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Iterative refinement for approximate eigenelements of compact operators

ITERATIVE REFINEMENT FOR APPROXIMATE EIGENELEMENTS OF COMPACT OPERATORS (*)

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Abstract. — *We present an iterative refinement method to approximate the eigenelements of a compact linear operator in a Banach space. Convergence is proved under a very weak hypothesis. Numerical comparisons with a similar method are illustrated on a compact integral operator of Fredholm's type.*

Résumé. — *On propose une méthode de raffinement itératif pour approcher les éléments propres d'un opérateur compact sur un espace de Banach. La convergence est démontrée sous une hypothèse très faible. Des essais numériques sur un opérateur intégral de Fredholm permettent de comparer avec une autre méthode du même genre.*

1. INTRODUCTION

We are concerned with the numerical solution of the eigenvalue problem

$$T\phi = \lambda\phi$$
$$\phi \neq 0 \qquad\qquad (1.1)$$
$$\phi \in X$$

where T is a compact operator defined on the Banach space X.

Often in practice X is $C[0, 1]$ or $L^2[0, 1]$ and T is an integral operator of the form

$$x(t) \rightarrow (Tx)(t) = \int_0^1 k(t, s)\, x(s)\, ds$$

where the kernel k is such that T is compact in X.

Problem (1.1) is approximated by a discretized version

$$T_n \phi_n = \lambda_n \phi_n$$
$$\phi_n \neq 0 \qquad\qquad (1.2)$$
$$\phi_n \in X_n$$

where X_n is a finite dimensional subspace of X and T_n results from a projection or an approximate quadrature method. Since the matrix representing $T_{n|X_n}$ is commonly full then the solution of (1.2) for large n may be very expensive in computer time or storage.

We present here two iterative methods to compute the eigenelements of T which require the solution of (1.2) only for a relatively small value of n, the operator T (or indeed a fine discretization T_M of it with $M \gg n$) being used only for evaluations (that is only for products of type $T_M x_M$ with $x_M \in X_M$).

2. MATHEMATICAL BACKGROUND

The reader is referred to Kato (1976) and Chatelin (1983) for a detailed treatment of spectral theory. Here we shall recall only the fundamental notions which will be used later on.

X is a Banach space over the complex field. $\mathcal{L}(X)$ is the algebra of bounded (that is continuous) linear operators with domain X. 1 denotes the identity operator on X and, for z in $\mathbb{C}$, z stands also for the scalar operator $z1$.

2.1. Convergence notions

Let $\{ T_n \}$ be a sequence in $\mathcal{L}(X)$ and T an operator in $\mathcal{L}(X)$. $\{ T_n \}$ is pointwise convergent to T (we write $T_n \to p \to T$) iff for any $x \in X$ $\{ T_n x \}$ is convergent to Tx. $\{ T_n \}$ converges in norm to T (we write then $T_n \to \| \ \| \to T$) iff $\{ \| T_n - T \| \}$ converges to 0 in $\mathbb{R}$. $\{ T_n \}$ is collectively compact convergent to T (we write $T_n \to cc \to T$) iff $\bigcup_n (T_n - T) B$ is relatively compact and $T_n \to p \to T$, B being the closed unit ball in X.

2.2. Spectral definitions

For T in $\mathcal{L}(X)$, $r_\sigma(T) = \lim_k \inf \| T^k \|^{1/k}$ is the spectral radius. The resolvent set is $\rho(T) = \{ z \in \mathbb{C} : (T-z)^{-1} \in \mathcal{L}(X) \}$ and the spectrum is $\sigma(T) = \mathbb{C} - \rho(T)$. For $z \in \rho(T)$ $R(z) = (T - z)^{-1}$ is called the resolvent operator (since it solves for x the equation $(T - z) x = f$). If T is compact $\sigma(T)$ consists of 0 and

countably many isolated eigenvalues of finite multiplicity. Let $\lambda \in \sigma(T)$ be an eigenvalue of T, T being a compact operator, let m be the algebraic multiplicity of λ and Γ a Jordan curve in the complex plane isolating λ. $P = -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma R(z)\,dz$ is the spectral projection and PX the invariant subspace (which, by definition of m has dimension m). $S = ((T - \lambda)_{|(1 - P)X})^{-1}(1 - P)$ belongs to $\mathscr{L}(X)$ and is called the reduced resolvent operator. S satisfies the identity

$$S(T - \lambda) = (T - \lambda)\,S = 1 - P \qquad (2.1)$$

which will be used in Section 4 to derive a numerical method. λ is called a simple eigenvalue iff $m = 1$.

2.3. Convergence notions related to spectral concepts

Let $\{T_n\}$ be a sequence in $\mathscr{L}(X)$ such that $T_n \to p \to T$ where $T \in \mathscr{L}(X)$. For $z \in \rho(T_n)$ we set $R_n(z) = (T_n - z)^{-1}$. $\{T_n\}$ is a stable approximation to T at $z \in \rho(T)$ (and we write $T_n - z \to s \to T - z$) iff for all n large enough $z \in \rho(T_n)$ and $R_n(z)$ is uniformly bounded in n.

Let λ be an isolated eigenvalue of T of finite multiplicity m, P the spectral projection, Γ a Jordan curve isolating λ and Δ the domain enclosed by Γ. We set $P_n = -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma R_n(z)\,dz$.

$\{T_n\}$ is a strongly stable approximation to T on Γ (resp. in Δ), and we then write $T_n - z \to ss \to T - z$ on Γ (resp. in Δ) iff $T_n - z \to s \to T - z$ for all $z \in \Gamma$ (resp. in $\Delta - \{\lambda\}$) and $\dim P_n X = m$ for all n large enough.

2.4. Convergence of the eigenelements

The proof of the following results can be found in Chatelin (1983).

LEMMA 1 : *If $T_n - z \to ss \to T - z$ on Γ then $P_n \to cc \to P$ and, for large enough n there are exactly m eigenvalues of T_n (counting their algebraic multiplicities) which converge to λ.*

LEMMA 2 : $T_n \to \|\ \|\ T$ *or* $T_n \to cc \to T$ *imply* $T_n - z \to ss \to T - z$ *in* $\mathbb{C} - \{0\}$ *when T is compact.*

LEMMA 3 : *If $T_n - z \to ss \to T - z$ on Γ and if λ is simple then there is an eigenvector ϕ of T normalized by $P_n\phi = \phi_n$ where ϕ_n is an eigenvector of T_n normalized by $\|\phi_n\| = 1$ and associated to λ_n, the eigenvalue of T_n that approxi-*

mates λ. *The following error bound holds for sufficiently large n :*

$$| \lambda - \lambda_n | + \| \phi - \phi_n \| \leqslant c \| (T - T_n) P \|$$

where c does not depend on n. The reduced resolvent of T_n *at* λ_n *is pointwise convergent to S and hence uniformly bounded in n :*

$$S_n = ((T_n - \lambda_n)_{|(1 - P_n)X})^{-1}(1 - P_n) \, .$$

3. THE EIGENVALUE PROBLEM

In what follows T is compact, λ is a nonzero simple isolated eigenvalue of T, ϕ is an eigenvector of T associated to λ and normalized by $P_n \phi = \phi_n$ accordingly with the notations of Lemma 3.

X^* will be the adjoint space, T_n^* the adjoint of T_n and $\bar{\lambda}_n$, ϕ_n^* the eigenelements of T_n^* such that $\langle \phi_n, \phi_n^* \rangle = 1$ where $\langle , \rangle$ denotes the duality between X and X^*. We then have $P_n x = \langle x, \phi_n^* \rangle \phi_n$ for all x in X.

We consider the general problem

$$(T - \lambda) x = f \tag{3.1}$$

where we suppose λ and ϕ to be known. The general solution of (3.1) is the affine manifold defined by the equation $x = \tilde{x} + \alpha\phi$ where α varies over $\mathbb{C}$ and $\tilde{x}$ is a particular solution of (3.1) which we can compute by means of the projection P. We project (3.1) on PX and on $(1 - P) X$ and we obtain respectively

$$(T - \lambda) Px = Pf \tag{3.2}$$

$$(T - \lambda) (1 - P) x = (1 - P) f \tag{3.3}$$

since T and P commute. Equation (3.2) shows that (3.1) is solvable if and only if $Pf = 0$ and from equation (3.3) we get $(1 - P) x = Sf$ hence we may set, for instance $\tilde{x} = Sf$ and the general solution of (3.1) may be written

$$x = Sf + \alpha\phi \quad \alpha \in \mathbb{C} \, . \tag{3.4}$$

The normalizing factor of ϕ may be written $\langle \phi, \phi_n \rangle = 1$ so we have that

$$\lambda = \langle T\phi, \phi_n^* \rangle \, . \tag{3.5}$$

4. ITERATIVE REFINEMENT OF THE APPROXIMATE EIGENELEMENTS

4.1. The iterations

Taking into account (3.4) and (3.5) we may suggest the following fixed point iteration for solving (1.1), where λ^{k+1} stands for $\langle Tx^k, \phi_n^* \rangle$.

METHOD I

$$x^0 = \phi_n$$
$$x^{k+1} = x^k - S_n(Tx^k - \lambda^{k+1} x^k). \tag{4.1}$$

The identity (2.1) may be written $S_n = \dfrac{1}{\lambda_n}(S_n T_n + P_n - 1)$ which, if we substitute T to T_n, leads to

METHOD II

$$x^0 = \phi_n$$
$$x^{k+1} = x^k + \frac{1}{\lambda_n}(1 - S_n T)(Tx^k - \lambda^{k+1} x^k) \tag{4.2}$$

since $P_n(Tx^k - \lambda^{k+1} x^k) = 0$.

Iteration (4.1) was proposed by Chatelin (1983) and related to multigrid algorithms. Iteration (4.2) is proposed by the authors.

4.2. Convergence and error bounds

For Method I convergence has been established by Chatelin (1983) under two different hypotheses which lead to two corresponding error bounds. We report them here below. Throughout this section c is a generic constant, that is, independent of both k and n and $\lambda^0 = \lambda_n$.

THEOREM 1 : a) *If $T_n \to \| \| \to T$ then Method I satisfies for all k*

$$| \lambda^k - \lambda | + \| x^k - \phi \| \leqslant c(c \| T_n - T \|)^{k+1}.$$

b) *If $T_n \to cc \to T$ then Method I satisfies for all k*

$$\text{Max} \{ | \lambda^{2k} - \lambda | + \| x^{2k} - \phi \|, | \lambda^{2k+1} - \lambda | + \| x^{2k+1} - \phi \| \} \leqslant c(\delta_n + \varepsilon_n)^k \delta_n$$

where $\delta_n = c \| (T_n - T) P \|$ and $\varepsilon_n = c \| (T_n - T) S_n(T_n - T) \|$.

COROLLARY 1 : *If* $T_n \to \| \ \| \to T$ *or if* $T_n \to cc \to T$ *then for large enough n Method I satisfies* $\lim_k x^k = \phi$ *and* $\lim_k \lambda^k = \lambda$.

THEOREM 2 : *If* $T_n - z \to ss \to T - z$ *on* Γ *then Method II satisfies for all k*

$$| \lambda^{k+1} - \lambda | + \| x^k - \phi \| \leqslant c(\| (T_n - T) T \|)^{k+1}.$$

Proof : Since $\| (T - T_n) P \| \leqslant c \| (T - T_n) T \|$ then Lemma 3 applies to prove the case $k = 0$.

The following identities end the proof by induction on k :

$$\lambda^{k+2} - \lambda = \langle T(x^{k+1} - \phi), \phi_n^* \rangle$$

$$x^{k+1} - \phi = \frac{\lambda - \lambda_n}{\lambda_n} (\phi - x^k - S_n T(\phi - x^k)) +$$

$$+ \frac{\lambda - \lambda^{k+1}}{\lambda_n} (x^k - \phi + \phi - \phi_n - S_n T(x^k - \phi)$$

$$+ (\lambda^{k+1} - \lambda) S_n \phi + \lambda^{k+1} S_n(\phi_n - \phi))$$

$$+ \frac{1}{\lambda_n} S_n(T_n - T) T(x^k - \phi). \quad /\!/\!/$$

COROLLARY 2 : *If* $T_n - z \to ss \to T - z$ *on* Γ *then for large enough n, Method II satisfies* $\lim_k x^k = \phi$ *and* $\lim_k \lambda^k = \lambda$.

Proof : Since $T_n \to p \to T$ and T is compact then $c \| (T_n - T) T \| < 1$ for n sufficiently large. $/\!/\!/$

5. NUMERICAL EXAMPLES

We report the results of numerical experiments done on the integral operator T defined by the kernel

$$k(t, s) = 2 \mid \sin 10 \pi t - \sin 10 \pi s \mid^{1/2} \quad 0 \leqslant t, s \leqslant 1.$$

The problem is set in $X = C[0, 1]$ and as approximations T_n we use the Fredholm's and Nyström's discretizations.

Let π_n be the projection which maps X onto X_n, the subspace of continuous piecewise linear functions associated to the uniform grid $\left\{ \dfrac{i - 1}{n - 1} \right\}_{i=1}^{n-1}$ defined as the piecewise linear interpolation in this grid.

In practice X is replaced by X_M with $M \gg n$. The Fredholm's and Nyström's discretizations are defined respectively, by

$T_n^F = \pi_n \, T \pi_n$ coupled with the M-points trapezoidal quadrature rule
$T_n^N = T \pi_n$ coupled with the M-points trapezoidal quadrature rule.

We have, see Chatelin (1983), $T_n^F \to cc \to T$ and $T_n^N \to cc \to T$.

The evaluations of T needed in computations are thus performed by means of the fine discretization $\tau_M = T_{M|X_M}^F$.

We are interested in the dominant eigenvalue and its invariant subspace. Table 1 shows the number of iterations performed by each method for different approximation policies (values of n and M). Iteration has been stopped when $\delta_k = \| x^k - x^{k-1} \| / \| x^k \| < 10^{-12}$.

Figure 1 shows the decimal logarithmic relative error of λ^k (crosses) and x^k (solid line) for the case $n = 5$, $M = 101$ and Method I. Figure 2 does it for Method II. Figure 3 shows ϕ_n, the starting vector, and the last iterate of Method II for the same case.

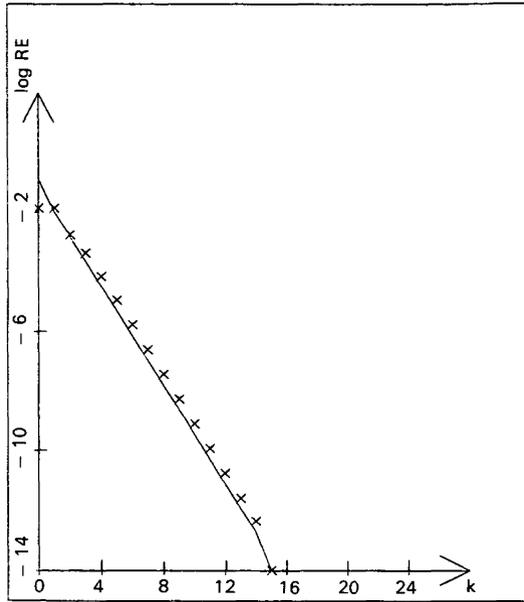| Approximation	Policy n M	Iterations needed to achieve $\delta_k < 10^{-12}$	
		Method I	Method II
F	3 51	26	19
N		25	18
F	5 101	15	9
N		13	9
F	10 100	14	7
N		13	7

TABLE 1

Figure 1. — Fredholm's Approximation $n = 5$ $M = 101$
METHOD I   × Relative Error of λ^k
           | Relative Error of x^k

6. CONCLUSIONS AND FINAL COMMENTS

We first remark that iterative refinement techniques need no eigensystem computation for the fine discretization of T. The eigenvalue problem (1.2) is solved once for all with a relatively small n.

From a computational point of view, we see that Method II is faster than Method I, but requires one more evaluation of T. The figures show us that the theoretical error bounds are respected in practice with moderate error constants.

Theoretically, the hypothesis $T_n - z \to ss \to T - z$ on Γ is clearly the weakest one we may impose to the sequence $\{ T_n \}$ if we want the eigenelements of T_n to converge to those of T with preservation of algebraic (finite) multiplicities, and we note that this hypothesis suffices to prove the convergence of Method II.

In Ahués *et al.* (1982a) these iterations are presented in the framework of Defect Correction Methods (cf. Stetter 1978) and the proofs of convergence are based on contraction properties of suitable nonlinear operators. Other related methods are recast in Ahués *et al.* (1982a, b), in particular Lin Qun's iteration

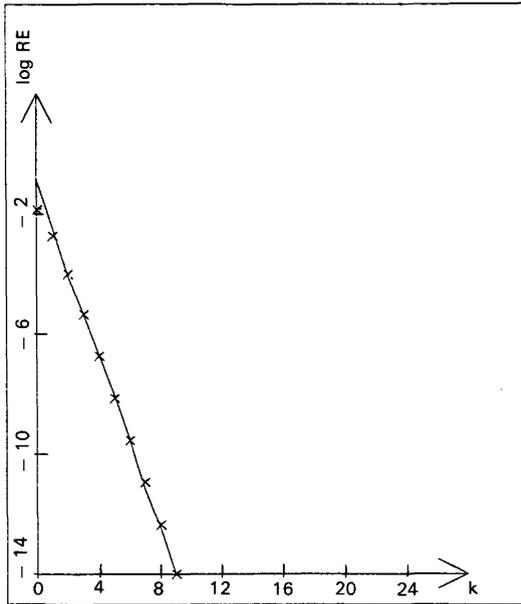**Figure 2. — Fredholm's Approximation** $n = 5\ M = 101$
METHOD II   × Relative Error of λ^k
| Relative Error of x^k.

(*cf.* Lin Qun, 1982) which was proposed for selfadjoint operators in a Hilbert space setting. All these techniques are compared in Ahués *et al.* (1983). Quasi-Newton iterative refinement techniques are developed and compared in Ahués and Telias (1982). These ones require the solution of an n-dimensional linear system in each iteration.

Method I may be interpreted as the analogous of Atkinson's method for the associated linear equation $(T - z)\, x = f(z \in \rho(T))$ and then Method II appears to be the natural analogous of Brakhage's method for the same problem (*cf.* Atkinson, 1973 and Brakhage, 1960).

A summary of all these methods is presented in Chatelin (1983) together with other methods obtained by a generalization of the results of Perturbation theory for linear operators (*cf.* Kato 1976 and Chatelin, 1983).
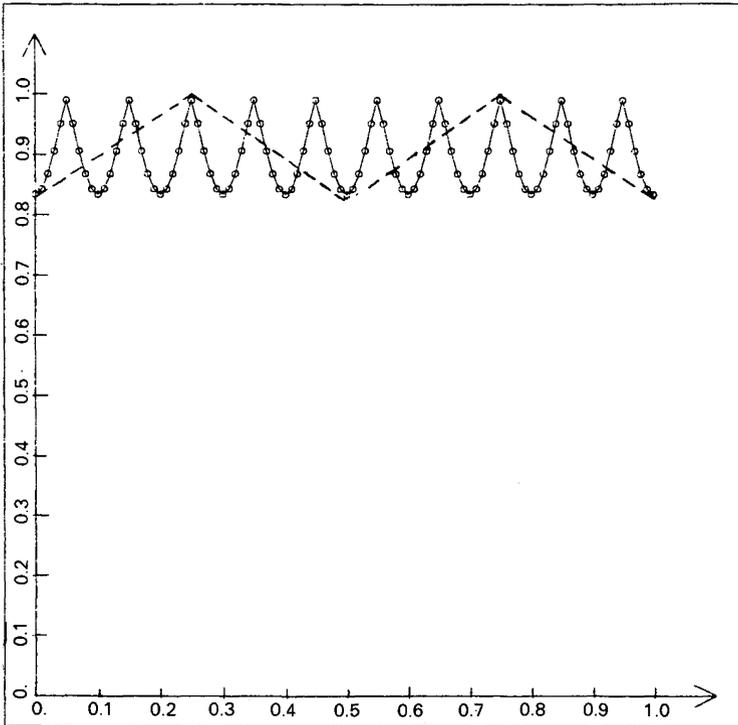
Figure 3. — **Fredholm's Approximation** $n = 5$, $M = 101$
- - - - - - - Starting vector $x^0 = \phi_n$
O-O-O-O-O Last Iterate.

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