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## EXTERNAL APPROXIMATION OF EIGENVALUE PROBLEMS IN BANACH SPACES (\*)

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*Abstract.* — We are concerned with approximate methods for solving the eigenvalue problem  $Tu = \lambda u$ ,  $u \neq 0$ , for the linear bounded operator  $T$  in a Banach space  $X$ . The problem is approximated by an appropriate family of eigenvalue problems for operators  $\{T_h\}$ . We present a theoretical framework which allows us to consider in the same way the methods for which  $T_h$  are defined on subspaces of  $X$  and those which are defined on spaces forming external approximation of  $X$ . Particularly, the paper contains theorems on sufficient conditions for stability and strong stability of  $\{T_h\}$ .

*Résumé.* — On considère ici une classe de méthodes de résolution approchée du problème spectral de la forme  $Tu = \lambda u$ , où  $T$  est un opérateur linéaire, borné dans un espace Banach  $X$ . Les méthodes présentées remplacent le problème original par une famille de problèmes spectraux pour des opérateurs  $T_h$ . Les résultats sont présentés d'une manière qui permet de considérer à la fois les méthodes où les  $T_h$  sont définis sur des sous-espaces de  $X$  et celles où les espaces de définition de  $T_h$  forment une approximation externe de  $X$ . L'ouvrage contient certaines conditions suffisantes de stabilité et de stabilité forte de la famille  $\{T_h\}$ .

### 1. INTRODUCTION

Let  $X$  be a Banach space and  $T \in \mathcal{L}(X)$  be a linear bounded operator on  $X$ . Let us consider the eigenvalue problem  $Tu = \lambda u$ ,  $u \neq 0$ . Most methods used to solve this problem consist in approximation of the initial problem by a sequence of eigenvalue problems for  $T_h \in \mathcal{L}(X_h)$ , where  $X_h$  are finite dimensional subspaces of  $X$  and  $T_h$  are certain approximantes of  $T$ . This approach has been used in many papers, among others by J. Decloux, N. Nassif, J. Rappaz in [5] and by F. Chatelin in [2]. However, there are methods which cannot be presented within this unifying theoretical framework (e.g. the Aronszajn's method, cf. [1, 12]). Therefore we consider the more general case of approximation when the operators  $T_h$  are defined in spaces not contained in  $X$ . Strictly speaking we use an external approximation of  $X$ . We present some theorems

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concerning the approximation of eigenelements of  $T$  by eigenelements of  $T_h$ . Particularly we formulate new theorems about sufficient conditions for stability and strong stability of  $\{T_h\}$ .

Let us introduce a family of Banach spaces  $\{X_h\}_{h \in \mathcal{H}}$  with the norms  $\|\cdot\|_h$ , where  $\mathcal{H} \subset \mathbb{R}^+$  has an accumulation point at 0. We assume that there exist uniformly bounded linear maps  $r_h : X \xrightarrow{\text{on}} X_h$ . Let  $F$  be a normed space such that there exist an isomorphism  $\omega : X \rightarrow F$  and uniformly bounded linear maps  $p_h : X_h \rightarrow F$ . We adopt the following definition :

**DEFINITION 1 :** *An approximation  $\{X_h, r_h, p_h\}$  of  $X$  is said to be an external approximation convergent in  $F$  if for any  $u \in X$*

$$\lim_{h \rightarrow 0} \|\omega u - p_h r_h u\|_F = 0.$$

The above definition is weaker than that used customarily (cf. [11, 6]).

Next, let us introduce a family  $\{T_h\}_{h \in \mathcal{H}}$  of linear operators where  $T_h \in \mathcal{L}(X_h)$ . We will assume that :

A1 : The approximation  $\{X_h, r_h, p_h\}$  of  $X$  is convergent in  $F$ ;

A2 : For any  $u \in X \lim_{h \rightarrow 0} \|r_h T u - T_h r_h u\|_h = 0$ .

**2. STABILITY OF  $\{T_h\}$**

Let  $\rho(T)$  and  $\rho(T_h)$  denote, as usually, the resolvent sets of operators  $T$  and  $T_h$  respectively. We additionally assume that either the operators  $T_h$  have no residual spectrum or that the residual spectrum of  $T_h$  does not contain the points of  $\rho(T)$  (since not only finite dimensional approximation is considered). We will use the following definition of stability cf. [4, 2] :

**DEFINITION 2 :** *The approximation  $\{T_h\}$  is stable at  $z \in \rho(T)$  iff  $\exists h(z)$ ,*

$$\forall h \leq h(z) : z \in \rho(T_h) \text{ and } \|(z - T_h)^{-1}\| \leq M(z) < \infty.$$

Now we are going to formulate some sufficient conditions for stability of  $\{T_h\}$  in terms of external approximation of  $T$ .

Let  $N(r_h)$  denote the null space of  $r_h$ . Let us introduce the set of families of complementary subspaces of  $N(r_h)$  in  $X$

$$\mathcal{F} = \{ \{V_h\}_{h \in \mathcal{H}}, V_h \subset X, V_h \oplus N(r_h) = X \}.$$

**LEMMA 1 :** *If there exists  $\{V_h\}_{h \in \mathcal{H}} \in \mathcal{F}$  such that*

$$\delta_h = \delta(V_h) := \sup_{\substack{v \in V_h \\ \|v\|=1}} \|\omega T v - p_h T_h r_h v\|_F \rightarrow 0, \tag{2.1}$$

$$\varepsilon_h = \varepsilon(V_h) := \sup_{\substack{v \in V_h \\ \|v\|=1}} \|\omega v - p_h r_h v\|_F \rightarrow 0, \tag{2.2}$$

then  $\{ T_h \}$  is stable at any  $\lambda \in \rho(T)$ .

*Proof:* Let  $\lambda \in \rho(T)$ . Hence, there exists  $c > 0$  such that

$$\|(\lambda - T) u\| \geq c \|u\| \quad \forall u \in X,$$

and for  $\tilde{c} = c/\|\omega^{-1}\|$ ,  $\|\omega(\lambda - T) u\|_F \geq \tilde{c} \|u\| \quad \forall u \in X$ . Let us take an arbitrary  $u_h \in X_h$ . Then there exists  $v_h \in V_h$  such that  $r_h v_h = u_h$ . We have  $\|v_h\| \geq (1/d) \|u_h\|_h$  and  $\forall x_h \in X_h \quad \|x_h\|_h \geq 1/d \|p_h x_h\|_F$ , where

$$d \geq \max(\|p_h\|, \|r_h\|)$$

for any  $h$ . Hence

$$\begin{aligned} \|(\lambda - T_h) u_h\|_h &= \|(\lambda - T_h) r_h v_h\|_h \geq \frac{1}{d} \|p_h(\lambda - T_h) r_h v_h\|_F = \\ &= \frac{1}{d} \|\omega(\lambda - T) v_h + \lambda(p_h r_h - \omega) v_h + (\omega T - p_h T_h r_h) v_h\|_F \geq \\ &\geq \frac{1}{d^2} \|u_h\|_h (\tilde{c} - |\lambda| \varepsilon_h - \delta_h). \end{aligned}$$

Thus, for given  $\lambda \in \rho(T)$  there exists  $h_0$  such that for  $h < h_0$

$$\|(\lambda - T_h) u_h\|_h \geq \frac{\tilde{c}}{2d^2} \|u_h\|_h,$$

what means, according to Definition 2, that  $\{ T_h \}$  is stable at  $\lambda$ .

*Remark 1:* In the case of an internal approximation of  $X$ , when  $F = X$ ,  $X_h = V_h \subset X$  and  $\omega$  and  $p_h$  are identity maps, and  $r_h$  are projections of  $X$  on  $X_h$ , the condition (2.2) is automatically satisfied with  $\varepsilon_h = 0$ . In turn, the condition (2.1) takes the form  $\|(T - T_h)| X_h\| \rightarrow 0$  i.e. the assumption of Lemma 1 in [5].

In the general case of an external approximation we have  $\varepsilon_h \neq 0$ . Thus, we must analyse how  $\varepsilon(V_h)$  depends on  $\{ V_h \} \in \mathcal{F}$ . To do this we introduce the following numbers characterizing the subspaces  $V_h$ :

$$\gamma(V_h) := \sup_{\substack{v \in V_h \\ \|v\|=1}} \|Q_h v\|, \tag{2.3}$$

where  $Q_h (h \in \mathcal{H})$  are some given linear and bounded projections of  $X$  onto  $N(r_h)$ .

Let  $\hat{V}_h = (1 - Q_h) X$ . In this case  $\gamma(\hat{V}_h) = 0$ .

We can state the following result :

**LEMMA 2 :** *Let us assume that  $\varepsilon(\hat{V}_h) \rightarrow 0$  as  $h \rightarrow 0$ . Then  $\varepsilon(V_h) \rightarrow 0$  for  $\{V_h\} \in \mathcal{F}$  if and only if  $\gamma(V_h) \rightarrow 0$ .*

*Proof :*

$$\begin{aligned} \varepsilon(V_h) &= \sup_{\substack{v \in \hat{V}_h \\ \|v\|=1}} \|\omega Q_h v + \omega(1 - Q_h)v - p_h r_h(1 - Q_h)v\|_F \geq \\ &\geq \sup_{\substack{v \in \hat{V}_h \\ \|v\|=1}} \left\{ \frac{1}{\|\omega^{-1}\|} \|Q_h v\| - \|(1 - Q_h)v\| \varepsilon(\hat{V}_h) \right\} \\ &\geq \frac{1}{\|\omega^{-1}\|} \gamma(V_h) - (1 + \gamma(V_h)) \varepsilon(\hat{V}_h). \end{aligned}$$

The implication “ $\Rightarrow$ ” follows from the above inequality.

It is easy to see that

$$\varepsilon(V_h) \leq \sup_{\substack{v \in \hat{V}_h \\ \|v\|=1}} \{ \|\omega\| \cdot \|Q_h v\| + \|(1 - Q_h)v\| \varepsilon(\hat{V}_h) \} \leq \gamma(V_h) \|\omega\| + \varepsilon(\hat{V}_h)$$

which ends the proof of Lemma 2.

In the case when the  $X_h$  are infinite dimensional spaces the condition (2.2) becomes very strong, so another version of Lemma 1 will be more useful in this special case. Let us introduce the following

**DEFINITION 3 :** *The family  $\{V_h\}$ ,  $V_h \subset X$  is asymptotically equivalent to  $\{X_h\}$  with respect to  $\{r_h\}$  ( $r_h \in \mathcal{L}(X, X_h)$ ,  $r_h X = r_h V_h = X_h$ ) if the  $r_h$  are uniformly bounded and  $\inf_{\substack{x \in V_h \\ \|x\|=1}} \|r_h x\|_h \geq c > 0$ ,  $\forall h \in \mathcal{H}$ .*

**LEMMA 3 :** *If there exist  $\{\hat{r}_h\}$  and  $\{\hat{V}_h\}$  asymptotically equivalent to  $\{X_h\}$  with respect to  $\{\hat{r}_h\}$  such that*

$$\hat{\delta}(\hat{V}_h) := \sup_{\substack{v \in \hat{V}_h \\ \|v\|=1}} \|(T - (r_h|_{V_h})^{-1} T_h r_h)v\| \rightarrow 0,$$

then  $\{T_h\}$  is stable at any  $\lambda \in \rho(T)$ .

*Proof :* Let us take  $u_h \in X_h$ . Let  $v_h \in V_h$  be such that  $\hat{r}_h v_h = u_h$  :

$$\begin{aligned} \|(\lambda - T_h)u_h\|_h &= \|(\lambda - T_h)\hat{r}_h v_h\|_h = \|\hat{r}_h(\hat{r}_h|_{V_h})^{-1}(\lambda - T_h)\hat{r}_h v_h\|_h \geq \\ &\geq c \|\lambda v_h - T v_h + (T - (\hat{r}_h|_{V_h})^{-1} T_h \hat{r}_h)v_h\| \\ &\geq c \|(\lambda - T)v_h\| - \hat{\delta}(\hat{V}_h) \|v_h\|. \end{aligned}$$

Since  $\lambda \in \rho(T)$ , there exists a constant  $c_1 > 0$  such that  $\|(\lambda - T)v_h\| \geq c_1 \|v_h\|$ . Moreover,  $\|v_h\| \geq \frac{1}{\|\hat{r}_h\|} \|u_h\|_h$ . If  $c_2 := \sup_h \|\hat{r}_h\|$ , then

$$\|(\lambda - T_h)u_h\| \geq \left\{ \frac{c \cdot c_1}{c_2} - \frac{\hat{\delta}(\hat{V}_h)}{c_2} \right\} \|u_h\|_h,$$

what proves Lemma 3.

Now, we are going to give a short analysis of the assumptions of the above lemma. To do this we restrict our considerations to the case of separable Hilbert spaces.

**LEMMA 4 :** *For an arbitrary separable Hilbert space  $X$  and a family of separable Hilbert spaces  $X_h$  there exist uniformly bounded maps  $r_h : X \rightarrow X_h$  such that the orthogonal complements of the null spaces of  $r_h$  form a family asymptotically equivalent to  $\{X_h\}$  with respect to  $\{r_h\}$ .*

*Proof :* Let  $\{u_n\}_{n=1}^\infty$  and  $\{u_n^h\}_{n=1}^\infty$  be orthonormal bases in  $X$  and  $X_h$  respectively. If  $X_h$  is  $k$ -dimensional, we put  $u_j^h = 0$  for  $j > k$ . Transformation  $\varphi : X \rightarrow l^2$  and  $\varphi_h : X_h \rightarrow l^2$  are defined as follows :

$$\begin{aligned} \varphi u &= \{(u, u_1), (u, u_2), \dots\} \quad \text{for } u \in X, \\ \varphi_h v &= \{(v, u_1^h)_h, (v, u_2^h)_h, \dots\} \quad \text{for } v \in X_h. \end{aligned}$$

Thus  $\forall u \in X \|\varphi u\|_{l^2} = \|u\|$  and  $\forall \{x_n\} \in l^2$

$$\|\varphi^{-1} \{x_n\}\|^2 = \left\| \sum_{n=1}^{\infty} x_n u_n \right\|^2 = \sum_{n=1}^{\infty} x_n^2 = \|\{x_n\}\|_{l^2}^2.$$

Similarly  $\|\varphi_h\| = 1$  and  $\varphi_h^{-1} : \varphi_h X_h \rightarrow X_h$ ,  $\|\varphi_h^{-1}\| = 1$ . Let  $P_h$  be the orthogonal projection from  $l^2$  onto  $\varphi_h X_h$ . Let

$$r_h := \varphi_h^{-1} P_h \varphi : X \rightarrow X_h, \quad (2.5)$$

$$V_h := \varphi^{-1} \varphi_h X_h. \quad (2.6)$$

For any  $v \in X$   $\|r_h v\|_h \leq \|v\|$  and since  $\varphi V_h = \varphi_h X_h$ ,  $r_h|_{V_h} = \varphi_h^{-1} \varphi|_{V_h}$  and  $(r_h|_{V_h})^{-1} = \varphi^{-1} \varphi_h$ . Thus  $\|(r_h|_{V_h})^{-1}\| = 1$ . Hence  $\{V_h\}$  is asymptotically equivalent to  $\{X_h\}$  with respect to  $\{r_h\}$ .

Now, let us take arbitrary elements  $v \in V_h$  and  $x \in N(r_h)$ . For  $v$  there exists  $u_v \in X_h$  such that  $(v, u_i) = (u_v, u_i^h)$ ,  $i = 1, 2, \dots$ . Hence  $(v, x) = \sum_{i=1}^{\infty} (u_v, u_i^h)(x, u_i)$ .

Since  $\varphi x \perp \varphi_h X_h$ ,  $\sum_{i=1}^{\infty} (x, u_i) (u, u_i^h) = 0$  for any  $u \in X_h$ , so it also holds for  $u = u_v$ . Thus  $(v, x) \stackrel{i}{=} 0$  for any  $v \in V_h$  and  $x \in N(r_h)$ , what means that  $V_h$  is orthogonal to  $N(r_h)$ .

Let  $Q_h$  be orthogonal projection onto  $N(r_h)$ , and  $V_h$  be complementary subspace of  $N(r_h)$  in  $X$ . Thus

$$\begin{aligned} \inf_{\substack{v \in V_h \\ \|v\|=1}} \|r_h v\|_h &= \inf_{\substack{v \in V_h \\ \|v\|=1}} \|r_h Q_h v + r_h(1 - Q_h) v\| = \\ &= \inf_{\substack{v \in V_h \\ \|v\|=1}} \|(1 - Q_h)v\| \cdot \left\| r_h \frac{(1 - Q_h)v}{\|(1 - Q_h)v\|} \right\| \geq \inf_{\substack{v \in V_h \\ \|v\|=1}} \|(1 - Q_h)v\| \cdot \inf_{\substack{x \perp N(r_h) \\ \|x\|=1}} \|r_h x\|_h. \end{aligned}$$

Using the notation (2.3) we obtain

$$\inf_{\substack{v \in V_h \\ \|v\|=1}} \|r_h v\|_h \geq (1 - \gamma(V_h)) \cdot \inf_{\substack{x \perp N(r_h) \\ \|x\|=1}} \|r_h x\|_h.$$

This leads us to the following remark :

*Remark 2 :* Let  $\{N(r_h)^\perp\}$  be asymptotically equivalent to  $\{X_h\}$  with respect to  $\{r_h\}$ . If  $\exists c_0 > 0$  such that  $\forall h < h_0$   $1 - \gamma(V_h) \geq c_0$  then the family  $\{V_h\}$  is also asymptotically equivalent to  $\{X_h\}$  with respect to  $\{r_h\}$ .

*Remark 3 :* If  $\{V_h\}$  satisfies the condition (2.2), then  $\{V_h\}$  is asymptotically equivalent to  $\{X_h\}$  with respect to  $\{r_h\}$ .

This follows from the inequalities :  $\forall v \in V_h, \|v\| = 1 :$

$$\|r_h v\|_h \geq \frac{1}{\|p_h\|} [\|\omega v\|_F - \varepsilon(V_h)].$$

Since  $\|p_h\| \leq \alpha$  and  $\|\omega v\|_F \geq \frac{1}{\|\omega^{-1}\|} \|v\|$ , we have

$$\|r_h v\| \geq \frac{1}{\alpha} \left[ \frac{1}{\|\omega^{-1}\|} - \varepsilon(V_h) \right]$$

for any  $v \in V_h$  and  $\|v\| = 1$ .

3. APPROXIMATION OF EIGENELEMENTS OF  $T$ 

In this section the proofs of the theorems are based on the ideas contained in [5] and [2].

Let  $\Gamma$  be a Jordan curve in the resolvent set  $\rho(T)$ . If  $\{T_h\}$  is stable for all  $\lambda \in \Gamma$ , then  $\Gamma \subset \rho(T_h)$  for sufficiently small  $h < h_0$ . Hence, we can define the spectral projectors  $E : X \rightarrow X$  and  $E_h : X_h \rightarrow X_h$  by

$$E = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T)^{-1} d\lambda, \quad E_h = \frac{1}{2\pi i} \int_{\Gamma} (\lambda - T_h)^{-1} d\lambda.$$

LEMMA 5 : If the assumption A2 is satisfied and  $\{T_h\}$  is stable on  $\Gamma$ , then  $\forall v \in X \lim_{h \rightarrow 0} \|r_h E v - E_h r_h v\|_h = 0$ .

*Proof* : From the definition of  $E$  and  $E_h$  and from the identity

$$r_h(\lambda - T)^{-1} - (\lambda - T_h)^{-1} r_h = (\lambda - T_h)^{-1} (T_h r_h - r_h T) (\lambda - T)^{-1}$$

it follows that for given  $v \in X$

$$\begin{aligned} \|r_h E v - E_h r_h v\| &\leq \frac{|\Gamma|}{2\pi} \sup_{\Gamma} \|(\lambda - T_h)^{-1} (T_h r_h - r_h T) (\lambda - T)^{-1} v\| = \\ &= \frac{|\Gamma|}{2\pi} \sup_{u \in U} \|(\lambda - T_h)^{-1} (T_h r_h - r_h T) u\|, \end{aligned}$$

where  $U = \{u \in X : u = (\lambda - T)^{-1} v, \lambda \in \Gamma\}$ .

The operators  $(\lambda - T_h)^{-1}$  are uniformly bounded for  $\lambda \in \Gamma$  and  $h < h_0$  since the stability of  $\{T_h\}$  on  $\Gamma$ . Thus, by the assumption A2,

$$\forall u \in X \quad \|(\lambda - T_h)^{-1} (T_h r_h - r_h T) u\| \rightarrow 0.$$

Moreover,

$$\|(\lambda - T_h)^{-1} (T_h r_h - r_h T)\| \leq \|(\lambda - T_h)^{-1} r_h T\| + \|\lambda (\lambda - T_h)^{-1} r_h\| + \|r_h\|,$$

so the operators  $(\lambda - T_h)^{-1} (T_h r_h - r_h T)$  are uniformly bounded for  $\lambda \in \Gamma$  and  $h < h_0$ . Thus, since the set  $U$  is compact,

$$\sup_{u \in U} \|(\lambda - T_h)^{-1} (T_h r_h - r_h T) u\| \rightarrow 0.$$



LEMMA 6 : If A1 and A2 are satisfied and  $\{ T_h \}$  is stable on  $\Gamma$ , then

$$\forall v \in EX \quad \inf_{y_h \in E_h X_h} \| \omega v - p_h y_h \|_F \rightarrow 0 .$$

*Proof* : Since

$$\inf_{y_h \in E_h X_h} \| \omega v - p_h y_h \|_F \leq \| \omega v - p_h r_h v \|_F + \| p_h \| \| r_h E v - E_h r_h v \|_h ,$$

the proof follows immediately from Lemma 5.

As usually,  $\sigma(T)$  denotes the spectrum of  $T$ . Let  $\Omega \subset \mathbb{C}$  be an open domain with the boundary  $\Gamma \subset \rho(T)$  which is a Jordan curve. Finally, let

$$K(\lambda, \delta) := \{ z \in \mathbb{C} : |z - \lambda| \leq \delta \} .$$

THEOREM 1 : If the assumptions A1 and A2 are satisfied and  $\{ T_h \}$  is stable in  $\rho(T)$  then :

- 1° if  $\Omega \cap \sigma(T) \neq \emptyset$  then  $\sigma(T_h) \cap \Omega \neq \emptyset$  for sufficiently small  $h$ ,
- 2° if  $\lambda_0 \in \sigma(T)$  and  $\exists \delta_0 > 0 : K(\lambda_0, \delta_0) \cap \sigma(T) = \{ \lambda_0 \}$  then  $\forall 0 < \delta < \delta_0$ ,  $0 \neq \sigma(T_h) \cap K(\lambda_0, \delta) \subset K(\lambda_0, \delta)$  for sufficiently small  $h$ ,
- 3° if  $\lambda_h \in \sigma(T_h)$  and  $\lambda_h \rightarrow \lambda_0$  then  $\lambda_0 \in \sigma(T)$ .

*Proof* : It follows from Lemma 5 that  $\forall v \in EX \quad \inf_{y_h \in E_h X_h} \| r_h v - y_h \|_h \rightarrow 0$ .

If  $v \neq 0$  then, since A1,  $r_h v \neq 0$  for sufficiently small  $h$ . Thus 1° is proved. For the proof of 2° it is enough to remark, that for

$$0 < \delta < \delta_0 \quad K(\lambda, \delta_0) \setminus \text{int } K(\lambda, \delta) \subset \rho(T)$$

and thus, by the stability of  $\{ T_h \}$ ,  $K(\lambda, \delta_0) \setminus \text{int } K(\lambda, \delta)$  is contained in  $\rho(T_h)$  for  $h < h_0$ . Assume now that  $\lambda_h \in \sigma(T_h)$  and  $\lambda_h \rightarrow \lambda_0 \notin \sigma(T)$ . Thus there exists  $\delta > 0$  such that  $K(\lambda_0, \delta) \subset \rho(T)$  and from the stability  $K(\lambda_0, \delta) \subset \rho(T_h)$  for  $h < h_0$ , what means that for  $h < h_1$ ,  $\lambda_h \in \rho(T_h)$ .

The above theorem gives convergence of eigenvalues, but without preservation of the algebraic multiplicities. Namely, we have only

THEOREM 2 : If A1 and A2 are satisfied and  $\{ T_h \}$  is stable on  $\Gamma$  then

- 1°  $\dim EX = \infty \Rightarrow \dim E_h X_h \rightarrow \infty$
- 2°  $\dim EX = n \Rightarrow \dim p_h E_h X_h \geq n$ .

*Proof* : Let  $\{ u_i \}_{i=1}^\infty$  be a linearly independent set of elements of  $EX$ . From Lemma 6 it follows that for every finite number

$$N \forall \varepsilon \exists h_\varepsilon \forall h < h_\varepsilon \forall i = 1, \dots, N \exists x_i^h \in E_h X_h : \| \omega u_i - p_h x_i^h \|_F \leq \varepsilon .$$

Thus  $\forall N < \infty \exists h_N \forall h < h_N \dim p_h E_h X_h \geq N$ , hence 1°.

Let now  $\dim EX = n$ . By Lemma 6 we have

$$\sup_{\substack{v \in EX \\ \|v\|=1}} \inf_{y_h \in E_h X} \|\omega v - p_h y_h\|_F \rightarrow 0.$$

Using the known notation (cf. [7] chap. IV) : for closed subspaces  $Y, Z$  of  $X$

$$\delta(Y, Z) = \sup_{\substack{y \in Y \\ \|y\|=1}} \inf_{z \in Z} \|y - z\|, \quad (3.1)$$

we have  $\delta(\omega EX, p_h E_h X_h) \rightarrow 0$ . It is known that if  $\delta(Y, Z) < 1$  then  $\dim Y \leq \dim Z$  (cf. [7] chap. IV, Corollary 2.6). Thus

$$n = \dim \omega EX \leq \dim p_h E_h X_h.$$

Under additional assumptions we can state the following result :

**THEOREM 3 :** *One supposes  $A_1, A_2$  and stability of  $\{T_h\}$  on  $\Gamma$ . Moreover let  $\|p_h u_h - f\|_F \rightarrow 0$ , where  $u_h \in X_h$ , imply that  $f$  belongs to  $\omega X$ , and let the norms in  $F$  and  $X_h$  be asymptotically equivalent (i.e. if  $u_h \in X_h$  and  $\|p_h u_h\|_F \rightarrow 0$  then  $\|u_h\|_h \rightarrow 0$ ). Then if  $x_h \in E_h X_h$  and  $\|p_h x_h - f\|_F \rightarrow 0$  then  $f \in \omega EX$ .*

*Proof :* If  $\|p_h x_h - f\|_F \rightarrow 0$  then there exists  $x_0 \in X$  such that  $f = \omega x_0$ . It remains to show that  $Ex_0 = x_0$ . From the inequality

$$\|\omega x - p_h x_h\|_F \geq \|\omega(Ex_0 - x_0)\| - \|\omega Ex_0 - p_h E_h r_h x_0\|_F - \|p_h E_h(r_h x_0 - x_h)\|_F$$

we get

$$\|Ex_0 - x_0\| \leq \|\omega^{-1}\| [\|\omega x_0 - p_h x_h\|_F + \|\omega Ex_0 - p_h r_h Ex_0\|_F + \|p_h\| \|r_h Ex_0 - E_h r_h x_0\|_h + \|p_h E_h\| \|r_h x_0 - x_h\|_h].$$

The convergence  $\|p_h x_h - \omega x_0\| \rightarrow 0$  implies  $\|p_h r_h x_0 - p_h x_h\|_F \rightarrow 0$  and thus, by the additional assumption on  $p_h$ ,  $\|r_h x_0 - x_h\|_h \rightarrow 0$ . By Lemma 5 and  $A_1$  we have :  $\forall \varepsilon \exists h_0 \forall h < h_0 \|Ex_0 - x_0\| \leq \varepsilon$ , thus  $Ex_0 = x_0$ .

#### 4. STRONG STABILITY OF $\{T_h\}$

Let  $\Omega \subset \mathbb{C}$  be a domain limited by the Jordan curve  $\Gamma \subset \rho(T)$ . Let  $E$  and  $E_h$  be the spectral projections associated with the spectrum of  $T$  and  $T_h$  inside  $\Gamma$ . We will assume that  $\dim EX < \infty$ . With respect to the convergence of eigenvectors it is very important to have the same dimensions of  $E_h X_h$  (or  $p_h E_h X_h$ )

and  $EX$ . We will use the notion of strongly stable approximation  $\{ T_h \}$  similar to that introduced by F. Chatelin in [4].

**DEFINITION 4 :** *An approximation  $\{ T_h \}$ , stable on  $\Gamma$ , is strongly stable on  $\Gamma$  if  $\dim EX = \dim p_h E_h X_h$  for  $h$  small enough.*

The convergence of external approximation (i.e. A1), the consistency of  $\{ T_h \}$  to  $T$  (i.e. A2) and the stability of  $\{ T_h \}$  are not sufficient for strong stability of  $\{ T_h \}$ , so we need a stronger assumption.

**LEMMA 7 :** *If  $\{ T_h \}$  is stable on  $\Gamma$  and*

$$\| (T_h r_h - r_h T)(\lambda - T)^{-1} \|_h \rightarrow 0 \text{ for } \lambda \in \Gamma \tag{3.2}$$

*then  $\| r_h E - E_h r_h \|_{\mathcal{L}(X, X_h)} \rightarrow 0$ .*

*Proof :* Repeating argumentation of the proof of Lemma 5 we get  $\| r_h E - E_h r_h \| \leq c_0 \| (T_h r_h - r_h T)(\lambda - T)^{-1} \|$  for a some constant  $c_0$ .

**LEMMA 8 :** *If there exists  $\{ V_h \} \in \mathcal{F}$  such that  $\forall h < h_0$*

$$\eta_h := \inf_{\substack{x \in V_h \\ \|x\|=1}} \| p_h r_h x \|_F \geq \varepsilon_0 > 0$$

*then*

$$\delta(p_h E_h X_h, \omega EX) \leq \frac{1}{\varepsilon_0} \| p_h E_h r_h - \omega E \| .$$

*Proof :* Let  $\tilde{V}_h$  be a subspace of  $V_h$  such that  $r_h \tilde{V}_h = E_h X_h$ . Then

$$\| p_h E_h r_h - \omega E \| \geq \sup_{\substack{x \in X \\ \|x\|=1}} \inf_{y \in EX} \| p_h E_h r_h x - \omega y \| \geq$$

$$\geq \sup_{\substack{x \in \tilde{V}_h \\ \|x\|=1}} \inf_{y \in EX} \| p_h r_h x - \omega y \| \geq \inf_{\substack{x \in \tilde{V}_h \\ \|x\|=1}} \| p_h r_h x \| \sup_{\substack{x_h \in E_h X_h \\ \|p_h x_h\|=1}} \inf_{y \in EX} \| p_h x_h - \omega y \| .$$

According to (3.1) the last factor is equal to  $\delta(p_h E_h X_h, \omega EX)$ .

**THEOREM 4 :** *If the assumptions A1, (2.1), (2.2), (3.2) are satisfied, then  $\{ T_h \}$  is strongly stable on  $\Gamma$ .*

*Proof :* It follows from (2.2) that

$$\eta_h \geq \inf_{\substack{x \in V_h \\ \|x\|=1}} \| \omega x \|_F - \sup_{\substack{x \in V_h \\ \|x\|=1}} \| p_h r_h x - \omega x \|_F \geq \frac{1}{\| \omega^{-1} \|} - \varepsilon_h ,$$

thus  $\eta_h \geq \varepsilon_0 > 0$  for sufficiently small  $h$ . Moreover, since  $\dim EX < \infty$ , by Lemma 7

$$\|p_h E_h r_h - \omega E\| \leq \|p_h\| \|E_h r_h - r_h E\| + \|(p_h r_h - \omega) E\| \rightarrow 0.$$

Hence, from Lemma 8 we get  $\delta(p_h E_h X_h, \omega EX) < 1$  for  $h$  small enough and thus  $\dim p_h E_h X_h \leq \dim \omega EX$ . The opposite inequality has been obtained in Theorem 2, thus  $\dim p_h E_h X_h = \dim EX$ .

The assumption (2.2), which is very strong in the case of infinite dimensional spaces  $X_h$ , can be omitted as it is shown in the following.

**THEOREM 5:** *Let A1 be satisfied. Moreover, let  $\{V_h\}$  be asymptotically equivalent to  $\{X_h\}$  with respect to  $\{r_h\}$  and  $\{X_h\}$  be asymptotically equivalent to  $\{p_h X_h\}$  with respect to  $\{p_h\}$ . If*

$$\|[T - (r_h|_{V_h})^{-1} T_h r_h](\lambda - T)^{-1}\| \rightarrow 0 \quad \text{for } \lambda \in \Gamma \quad (3.3)$$

then  $\{T_h\}$  is strongly stable on  $\Gamma$ .

*Proof:* It follows from (3.3) that

$$\exists c > 0 \forall h < h_0 \forall \lambda \in \Gamma \|(r_h|_{V_h})^{-1}(\lambda - T_h)r_h(\lambda - T)^{-1}\| \geq c.$$

On the other hand

$$\|(r_h|_{V_h})^{-1}(\lambda - T_h)r_h(\lambda - T)^{-1}\| \leq \|\lambda - T_h\| \|(r_h|_{V_h})^{-1}\| \|r_h\| \|(\lambda - T)^{-1}\|.$$

Thus, by the uniform boundness of  $\|(r_h|_{V_h})^{-1}\|$  and  $\|r_h\|$  we obtain that  $\|\lambda - T_h\| \geq c_1 > 0$  for  $h < h_0$  and  $\lambda \in \Gamma$ , what gives the stability of  $\{T_h\}$  on  $\Gamma$ .

Moreover, (3.3) implies (3.2). Thus, by Lemma 7,  $\|r_h E - E_h r_h\| \rightarrow 0$ , what implies  $\|p_h E_h r_h - \omega E\| \rightarrow 0$ , since  $\dim EX < \infty$ . The assumption on asymptotic equivalence of  $\{V_h\}$ ,  $\{X_h\}$  and  $\{p_h X_h\}$  guarantees the existence of positive lower bound for  $\eta_h$ . Hence, by Lemma 8,  $\delta(p_h E_h X_h, \omega EX) \rightarrow 0$ . Thus  $\dim p_h E_h X_h \leq \dim \omega EX$  what together with Theorem 2 gives:  $\dim p_h E_h X_h = \dim E_h X_h = \dim EX$  for sufficiently small  $h$ .

The condition (3.3) imposed on the approximation is some modification of radial convergence introduced in [2, 3] for the case of internal approximation.

## 5. APPLICATION

Let  $X$  be a Hilbert space with the scalar product  $a(\cdot, \cdot)$ . Let  $b$  be a bounded sesquilinear form defined on  $X \times X$ . The eigenvalue problem for two forms

$$b(u, v) = \lambda a(u, v) \quad \forall v \in X \quad (5.1)$$

is considered. This problem is equivalent to the eigenproblem for an operator  $T$  defined by  $b(u, v) = a(Tu, v) \forall u, v \in X$ . Let  $V$  be a dense subspace of  $X$ . We will consider approximate methods of solving the problem (5.1) which are generated by sequences of sesquilinear forms  $a_n$  and  $b_n$  defined on  $V \times V$ . It is assumed that  $a_n, n = 0, 1, \dots$  are symmetric and positive definite and  $b_n$  are bounded with respect to  $a_n$ , i.e.  $\forall u, v \in V \mid b_n(u, v) \mid \leq c_n a_n^{1/2}(u, u) a_n^{1/2}(v, v)$ . Let  $X_n$  be the closure of  $V$  in the norm  $a_n^{1/2}, n = 0, 1, \dots$ . The  $n$ -th approximate eigenvalue problem has the form

$$\begin{aligned} \text{find } \lambda \in \mathbb{C} \quad \text{and} \quad 0 \neq u \in X_n \text{ such that} \\ b_n(u, v) = \lambda a_n(u, v) \quad \forall v \in V, \end{aligned} \quad (5.2)$$

which is equivalent to the eigenproblem for an operator  $T_n$  defined by  $a_n$  and  $b_n : b_n(u, v) = a_n(T_n u, v) \forall v \in V, u \in X_n$ . Under the assumptions

$$a_0 \leq a_n \leq a, \quad (5.3)$$

$a$  is quasi-bounded with respect to  $a_0$ , i.e. there exists a symmetric operator  $\hat{L}$  in  $X_0$ , with dense domain  $V$ , such that  $a(u, v) = a_0(\hat{L}u, v) \forall u, v \in V$  (cf. [1]),

$$(5.4)$$

the approximation (5.2) can be described in terms of external approximation (for details see [8]).

From (5.3) and (5.4) it follows that  $a$  is quasi-bounded with respect to  $a_n, n = 1, 2, \dots$ . Let  $\hat{L}_n$  be the symmetric operator defined by  $a(u, v) = a_n(\hat{L}_n u, v) \forall u, v \in V$ , and let  $L_n$  denote its selfadjoint extension in  $X_n$ .  $L_n$  is positive definite. Thus, there is a unique positive definite and self-adjoint square root  $L_n^{1/2}$  of  $L_n$  and the domain  $D(L_n)$  of  $L_n$  is dense in  $D(L_n^{1/2})$ . It can be proved (see [8]) that  $D(L_n^{1/2}) = X$  and  $\forall u, v \in X \ a(u, v) = a_n(L_n^{1/2} u, L_n^{1/2} v)$ . Let us put  $r_n := L_n^{1/2}$ . It is easy to show (see [8]) that  $\|r_n\|_{\mathcal{L}(X, X_n)} = \|r_n^{-1}\|_{\mathcal{L}(X_n, X)} = 1$ . We define  $p_n := r_n^{-1}$ . The approximation  $\{X_n, r_n, p_n\}$  is convergent in  $X$  due to Definition 1. The following property can be proved (see [8]) :

LEMMA 9 : Let (5.3) and (5.4) be satisfied and moreover

$$\forall u \in V \sup_{\substack{v \in V \\ \|v\|=1}} |a_n(u, v) - a(u, v)| \rightarrow 0, \quad (5.5)$$

$$\sup_{\substack{u, v \in V \\ \|u\| = \|v\| = 1}} |b_n(u, v) - b(u, v)| \rightarrow 0. \quad (5.6)$$

Let  $\|u_n\|_n \leq M$  and  $\|v_n\|_n \leq M$   $n = 0, 1, \dots$  for some  $M$ .

If  $a_n(u_n, w) \rightarrow a(u, w) \forall w \in V$ , and  $a_n(v_n, w) \rightarrow a(v, w) \forall w \in V$  imply

$$b_n(u_n, v_n) \rightarrow b(u, v), \quad (5.7)$$

then  $\{T_n\}$  is stable at any  $\lambda \in \rho(T)$ .

Let us remark, that in the considered case the condition (2.1) of Lemma 1 implies A2 and (3.2). Thus we have

COROLLARY 1 : If the assumptions (5.3)-(5.7) are satisfied then the method is convergent in the sense of Theorems 1 to 4.

The class of methods described above has been investigated by R. D. Brown in [1] by using the another theory. He adopts the theory of discrete convergence of Banach spaces in the form developed by Stummel [10]. His results are similar to those obtained above.

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