(R)) \mapsto S_h$ defined by $K_h(R) f = v_h$,

$$(\nabla v_h \mid \nabla\varphi) + (v_h \mid \varphi) = -(f \mid \varphi) , \quad \forall \varphi \in S_h ,$$

then the equation

$$(\nabla (u_\varepsilon^R)_h \mid \nabla\varphi) - (k + i\varepsilon)^2 ((u_\varepsilon^R)_h \mid \varphi) = -(f \mid \varphi) , \quad \forall \varphi \in S_h ,$$

is equivalent to

$$(I + [1 + (k + i\varepsilon)^2] K_h(R)) (u_\varepsilon^R)_h = K_h(R) f, \quad (u_\varepsilon^R)_h \in L_2(\Omega(R)) . \quad (45)$$

For brevity we write

$$U = I + [1 + (k + i\varepsilon)^2] K(R),$$
$$U_h = I + [1 + (k + i\varepsilon)^2] K_h(R) .$$

The operator K is compact with respect both of the norms $\| \cdot \|_{l,\Omega(R)}$, $l = 0, 1$ (for $l = 1$ see [22]). Let $\| \cdot \|_l$, $l = 0, 1$ be the operator norm in $L_2(\Omega(R))$ for $l = 0$ and in $H_1(\Omega(R))$ for $l = 1$. Since U is one-to-one, the inverse exists. According to the Assumption 1

$$\| U - U_h \|_l \leqslant ch^{k(l)}, \quad l = 0, 1 . \quad (46)$$

Therefore, the inverse U_h^{-1} exists if h is sufficiently small. Moreover, we get from a Neumann-expansion

$$\| U^{-1} - U_h^{-1} \|_l \leqslant \| U^{-1}(U - U_h) \|_l (1 - \| U^{-1}(U - U_h) \|_l)^{-1} \| U^{-1} \|_l \quad (47)$$

if

$$\| U^{-1}(U - U_h) \|_l < 1 . \quad (48)$$

To obtain the inequality (48) an estimate for the norm $\| U^{-1} \|_l$ is needed. Let us first consider the case $l = 0$. Define $\mu = 1 + (k + i\varepsilon)^2$. Because $K(R)$ is selfadjoint in $L_2(\Omega(R))$ the inequality

$$\| U^{-1} \|_0 = | \mu |^{-1} \| (\mu^{-1} + K(R))^{-1} \|_0 \leqslant | \mu |^{-1} | \operatorname{Im} \mu^{-1} |^{-1}$$
$$\leqslant c\varepsilon^{-1} = ch^{-\delta} \quad (49)$$

is true [13 : p. 272]. Since $\delta < k(l) \leqslant k(0)$ the inequality (48) ($l = 0$) is satisfied if h is small enough. From (47) we then have

$$\| U^{-1} - U_h^{-1} \|_0 \leqslant ch^{k(0) - 2\delta} . \quad (50)$$

The norm $\| U^{-1} \|_1$ can be estimated as follows. Let $u, v \in H_1(\Omega(R))$ and let $v = U^{-1} u$. Then

$$(\Delta_0 - 1)(u - v) = (1 + (k + i\varepsilon)^2) v .$$

According to (49) this implies that

$$\| u - v \|_{1,\Omega(R)} \leqslant (1 + | k + i\varepsilon |^2) \| v \|_{0,\Omega(R)}$$
$$\leqslant ch^{-\delta} \| u \|_{0,\Omega(R)} .$$

Hence

$$\| v \|_{1,\Omega(R)} \leqslant \| u \|_{1,\Omega(R)} + \| u - v \|_{1,\Omega(R)} \leqslant ch^{-\delta} \| u \|_{1,\Omega(R)} .$$

Therefore the bound (49) is valid for the norm $\| U^{-1} \|_1$, too. By analogy with (50) the inequality

$$\| U^{-1} - U_h^{-1} \|_1 \leqslant ch^{k(1)-2\delta} \tag{51}$$

is obtained if $\delta < k(1)$. Finally,

$$\begin{aligned}\| u_\varepsilon^R - (u_\varepsilon^R)_h \|_l &= \| U^{-1}(K(R) f) - U_h^{-1}(K_h(R) f) \|_l \\ &\leqslant \| U^{-1}((K(R) - K_h(R)) f) \|_l + \| (U^{-1} - U_h^{-1}) K_h(R) f \|_l \\ &\leqslant ch^{k(l) - 2\delta} \| f \| . \quad \square \end{aligned} \tag{52}$$

We now choose $R(h) = h^{-(\delta+\alpha)}$, $\alpha > 0$. Write $u_h = (u_{\varepsilon(h)}^{R(h)})_h$. If $R_2 \geqslant R_1$ is fixed, we have $\Omega(R_2) \subset \Omega\left(\frac{1}{4} R(h)\right)$ for sufficiently small h. Then Theorems 1-3 imply that

$$\begin{aligned}&\| u_0 - u_h \|_{1,\Omega(R_2)} \leqslant \\ &\leqslant \| u_0 - u_{\varepsilon(h)} \|_{1,\Omega(R_2)} + \| u_{\varepsilon(h)} - u_{\varepsilon(h)}^{R(h)} \|_{1,\Omega(R_2)} + \| u_{\varepsilon(h)}^{R(h)} - (u_{\varepsilon(h)}^{R(h)})_h \|_{1,\Omega(R_2)} \\ &\leqslant c(R_2) [h^\delta + h^{-\delta}.h^{-(\delta+\alpha)\frac{n-3}{2}} e^{-\frac{1}{4}h^{-\alpha}} + h^{k(l)-2\delta}] \| f \|\end{aligned}$$

if $0 < \delta < k(l)$. If h is small enough ; $0 < h \leqslant h_1$ (where h_1 depends on the choice of δ, α), then the middle term has the upper bound $h^{k(l)}$. Therefore

$$\| u_0 - u_h \|_{l,\Omega(R_2)} \leqslant c(R_2) (h^\delta + h^{k(l)-2\delta}) \| f \| . \tag{53}$$

The best error bound

$$\| u_0 - u_h \|_{l,\Omega(R_2)} \leqslant c(R_2) h^{\frac{1}{3}k(l)} \| f \| \tag{54}$$

is achieved by choosing $\delta = k(l)/3$.

THEOREM 4 : *Let the Assumption* 1 *be satisfied. If* $u_h = (u^{R(h)}_{\varepsilon(h)})_h$, *where* $\varepsilon(h) = h^{k(l)/3}$,

$$R(h) = h^{-\left(\frac{k(l)}{3}+\alpha\right)}, \quad \alpha > 0,$$

then the error estimate (54) *is valid.*

We note once more that it is a different choice of $\varepsilon(h)$, $R(h)$, which gives the best bound for the error with respect of the $\| \cdot \|_{0,K}$ and $\| \cdot \|_{1,K}$ norms. For example in the Examples 1 and 2 the choice $\delta = 2/3$ gives the best error bound

$$\| u_0 - u_h \|_{0,K} \leqslant c(K)\, h^{2/3} \, \| f \|$$

for the $\| \cdot \|_{0,K}$ norm but no convergence with respect of the $\| \cdot \|_{1,K}$ norm. On the other hand $\delta = 1/3$ gives the best bound

$$\| u_0 - u_h \|_{1,K} \leqslant c(K)\, h^{1/3} \, \| f \|$$

with respect of the $\| \cdot \|_{1,K}$ norm, but no better estimate with respect of the $\| \cdot \|_{0,K}$ norm.

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