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## SPECTRAL PROPERTIES OF A TYPE OF INTEGRO-DIFFERENTIAL STIFF PROBLEMS (\*)

by J. CAINZOS (<sup>1</sup>)

Communiqué par E. SANCHEZ-PALENCIA

*Abstract — We consider an integro-differential stiff problem, which can be used as a model for the vibrations of a body with linear viscoelasticity at large times. First, the problem is introduced in the framework of the theory of semigroups in Hilbert spaces and then, an asymptotic expansion of the solution is obtained. Finally, it is shown the double convergence, when  $\varepsilon \rightarrow 0^+$ , of the eigenvalues of the problem to the eigenvalues of two associated problems. These problems are related to the behaviour of the stiff problem on two different domains.*

*Résumé — On considère un problème raide intégro-différentiel qui modélise les vibrations d'un corps viscoélastique à mémoire longue. On introduit le problème dans le cadre de la théorie des semigroupes de contraction et on obtient un développement asymptotique de la solution. On étudie le spectre du générateur et on démontre une double convergence, lorsque  $\varepsilon \rightarrow 0^+$ , des valeurs propres du problème vers les valeurs propres de deux problèmes associés qui rendent compte du comportement du problème raide dans deux domaines différents.*

### 0. INTRODUCTION

Let  $\Omega_1, \Omega_2$  be two connected bounded domains of  $\mathbb{R}^n$  with smooth boundaries, located as shown in figure 1. We also consider the "total domain"  $\Omega = \Omega_1 \cup \Gamma_1 \cup \Omega_2$

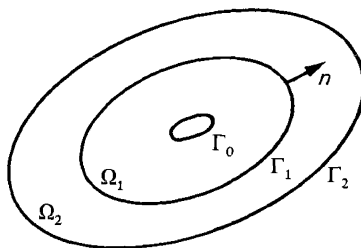


Figure 1.

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We define  $n$  as the outer unit normal to  $\Gamma_1$ .

Let us introduce the spaces :

$$H_0^1(\Omega_1, \Gamma_0) = \{ v \in H^1(\Omega_1) / v|_{\Gamma_0} = 0 \}$$

and

$$H_0^1(\Omega_2, \Gamma_2) = \{ v \in H^1(\Omega_2) / v|_{\Gamma_2} = 0 \}.$$

We consider the following stiff problem :

*Problem  $P_\varepsilon$*  : Find two functions of  $(t, x)$ , defined on  $(-\infty, \infty) \times \Omega$  :

$$u_1 : (-\infty, \infty) \rightarrow H_0^1(\Omega_1, \Gamma_0)$$

$$u_2 : [0, \infty) \rightarrow H_0^1(\Omega_2, \Gamma_2)$$

satisfying :

$$\frac{\partial^2 u_1}{\partial t^2}(t) - \Delta u_1(t) + \int_{-\infty}^t g(t - \tau) \Delta u_1(\tau) d\tau = f_1(t) \quad \text{in } \Omega_1$$

$$\varepsilon \frac{\partial^2 u_2}{\partial t^2} - \varepsilon \Delta u_2 = f_2 \quad \text{in } \Omega_2$$

with the transmission conditions :

$$u_1(t) = u_2(t) \quad \text{on } \Gamma_1$$

$$\frac{\partial u_1}{\partial n}(t) - \int_{-\infty}^t g(t - \tau) \frac{\partial u_1}{\partial n}(\tau) d\tau = \varepsilon \frac{\partial u_2}{\partial n}(t) \quad \text{on } \Gamma_1$$

and the initial conditions :

$$u_1(t) = \phi(t) \in H_0^1(\Omega_1, \Gamma_0) \quad \forall t \in (-\infty, 0]$$

$$u_2(0) = \phi_0 \in H_0^1(\Omega_2, \Gamma_2), \quad \frac{\partial u_2}{\partial t}(0) = \phi_1 \in L^2(\Omega_2).$$

The integro-differential operator which acts on the domain  $\Omega_1$  has the same characteristics as the one used by Dafermos in his papers on viscoelasticity (See Dafermos [2, 3 and 4]).

In § 1, the problem is reduced to the theory of contraction semigroups in Hilbert spaces, using the method of Dafermos [4]. Furthermore, following Lions [5], some results on the existence of a regular asymptotic expansion are given.

Section 2 is devoted to study the spectrum of the problem  $P_\varepsilon$ . It is well known that the eigenvalues  $\zeta$  are related to solutions of the following form :

$$u(x, t) = u(x) e^{-\zeta t}$$

i.e. steady solutions of exponential type.

This study is carried out using a modified version of Lobo-Sanchez's method [7]. It is shown that there is a double convergence of the eigenvalues of the problem  $P_\varepsilon$ , when  $\varepsilon \rightarrow 0^+$ , to the eigenvalues of two related problems. The first of these two problems, satisfying a Neumann type condition, shows a viscoelastic behaviour on the domain  $\Omega_1$ . The other problem, satisfying a Dirichlet condition, exhibits a merely elastic behaviour on the domain  $\Omega_2$ .

Similar problems without the integro-differential term have been studied by Lions [5] and their spectral study has been carried out by Lobo-Sanchez [7].

**1. EXISTENCE AND UNIQUENESS OF THE SOLUTION. ASYMPTOTIC EXPANSION**

In the sequel, we shall assume that the influence function  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$  satisfies :

$$g \in C^1[0, \infty), \quad g' \leq 0, \quad g \in L^1(0, \infty), \quad c = 1 - \|g\|_{L^1(0, \infty)} > 0 \quad (1.1)$$

as well as the following exponential majorization :

$$\lambda_1 e^{-\mu \xi} \leq g(\xi) \leq \lambda_2 e^{-\mu \xi} \quad \lambda_1, \lambda_2, \mu > 0. \quad (1.2)$$

**1.1. Existence and uniqueness of the solution**

Following the method used by Dafermos [4], it is possible to reduce the problem  $P_\varepsilon$  to the theory of contraction semigroups in Hilbert spaces. So, by making the following change of variable :  $t - \tau = \xi$ , and introducing the auxiliar variables  $v = u'$  and  $w(t, \xi) = u(t - \xi) - u(t)$ , which sums up the history of  $u$ , we can formulate the following equivalent problem :

$$\frac{dU}{dt} + \mathcal{A}_\varepsilon U = F \quad (1.3)$$

$$U(0) = U_0$$

where  $U = (u, v, w)$  and  $U_0 = (u_0, v_0, w_0)$ , with  $u_0|_{\Omega_1} = \phi(0)$

$$u_0|_{\Omega_2} = \phi_0, \quad v_0|_{\Omega_1} = \phi'(0), \quad v_0|_{\Omega_2} = \phi_1 \quad \text{and} \quad w_0(\xi) = \phi(-\xi) - \phi(0).$$

Both,  $U$  and  $U_0$  belong to the following Hilbert space :

$$H_\varepsilon = H_0^1(\Omega) \times L^2(\Omega) \times L_g^2(0, \infty; H_0^1(\Omega_1, \Gamma_0)) \tag{1.4}$$

endowed with the inner product :

$$\begin{aligned} \langle U, U^* \rangle_\varepsilon = & c \int_{\Omega_1} \nabla u \nabla \bar{u}^* dx + \varepsilon \int_{\Omega_2} \nabla u \nabla \bar{u}^* dx + \int_{\Omega_1} v \bar{v}^* dx + \\ & + \varepsilon \int_{\Omega_2} v \bar{v}^* dx + \int_0^\infty g(\xi) \int_{\Omega_1} \nabla w(\xi) \nabla \bar{w}^*(\xi) dx d\xi \end{aligned} \tag{1.5}$$

The operator  $\mathcal{A}_\varepsilon$  is defined by :

$$\mathcal{A}_\varepsilon U = \begin{bmatrix} -v \\ A_\varepsilon u - \int_0^\infty g(\xi) A w(\xi) d\xi \\ w' + v|_{\Omega_1} \end{bmatrix} \tag{1.6}$$

where  $A_\varepsilon \in L(H_0^1(\Omega), H^{-1}(\Omega))$  is the operator associated with the form :

$$\begin{aligned} a_\varepsilon : (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow a_\varepsilon(u, v) = \\ = c \int_{\Omega_1} \nabla u \nabla \bar{v} dx + \varepsilon \int_{\Omega_2} \nabla u \nabla \bar{v} dx \in \mathbb{C} \end{aligned} \tag{1.7}$$

and  $A \in L(H_0^1(\Omega_1, \Gamma_0), H^{-1}(\Omega))$  is the one associated with the form :

$$a : (u, v) \in H_0^1(\Omega_1, \Gamma_0) \times H_0^1(\Omega) \rightarrow a(u, v) = \int_{\Omega_1} \nabla u \nabla \bar{v} dx \in \mathbb{C}. \tag{1.8}$$

Finally, the domain of  $\mathcal{A}_\varepsilon$  is :

$$\begin{aligned} D(\mathcal{A}_\varepsilon) = \left\{ U \in H_\varepsilon / v \in H_0^1(\Omega), A_\varepsilon u - \int_0^\infty g(\xi) A w(\xi) d\xi \in L^2(\Omega), \right. \\ \left. w' \in L_g^2(0, \infty; H_0^1(\Omega_1, \Gamma_0)), w(0) = 0 \right\}. \end{aligned} \tag{1.9}$$

*Remark 1.1 :* When we define  $A_\varepsilon$ , we are considering that  $H_0^1(\Omega)$  is a subspace of  $L_\varepsilon^2(\Omega)$  i.e.  $L_\varepsilon^2(\Omega)$  is the space  $L^2(\Omega)$  endowed with the following inner

product :

$$(u, v)_\varepsilon = \int_{\Omega_1} u\bar{v} \, dx + \varepsilon \int_{\Omega_2} u\bar{v} \, dx \quad \forall u, v \in L^2(\Omega).$$

The previous choice of the space  $H_\varepsilon$  and the definition of the operator  $\mathcal{A}_\varepsilon$  allow us to prove the following proposition, using already known results (see Brezis [1] and other authors).

**PROPOSITION 1.1 :** *The operator  $-\mathcal{A}_\varepsilon$  is the infinitesimal generator of a contraction semigroup  $\{T_\varepsilon(t)\}_{t \geq 0}$  that allows us to obtain the unique solution of the problem (1.3).*

### 1.2. Asymptotic expansion

Let us denote by  $u_1$  (resp.  $u_2$ ) the restriction of  $u$  to  $\Omega_1$  (resp.  $\Omega_2$ ).

The search for an asymptotic expansion of the form :

$$\frac{1}{\varepsilon} u^{-1} + u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \dots + \varepsilon^k u^k + \dots \tag{1.10}$$

leads us, by formal substitution in the variational formulation of the problem  $P_\varepsilon$ , to obtain coupled equations. These equations, independent of  $\varepsilon$ , allow us to calculate the terms of the above expansion (1.10).

In short, the functions  $u_1^k$  (resp.  $u_2^k$ ), restriction of  $u^k$  to  $\Omega_1$  (resp.  $\Omega_2$ ), are the solution of :

Case  $k = -1$  :  $u_1^{-1}$  is the solution of the problem :

$$\frac{\partial^2 u_1^{-1}}{\partial t^2}(t) - \Delta u_1^{-1}(t) + \int_{-\infty}^t g(t - \tau) \Delta u_1^{-1}(\tau) \, d\tau = 0 \quad \text{in } \Omega_1 \tag{1.11}$$

$$u_1^{-1}|_{\Gamma_0} = 0, \quad \frac{\partial u_1^{-1}}{\partial n}(t) - \int_{-\infty}^t g(t - \tau) \frac{\partial u_1^{-1}}{\partial n}(\tau) \, d\tau = 0 \quad \text{on } \Gamma_1$$

$$u_1^{-1}(t) = 0 \quad \forall t \in (-\infty, 0].$$

Therefore,  $u_1^{-1} = 0$ .

On the other hand,  $u_2^{-1}$  is the solution of :

$$\frac{\partial^2 u_2^{-1}}{\partial t^2} - \Delta u_2^{-1} = f_2 \quad \text{in } \Omega_2 \tag{1.12}$$

$$u_2^{-1}|_{\Gamma_1} = 0, \quad u_2^{-1}|_{\Gamma_2} = 0.$$

Case  $k = 0$  :  $u_1^0$  is the solution of :

$$\frac{\partial^2 u_1^0}{\partial t^2}(t) - \Delta u_1^0(t) + \int_{-\infty}^t g(t - \tau) \Delta u_1^0(\tau) d\tau = f_1(t) \quad \text{in } \Omega_1 \quad (1.13)$$

$$u_1^0(t) = \phi(t) \quad \forall t \in (-\infty, 0]$$

$$u_1^0|_{\Gamma_0} = 0, \quad \frac{\partial u_1^0}{\partial n}(t) - \int_{-\infty}^t g(t - \tau) \frac{\partial u_1^0}{\partial n}(\tau) d\tau = \frac{\partial u_2^{-1}}{\partial n}(t) \quad \text{on } \Gamma_1$$

As a consequence,  $u_2^0$  is the solution of :

$$\begin{aligned} \frac{\partial^2 u_2^0}{\partial t^2} - \Delta u_2^0 &= 0 \quad \text{in } \Omega_2 \\ u_2^0(0) &= \phi_0, \quad (u_2^0(0))' = \phi_1 \\ u_2^0|_{\Gamma_1} &= u_1^0|_{\Gamma_1}, \quad u_2^0|_{\Gamma_2} = 0. \end{aligned} \quad (1.14)$$

*General case*  $k = 1, 2, \dots$  : The functions  $u_1^k$ , restriction of  $u^k$  to  $\Omega_1$ , are the solutions of the following problems :

$$\frac{\partial^2 u_1^k}{\partial t^2}(t) - \Delta u_1^k(t) + \int_{-\infty}^t g(t - \tau) \Delta u_1^k(\tau) d\tau = 0 \quad \text{in } \Omega_1 \quad (1.15)$$

$$u_1^k(t) = 0 \quad \forall t \in (-\infty, 0]$$

$$\frac{\partial u_1^k}{\partial n}(t) - \int_{-\infty}^t g(t - \tau) \frac{\partial u_1^k}{\partial n}(\tau) d\tau = \frac{\partial u_2^{k-1}}{\partial n}(t) \quad \text{on } \Gamma_1, \quad u_1^k|_{\Gamma_0} = 0.$$

On the other hand, the restrictions to  $\Omega_2$  are the solutions of :

$$\frac{\partial^2 u_2^k}{\partial t^2} - \Delta u_2^k = 0 \quad \text{in } \Omega_2 \quad \forall k = 1, 2, \dots \quad (1.16)$$

$$u_2^k(0) = 0, \quad (u_2^k(0))' = 0$$

$$u_2^k|_{\Gamma_1} = u_1^k|_{\Gamma_1}, \quad u_2^k|_{\Gamma_2} = 0.$$

*Remark 1.2* : As we progress in the actual calculation of the terms of the expansion it becomes, as it is usual, increasingly difficult to establish the sufficient conditions to justify the calculations. In fact, it is only possible to calculate the term  $u^{-1}$  and the restriction to  $\Omega_1$  of the term  $u^0$ . The lack of regularity results prevents us from going on.

It can be shown that, under convenient regularity hypotheses, this asymptotic expansion is rigorous. Through similar procedures to those used by Lions [5], we can prove the following proposition.

**PROPOSITION 1.2 :** *If  $u^k \in C^0([0, T]; H_0^1(\Omega))$  and  $(u^k)' \in C^0([0, T]; L^2(\Omega))$   $\forall k = -1, 0, 1, \dots, n + 1$  and if also  $(u^{n+1})' \in L^2(0, T; H_0^1(\Omega))$  and*

$$(u^{n+1})'' \in L^2(0, T; L^2(\Omega)) ;$$

then :

$$\left\| u_\varepsilon - \left( \frac{1}{\varepsilon} u^{-1} + u^0 + \dots + \varepsilon^n u^n \right) \right\|_{L^2(0, T; H_0^1(\Omega))} \leq C\varepsilon^{n+1}$$

$$\left\| u'_\varepsilon - \left( \frac{1}{\varepsilon} u^{-1} + u^0 + \dots + \varepsilon^n u^n \right)' \right\|_{L^2(0, T; L^2(\Omega))} \leq C\varepsilon^{n+1} .$$

**2. SPECTRAL STUDY**

The study of the convergence of the restrictions of  $u_\varepsilon$  to each domain  $\Omega_i$   $i = 1, 2$  may be described in terms of two associated operators  $A_1$  and  $A_2$ , closely related to  $\mathcal{A}_\varepsilon$ .

**2.1. Location of the spectra of  $A_1$ ,  $A_2$  and  $\mathcal{A}_\varepsilon$**

a) *The operator  $A_1$*

The operator  $A_1$  is defined from the Hilbert space

$$H = H_0^1(\Omega_1, \Gamma_0) \times L^2(\Omega_1) \times L_y^2(0, \infty; H_0^1(\Omega_1, \Gamma_0))$$

into itself and it is given by :

$$A_1 U = \begin{bmatrix} -v \\ cAu - \int_0^\infty g(\xi) Aw(\xi) d\xi \\ w' + v \end{bmatrix} \tag{2.1}$$

Being  $A$  a linear bounded operator from  $H_0^1(\Omega_1, \Gamma_0)$  into its dual, associated



with the form :

$$a : (u, v) \in H_0^1(\Omega_1, \Gamma_0) \times H_0^1(\Omega_1, \Gamma_0) \rightarrow a(u, v) = \int_{\Omega_1} \nabla u \nabla \bar{v} \, dx \in \mathbb{C} . \quad (2.2)$$

Finally, the domain of  $A_1$  is :

$$D(A_1) = \left\{ (u, v, w) \in H/v \in H_0^1(\Omega_1, \Gamma_0), cAu - \int_0^\infty g(\xi) Aw(\xi) \, d\xi \in L^2(\Omega_1), \right. \\ \left. w' \in L_y^2(0, \infty; H_0^1(\Omega_1, \Gamma_0)), w(0) = 0 \right\} .$$

The operator  $A_1$ , studied by Lobo [6], has a resolvent set  $\rho(A_1)$  containing the whole semiplane  $\text{Re } \eta < 0$  and the points of the strip  $0 \leq \text{Re } \eta < \mu/2$  which satisfy :

$$- \frac{\eta^2}{1 - \int_0^\infty g(\xi) e^{\eta \xi} \, d\xi} \neq b_k^2 \quad \forall k \in N \quad (2.3)$$

where the  $b_k^2$ , which belong to the  $A$  spectrum, form an unbounded sequence of real positive numbers. On the other hand,  $A_1$  has a point spectrum  $\sigma_p(A_1)$  formed by the points of the above mentioned strip which do not satisfy the inequality (2.3). Its residual spectrum  $\sigma_R(A_1)$  is the semiplane  $\text{Re } \eta \geq \mu/2$ .

b) *The operator  $A_2$*

The operator  $A_2$  from  $H_0^1(\Omega_2) \times L^2(\Omega_2)$  into itself is defined by :

$$A_2 U = \begin{bmatrix} -v \\ Au \end{bmatrix} \quad (2.4)$$

where  $A \in L(H_0^1(\Omega_2), H^{-1}(\Omega_2))$  is associated with the form :

$$a : (u, v) \in H_0^1(\Omega_2) \times H_0^1(\Omega_2) \rightarrow a(u, v) = \int_{\Omega_2} \nabla u \nabla \bar{v} \, dx \in \mathbb{C} .$$

As it is well known,  $A_2$  is a skew selfadjoint operator whose spectrum consists of the points  $\pm id_k$  such that  $\{d_k^2\}$  form the spectrum of  $A$ .

c) *The operator  $\mathcal{A}_\varepsilon$*

Now, let us study the location of the spectrum of the operator  $\mathcal{A}_\varepsilon$ .

**PROPOSITION 2.1 :** *The eigenvalues of  $\mathcal{A}_\varepsilon$  lie on the strip of the complex plane  $0 \leq \text{Re } \eta < \mu/2$ .*

*Proof :* First, because of proposition 1.1, the whole semiplane  $\text{Re } \eta < 0$  is contained in the resolvent set.

On the other hand, the discussion on the eigenvalue problem allows us to assert that the third component  $w$  of the eigenvector  $U$  must have the form :

$$w(\xi) = u(e^{\eta\xi} - 1) \tag{2.5}$$

where  $u$  is the first component of  $U$ . A necessary condition for  $w$  to belong to  $L^2_g(0, \infty; H^1_0(\Omega_1, \Gamma_0))$  is that  $\text{Re } \eta < \mu/2$ , thus the point spectrum is contained in the strip  $0 \leq \text{Re } \eta < \mu/2$ .

Finally, in a similar way, it can easily be shown that if  $\text{Re } \eta \geq \mu/2$  then  $\eta$  belongs to the residual spectrum.

**2.2. Existence of eigenvalues of the operator  $\mathcal{A}_\varepsilon$**

Now, let us discuss an equivalent formulation of the eigenvalue problem that allow us to proof the existence of eigenvalues of the operator  $\mathcal{A}_\varepsilon$  when  $\varepsilon$  is sufficiently small.

Moreover, we proof that the eigenvalues of  $A_1$  and  $A_2$  are accumulation points of the eigenvalues of  $\mathcal{A}_\varepsilon$  i.e. it is possible to obtain a sequence  $\{\zeta_\varepsilon\}$ , so that  $\zeta_\varepsilon$  is an eigenvalue of  $\mathcal{A}_\varepsilon$  and the sequence  $\{\zeta_\varepsilon\}$  converges to  $\zeta$  when  $\varepsilon \rightarrow 0^+$ , where  $\zeta$  is an eigenvalue of  $A_1$  or  $A_2$ .

Let us write the eigenvalue problem :

Find  $\eta \in \mathbb{C}$  and  $U = (u, v, w) \in D(\mathcal{A}_\varepsilon)$ ,  $U \neq 0$  such that :

$$-v - \eta u = 0 \tag{2.6}$$

$$A_\varepsilon u - \int_0^\infty g(\xi) Aw(\xi) d\xi - \eta v = 0 \tag{2.7}$$

$$w' + v_1 - \eta w = 0. \tag{2.8}$$

If we obtain  $v$  from (2.6) and substitute into (2.8), we can assert that  $w$  is the solution of :

$$\begin{aligned} w' - \eta w &= \eta u \\ w(0) &= 0. \end{aligned} \tag{2.9}$$

As a consequence,  $w$  has the following form :

$$w(\xi) = u(e^{\eta\xi} - 1). \tag{2.10}$$

Now, the substitution of  $v$  and  $w$  into (2.7) allow us to obtain the following equivalent eigenvalue problem :

Find  $\eta \in \mathbb{C}$  and  $u \in H_0^1(\Omega)$ ,  $u \neq 0$  such that :

$$a_1(\eta, u, v) + \varepsilon a_2(u, v) + \eta^2(b_1(u, v) + \varepsilon b_2(u, v)) = 0 \quad \forall v \in H_0^1(\Omega) \quad (2.11)$$

where :

$$a_1(\eta, u, v) = \left(1 - \int_0^\infty g(\xi) e^{\eta \xi} d\xi\right) \int_{\Omega_1} \nabla u \nabla \bar{v} dx \quad (2.12)$$

$$a_2(u, v) = \int_{\Omega_2} \nabla u \nabla \bar{v} dx \quad (2.13)$$

$$b_1(u, v) = \int_{\Omega_1} u \bar{v} dx \quad (2.14)$$

$$b_2(u, v) = \int_{\Omega_2} u \bar{v} dx . \quad (2.15)$$

In the sequel, we denote  $1 - \int_0^\infty g(\xi) e^{\eta \xi} d\xi$  by  $G(\eta)$ .

In order to solve the problem (2.11), we use a modified version of the method of expansion in powers of  $\varepsilon$  used by Lobo-Sanchez [7].

In this way, the following problem is considered :

Let be  $F \in L^2(\Omega)$ . Find  $u_\varepsilon \in H_0^1(\Omega)$ , depending on  $\eta$  and  $F$ , such that :

$$a_1(\eta, u, v) + \varepsilon a_2(u, v) + \eta^2(b_1(u, v) + \varepsilon b_2(u, v)) = \int_{\Omega} F \bar{v} dx, \quad \forall v \in H_0^1(\Omega) \quad (2.16)$$

First, let us find and asymptotic expansion of the kind :

$$\frac{1}{\varepsilon} u^{-1} + u^0 + \varepsilon u^1 + \dots + \varepsilon^k u^k + \dots, \quad u^k \in H_0^1(\Omega) \quad \forall k = -1, 0, \dots \quad (2.17)$$

By formal substitution in (2.11), we obtain the equations that allow us to calculate the terms of the above expansion :

$$a_1(\eta, u^{-1}, v) + \eta^2 b_1(u^{-1}, v) = 0 \quad \forall v \in H_0^1(\Omega) \quad (2.18)$$

$$a_2(u^{-1}, v) + \eta^2 b_2(u^{-1}, v) + a_1(\eta, u^0, v) + \eta^2 b_1(u^0, v) = \int_{\Omega} F \bar{v} dx \quad (2.19)$$

$$a_2(u^{k-1}, v) + \eta^2 b_2(u^{k-1}, v) + a_1(\eta, u^k, v) + \eta^2 b_1(u^k, v) = 0, \quad k = 1, 2, \dots \tag{2.20}$$

**PROPOSITION 2.2 :** *Let be  $F \in L^2(\Omega)$ . Let  $V$  be an open set of  $\mathbb{C}$  contained in  $\rho(A_1) \cap \rho(A_2)$ . Then, the functions  $u^k : \eta \in V \subset \mathbb{C} \rightarrow u^k(\eta) \in H_0^1(\Omega)$   $k = -1, 0, 1, 2, \dots$  exist and are holomorphic.*

*Proof :* First, taking into account the equation (2.18),  $u_1^{-1}$  is the solution of :

$$\begin{aligned} -G(\eta) \Delta u_1^{-1} + \eta^2 u_1^{-1} &= 0 \quad \text{in } \Omega_1 \\ u_1^{-1}|_{\Gamma_0} &= 0, \quad G(\eta) \frac{\partial u_1^{-1}}{\partial n} = 0 \quad \text{on } \Gamma_1. \end{aligned} \tag{2.21}$$

Since  $\eta \in \rho(A_1)$ ,  $-\eta^2$  belongs to the resolvent set of the above problem, therefore  $u_1^{-1} = 0$ .

On the other hand, if in the equation (2.19) we take  $v \in H_0^1(\Omega)$  such that  $v = 0$  in  $\Omega_1$ ,  $u_2^{-1}$  is the solution of the following problem :

$$\begin{aligned} -\Delta u_2^{-1} + \eta^2 u_2^{-1} &= F_2 \quad \text{in } \Omega_2 \\ u_2^{-1}|_{\Gamma_1} &= 0, \quad u_2^{-1}|_{\Gamma_2} = 0. \end{aligned} \tag{2.22}$$

Since  $\eta \in \rho(A_2)$ ,  $-\eta^2$  belongs to the resolvent set of the Laplace operator with homogeneous boundary conditions, therefore  $u_2^{-1} \in H_0^1(\Omega_2) \cap H^2(\Omega_2)$  and is holomorphic. Moreover,  $u_2^{-1}$  satisfies :

$$\|u_2^{-1}(\eta)\|_{H^2(\Omega_2)} \leq C_{-1}(\eta) \tag{2.23}$$

where  $C_{-1}$  is a continuous function of  $\eta$ .

In short,  $u^{-1}$  is holomorphic and satisfies :

$$\|u^{-1}(\eta)\|_{H_0^1(\Omega)} \leq C_{-1}(\eta). \tag{2.24}$$

Now, using the equation (2.19) we obtain that  $u_1^0$  is the solution of :

$$\begin{aligned} -G(\eta) \Delta u_1^0 + \eta^2 u_1^0 &= F_1 \quad \text{in } \Omega_1 \\ u_1^0|_{\Gamma_0} &= 0, \quad G(\eta) \frac{\partial u_1^0}{\partial n} = \frac{\partial u_2^{-1}}{\partial n} \quad \text{on } \Gamma_1. \end{aligned} \tag{2.25}$$

As a consequence of the previous calculation of  $u_2^{-1}(\eta)$ ,  $\partial u_2^{-1} / \partial n$  is an holomorphic function of  $\eta$  into  $H^{1/2}(\Gamma_1)$ . Thus,  $u_1^0$  belongs to  $H_0^1(\Omega_1, \Gamma_0) \cap H^2(\Omega_1)$  and is holomorphic.

In a similar way, we obtain that  $u_2^0$  is the solution of the problem :

$$\begin{aligned} -\Delta u_2^0 + \eta^2 u_2^0 &= 0 \quad \text{in } \Omega_2 \\ u_2^0|_{\Gamma_2} &= 0, \quad u_2^0|_{\Gamma_1} = u_1^0|_{\Gamma_1} \end{aligned} \tag{2.26}$$

and, as a consequence,  $u_2^0 \in H_0^1(\Omega_2, \Gamma_2) \cap H^2(\Omega_2)$  and is holomorphic.

In short, we deduce that  $u^0$  is holomorphic and satisfies :

$$\|u^0(\eta)\|_{H_0^1(\Omega)} \leq C_0(\eta) \tag{2.27}$$

where  $C_0$  is a continuous function of  $\eta$ .

In an analogous way, we obtain that  $u^k$  is holomorphic from  $V \subset \mathbb{C}$  into  $H_0^1(\Omega)$  and satisfies :

$$\|u^k(\eta)\|_{H_0^1(\Omega)} \leq [C_1(\eta)]^k \quad \forall k \geq 1. \tag{2.28}$$

**PROPOSITION 2.3 :** *Let  $K$  be a compact set contained in  $\rho(A_1) \cap \rho(A_2)$ . Let be  $F \in L^2(\Omega)$ . Then, for  $\varepsilon$  sufficiently small, the asymptotic expansion :*

$$\frac{1}{\varepsilon} u^{-1}(\eta) + u^0(\eta) + \varepsilon u^1(\eta) + \dots$$

*is uniformly convergent in  $H_0^1(\Omega)$  to  $u_\varepsilon(\eta)$ , i.e. to the solution of the problem (2.11).*

*Proof :* Since  $K$  is a compact set and the functions  $C_{-1}$ ,  $C_0$  and  $C_1$  are continuous, they are bounded.

Thus :

$$\left\| \sum_{n=-1}^N \varepsilon^n u^n(\eta) \right\|_{H_0^1(\Omega)} \leq \frac{1}{\varepsilon} k + k + \sum_{n=1}^N \varepsilon^n k^n.$$

As a consequence, if  $\varepsilon < 1/k$  the conclusion follows.

Now the principal result of this paper is stated.

**THÉORÈME 2.1 :** *Let be  $\zeta \in \sigma_p(A_1) \cup \sigma(A_2)$ . If  $\varepsilon$  is sufficiently small, then there exist eigenvalues of  $\mathcal{A}_\varepsilon$  near to  $\zeta$ .*

*Proof :* We consider a simple closed curve  $\gamma$  enclosing  $\zeta$ . It is possible to choose  $\gamma$  such that  $\gamma$  is contained in  $\rho(A_1) \cap \rho(A_2)$  and  $\gamma$  does not enclose any other point of  $\sigma_p(A_1) \cup \sigma(A_2)$  because  $\sigma(A_2)$  is contained in the complex axis and if  $g \neq 0$  then the eigenvalues of  $A_1$  are not purely imaginary numbers.

First, we assume that  $\zeta \in \sigma_p(A_1)$ . Let  $F$  be an element of  $L^2(\Omega)$  such that  $F_1$  is an eigenvector of  $A_1$  associated with  $\zeta$  and  $F_2 = 0$ . The existence of a solution

of the problem (2.11) can be easily deduced from the previous results when  $\eta \in \gamma$ . Then  $u_\varepsilon(\eta)$  becomes :

$$u_\varepsilon(\eta) = \sum_{n=-1}^{\infty} \varepsilon^n u^n(\eta). \tag{2.29}$$

Integrating along  $\gamma$ , we obtain :

$$\int_\gamma u_\varepsilon(\eta) d\eta = \frac{1}{\varepsilon} \int_\gamma u^{-1}(\eta) d\eta + \int_\gamma u^0(\eta) d\eta + \dots \tag{2.30}$$

Taking into account (2.22),  $u^{-1}$  has no singularity in the region enclosed by  $\gamma$  and using the Cauchy's integral theorem we can assert :

$$\int_\gamma u^{-1}(\eta) d\eta = 0. \tag{2.31}$$

In a similar way, considering (2.25), we obtain :

$$\int_\gamma u^0(\eta) d\eta \neq 0. \tag{2.32}$$

Then, by taking the limit value as  $\varepsilon \rightarrow 0^+$  in (2.30), we get :

$$\lim_{\varepsilon \rightarrow 0^+} \int_\gamma u_\varepsilon(\eta) d\eta \neq 0. \tag{2.33}$$

$u_\varepsilon$  is holomorphic because it is the uniform limit of holomorphic functions, hence we can easily verify that, for sufficiently small  $\varepsilon$ ,  $u_\varepsilon$  has a singularity in the region enclosed by  $\gamma$  and consequently, this singularity is an eigenvalue.

On the other hand, let be  $\zeta \in \sigma(A_2)$ . If we choose  $F \in L^2(\Omega)$  such that  $F_1 = 0$  and  $F_2$  is an eigenvector of  $A_2$  associated with  $\zeta$ , we have that  $u_\varepsilon(\eta)$  exists and it is defined by the series (2.29). Integrating along  $\gamma$ , we have that :

$$\int_\gamma u_\varepsilon(\eta) d\eta = \frac{1}{\varepsilon} \int_\gamma u^{-1}(\eta) d\eta + \int_\gamma u^0(\eta) d\eta + \dots \tag{2.34}$$

Now, taking into account (2.22), we can verify that :

$$\int_\gamma u^{-1}(\eta) d\eta \neq 0. \tag{2.35}$$

As a consequence

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_{\gamma} u_{\varepsilon}(\eta) \, d\eta \neq 0 \tag{2.36}$$

and, following the procedure above, the conclusion follows.

**COROLLARY 2.1 :** *Let be  $\zeta \in \sigma_p(A_1) \cup \sigma(A_2)$ . There exist a sequence  $\{\zeta_n\}$ , where  $\zeta_n$  is an eigenvalue of  $\mathcal{A}_{\varepsilon_n}$ , such that  $\zeta_n$  converges to  $\zeta$ , when  $\varepsilon_n \rightarrow 0^+$ .*

### 3. COMPLEMENTS

Similar results are obtained by considering some problems closely related to the one here discussed. For example, if the roles of the domains  $\Omega_1$  and  $\Omega_2$  are interchanged (i.e. the parameter  $\varepsilon$  acts on the outer domain) or if the integro-differential operator is defined on the domain  $\Omega_2$ .

The problem discussed in this paper may be taken as a model applicable to several situations appearing in other fields of science. In this way, it may be applied to the study of the elastic vibrations of a non-homogeneous bounded body, with a viscoelastic layer immersed in other merely elastic, layer having a very small density and rigidity with respect to the first one.

An adequate dimensional analysis leads us to the following problem, where the displacement vector  $u$  and the strain tensor  $\sigma_{ij}$  are used.

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

where  $\sigma_{ij}$  is defined in  $\Omega_1$  by :

$$\sigma_{ij} = C_{ijkl}^1(x) \frac{\partial u_k}{\partial x_l} - \int_{-\infty}^t G_{ijkl}(x, t - \tau) \frac{\partial u_k}{\partial x_l} \, d\tau$$

and in  $\Omega_2$  it is given by :

$$\sigma_{ij} = C_{ijkl}^2(x) \frac{\partial u_k}{\partial x_l}$$

The transmission conditions are :

$$\begin{aligned} u_i^1 &= u_i^2 \quad \text{on } \Gamma_1 \quad i = 1, 2, 3 \\ \sigma_{ij}^1 \cdot n_j &= \varepsilon \sigma_{ij}^2 \cdot n_j \quad \text{on } \Gamma_1 . \end{aligned}$$

The numerical study of several particular cases, such as the acoustic waves interaction with layered elastic or viscoelastic obstacles in water can be seen in Peterson-Varadan-Varadan [8].

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