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THE COMBINATION OF APPROXIMATE SOLUTIONS FOR ACCELERATING THE CONVERGENCE (*)

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Communiqué par A. BENSOUSSAN

Résumé. — *D'abord nous mentionnons le principe de combinaison des solutions approchées pour l'accélération de la convergence. Et puis nous donnons deux types de combinaisons, le premier est l'interpolation, l'autre est l'extrapolation de décomposition, accélérant la convergence des solutions numériques des équations intégrales et des équations aux dérivées partielles.*

Abstract. — *A principle for the combination of approximate solutions for accelerating the convergence is first proposed, and then two types of combination, one of interpolation and the other of the splitting extrapolation, are given for accelerating the numerical solution of integral equations and partial differential equations.*

1. COMBINATION PRINCIPLE

Let us consider the linear operator equation

$$Lu = f. \tag{1}$$

The operator L will map a subspace of a Banach space U into a Banach space F. Problem (1) will be assumed to have an unique solution u.

Usually this problem cannot be solved in a closed form and it is replaced by some associate, simpler problems depending on a small parameter h :

$$L_i^h u_i^h = f_i^h , \qquad o \leqslant i \leqslant M . \tag{2}$$

For ease of understanding we first consider a simple case in which the operators L_i^h map the Banach space U into the Banach space F just as L does and $f_i^h = f$.

The operators L_i^h, $0 \leqslant i \leqslant M$, in (2) will be assumed to have the following properties :

(A) $\| (L_i^h)^{-1} \| \leqslant c_i$,

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(*B*) for u, the solution of (1), there exists $l_i(u) \in F$ such that

$$L_i^h u - Lu = h^p(l_i(u) + r_i^h) \to 0 \qquad (h \to 0),$$

(*C*) the problem

$$Lw_i = l_i(u)$$

has a solution w_i and the associate approximate problem

$$L_i^h w_i^h = l_i(u)$$

has a solution w_i^h which converges to w_i :

$$e_i^h = w_i^h - w_i \to 0 \quad (h \to 0)$$

LEMMA 1 (Combination principle) : *Under the assumptions* (*A*), (*B*), (*C*), *if there exist constants* a_i, $0 \leqslant i \leqslant M$, *such that*

$$(D) \qquad \sum_0^M a_i\, l_i(u) = 0\,, \quad \textit{where} \quad \sum_0^M a_i \neq 0$$

then

$$u - \left(\sum_0^M a_i\right)^{-1} \sum_0^M a_i\, u_i^h = h^p \left(\sum_0^M a_i\right)^{-1} \sum_0^M a_i (e_i^h + (L_i^h)^{-1}\, r_i^h) \tag{3}$$

Proof : By assumption (*B*),

$$L_i^h(u - u_i^h - h^p\, w_i) = L_i^h\, u - Lu - h^p\, L_i^h\, w_i = h^p(l_i(u) - L_i^h\, w_i + r_i^h)\,.$$

This leads to, by assumption (*C*),

$$u - u_i^h - h^p\, w_i = h^p(w_i^h - w_i + (L_i^h)^{-1}\, r_i^h) = h^p(e_i^h + (L_i^h)^{-1}\, r_i^h)\,.$$

Hence (3) follows from assumption (*D*) :

$$\sum a_i\, w_i = L^{-1} \sum a_i\, l_i(u) = 0\,.$$

In order to include the finite difference method we have to describe the spaces and operators in (1), (2) in detail. The spaces U and F will consist of functions defined on the domain Ω, and $Lu \in F$ for $u \in U$. We define U_i^h and F_i^h, the spaces of the mesh functions defined on the mesh $\Omega_i^h \subset \Omega$ and $\overset{\circ}{\Omega}{}_i^h \subset \Omega$ respectively; and the operator L_i^h will map U_i^h into F_i^h.

If $u \in U, f \in F$, then $u, f, Lu, l_i(u), r_i^h, w_i$ can be considered as a mesh function defined on the mesh Ω_i^h (or $\overset{\circ}{\Omega}{}_i^h$), and $L_i^h\, u$, $L_i^h\, u - Lu$ is defined, and (3) will hold true on the mesh points $x \in \bigcap_0^M \Omega_i^h$.

We note that the combination method stated above seems appropriate for the parallel algorithm since the approximations u_i^h, $0 \leqslant i \leqslant M$, are independent.

2. INTERPOLATION OF APPROXIMATE SOLUTIONS

We want to solve the problem

$$u(x) - \int_0^1 K(x, y)\, u(y)\, dy = f(x)\,, \tag{4}$$

where the kernel $K(x, y)$ and the given function f are assumed to be sufficiently smooth and 1 is not an eigenvalue of (4).

The integral operator

$$Ku = \int_0^1 K(x, y)\, u(y)\, dy$$

in (4) will be replaced by the rectangle cubature operator K_0^h defined by

$$K_0^h u = h \sum_0^{n-1} K\left(x, \left(i + \frac{1}{2}\right) h\right) u\left(\left(i + \frac{1}{2}\right) h\right)$$

and by the trapezoid cubature operator K_1^h defined by

$$K_1^h u = \frac{h}{2} \sum_0^{n-1} \left(K(x, ih)\, u(ih) + K(x, (i + 1)\, h)\, u((i + 1)\, h)\right)$$

respectively, where $h = 1/n$. Correspondingly problem (4) will be replaced by the rectangle Nyström solution u_0^h defined by

$$u_0^h - K_0^h u_0^h = f$$

and by the trapezoid Nyström solution u_1^h defined by

$$u_1^h - K_1^h u_1^h = f$$

respectively.

We now explain how this problem can be embedded in the framework of Section 1. Set.

$$U = F = C[0, 1]\,, \qquad L = I - K\,,$$
$$L_i^h = I - K_i^h\,, \qquad i = 0, 1\,.$$

From the collectively compact operator theory of Anselone [1],

$$\| (K_i^h - K) K_i^h \| \to 0$$

which will lead to assumption (A). Assumptions (B), (C) can be derived by Taylor expressions :

$$K_i^h u - Ku = h^2 l_i(u) + 0(h^4), \quad i = 0, 1$$

with

$$l_0(u) = -\frac{1}{24}\int_0^1 \frac{\partial^2}{\partial y^2} K(x, y)\, u(y)\, dy\,, \quad l_1(u) = \frac{1}{12}\int_0^1 \frac{\partial^2}{\partial y^2} K(x, y)\, u(y)\, dy\,.$$

And the combination condition (D) is satisfied by

$$l_1(u) + 2\, l_0(u) = 0\,.$$

Hence, by (3),

$$u - \frac{1}{3}(u_1^h + 2\, u_0^h) = 0(h^4) \tag{5}$$

on [0, 1]. Compare it with the approximations u_1^h, u_0^h which are convergent of order h^2.

The combination principle also applies for the Poisson equation

$$\begin{aligned} Lu &= \sum_{i=1}^{3} u_{x_i x_i} = f \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \partial\Omega \end{aligned} \tag{6}$$

with domain $\Omega = (-1, 1)^3$ for simplicity and the solution u being sufficiently smooth.

The Laplacian operator L in (6) will be replaced by the 7-point difference operator L_7^h defined by

$$\begin{aligned} L_7^h u(x_1, x_2, x_3) = {} & (u(x_1 + h, x_2, x_3) + u(x_1 - h, x_2, x_3) + u(x_1, x_2 + h, x_3) + \\ & + u(x_1, x_2 - h, x_3) + u(x_1, x_2, x_3 + h) + u(x_1, x_2, x_3 - h) \\ & - 6\, u(x_1, x_2, x_3))/h^2 \end{aligned}$$

and by the 9-point difference operator L_9^h defined by

$$L_9^h\, u(x_1, x_2, x_3) = \left(\sum u(x_1 \pm h, x_2 \pm h, x_3 \pm h) - 8\, u(x_1, x_2, x_3)\right)/4\, h^2,$$

$$\sum u(x_1 \pm h, x_2 \pm h, x_3 \pm h) = u(x_1 + h, x_2 + h, x_3 + h) + u(x_1 - h, x_2 + h, x_3 + h) +$$

$$+u(x_1+h, x_2-h, x_3+h)+u(x_1-h, x_2-h, x_3+h)+u(x_1+h, x_2+h, x_3-h)$$
$$+u(x_1-h, x_2+h, x_3-h)+u(x_1+h, x_2-h, x_3-h)+u(x_1-h, x_2-h, x_3-h)\,.$$

It is easy to see that the truncation error will be

$$L_i^h u - Lu = h^2 l_i(u) + 0(h^4)\,, \quad i = 7, 9$$

with

$$l_7(u) = \frac{1}{12}\sum u_{x_i x_i x_i x_i}\,,$$
$$l_9(u) = \frac{1}{12}\left(\sum u_{x_i x_i x_i x_i} + 6\sum_{i<j} u_{x_i x_i x_j x_j}\right)$$

and

$$\frac{2}{3}l_7(u) + \frac{1}{3}l_9(u) = \frac{1}{12}L^2 u = L\left(\frac{1}{12}f\right). \tag{7}$$

Correspondingly the problem (6) will be replaced by the 7-point difference solution u_7^h defined by

$$\begin{aligned} L_7^h u_7^h &= f \quad \text{in} \quad \mathring{\Omega} \\ u_7^h &= g - \frac{h^2}{12}f \quad \text{on} \quad \partial\Omega_h \end{aligned} \tag{8}$$

and by the 9-point difference solution u_9^h defined by

$$\begin{aligned} L_9^h u_9^h &= f \quad \text{in} \quad \mathring{\Omega}_h \\ u_9^h &= g - \frac{h^2}{12}f \quad \text{on} \quad \partial\Omega_h \end{aligned}$$

with lattice domain

$$\mathring{\Omega}_h = \{ (x_1, x_2, x_3), x_i = m_i h, m_i = 0, \pm 1, ..., \pm n - 1, nh = 1 \}\,.$$

Then the combination principle will lead to

$$u - \frac{1}{3}(2\,u_7^h + u_9^h) - \frac{h^2}{12}f = 0(h^4) \quad \text{in} \quad \mathring{\Omega}_h\,. \tag{10}$$

In fact letting w_i be the solution of an auxiliary problem

$$\begin{aligned} Lw_i &= l_i(u) \quad \text{in} \quad \Omega \\ w_i &= \frac{1}{12}f \quad \text{on} \quad \partial\Omega \end{aligned}$$

for $i = 7, 9$, we have

$$L_i^h(u - u_i^h - h^2 w_i) = L_i^h u - Lu - h^2 L_i^h w_i$$
$$= h^2 l_i(u) - h^2 Lw_i + h^2(Lw_i - L_i^h w_i) + 0(h^4) = 0(h^4) \text{ in } \mathring{\Omega}_h$$
$$u - u_i^h - h^2 w_i = o \quad \text{on} \quad \partial\Omega_h$$

and, by the maximum principle,

$$u - u_i^h - h^2 w_i = 0(h^4) \quad \text{in} \quad \mathring{\Omega}_h .$$

Hence

$$u - \frac{1}{3}(2 u_7^h + u_9^h) - \frac{h^2}{3}(2 w_7 + w_9) = 0(h^4) \quad \text{in} \quad \mathring{\Omega}_h$$

where $w = \frac{1}{3}(2 w_7 + w_9)$ satisfies, by (7),

$$Lw = \frac{2}{3} l_7(u) + \frac{1}{3} l_9(u) = L\left(\frac{1}{12} f\right) \quad \text{in} \quad \Omega$$
$$w = \frac{1}{12} f \qquad \text{on} \quad \partial\Omega$$

i.e. $w = \frac{1}{12} f$ and (10) is proved.

Remark : Bramble has proposed a 19-point difference operator L_{19}^h and a difference solution u_{19}^h defined by

$$L_{19}^h u_{19}^h = f + \frac{h^2}{12}\Delta f \quad \text{in} \quad \mathring{\Omega}$$
$$u_{19}^h = g \quad \text{on} \quad \partial\Omega_h \tag{11}$$

and proved that

$$u - u_{19}^h = 0(h^4) \quad \text{in} \quad \mathring{\Omega}_h .$$

It seems that (8) with (9) is easier to solve than the Bramble scheme (11) not only because of the 19-point operator L_{19}^h but also because of the Δf appearing in (11).

We would like to mention that the combination principle can also be used to deal with the eigenvalue problems and the midly nonlinear elliptic problems by combining with the correction procedure (see Fox, Pereyra). We will discuss it in detail in a separate paper.

3. SPLITTING EXTRAPOLATION

The combination principle stated above is very similar to the classical extrapolation method due to Richardson (see Laurent).

The disadvantage of the extrapolation method is the computation of another approximation with a small parameter, say $h/2$, which involves computing once again an approximation of a much larger size than the one corresponding to the original h for multidimensional problems. To remove this imperfection, we present in the following a splitting extrapolation procedure which will save computational work and storage.

It is known (see Lin Qun and Lu Tao) that there exists an asymptotic expansion in powers of $h = (h_1, \dots, h_s)$ for the numerical integral $u(h)$ or the difference solution $u(h)$:

$$u(h) = u + \sum_{1 \leqslant |\alpha| \leqslant m} c_\alpha h^{2\alpha} + 0(|h|^{2m+1}) \tag{12}$$

where

$$\alpha = (\alpha_1, \dots, \alpha_s), \quad |\alpha| = \alpha_1 + \cdots + \alpha_s, \quad h^\alpha = h_1^{\alpha_1} \dots h_s^{\alpha_s}.$$

Let N_m be the number of the elements in the index set

$$S_m = \{ \alpha : 1 \leqslant |\alpha| \leqslant m \}.$$

The left side $u(h)$ of (12) is known while we take u and c_α ($\alpha \in S_m$) as unknowns. Let us now make up the following N_m equations

$$u(h/2^\beta) = u + \sum_{1 \leqslant |\alpha| \leqslant m} c_\alpha h^{2\alpha}/2^{2(\beta,\alpha)} \quad \forall \beta \in S_m$$

where

$$h/2^\beta = (h_1/2^{\beta_1}, \dots, h_s/2^{\beta_s}), \quad (\beta, \alpha) = \sum_1^s \beta_i \alpha_i.$$

Then we can solve u by using the general extrapolation algorithm presented by Brezinski. Note that the number of the points for numerical integral is $\binom{m+s}{m} 2^m$.

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