MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

JUAN ENRIQUE SANTOS
ERNESTO JORGE OREÑA

Elastic wave propagation in fluid-saturated porous media. Part II. The Galerkin procedures

Modélisation mathématique et analyse numérique, tome 20, nº 1 (1986), p. 129-139

<http://www.numdam.org/item?id=M2AN_1986__20_1_129_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 20, n° 1, 1986, p. 129 à 139)

ELASTIC WAVE PROPAGATION IN FLUID-SATURATED POROUS MEDIA PART II. THE GALERKIN PROCEDURES (*)

by Juan Enrique SANTOS (1) and Ernesto Jorge OREÑA (2)

Commuhicated by J. DOUGLAS Jr.

Abstract. — *Biot's dynamic equations describing elastic wave propagation in fluid-saturated porous media were analyzed in Part I where some results on the existence and uniqueness of such equations were derived. Here in Part II the continuous and discrete-time Galerkin methods for obtaining approximate solutions of Biot's equations are defined and the corresponding error analysis is performed.*

Résumé. — *Les équations dynamiques de Biot qui décrivent la propagation d'onde élastique dans un milieu poreux saturé de liquide ont été analysées dans la 1re partie de ce travail où des résultats d'existence et d'unicité de solution de telles équations ont été obtenus. Dans cette 2e partie nous donnons les méthodes de Galerkin continues et discrètes en temps pour l'obtention des solutions approchées des équations de Biot et nous donnons aussi les calculs d'erreur correspondants.*

1. INTRODUCTION

This work is the second part of a two-part paper dedicated to the analysis and numerical solution of Biot's dynamic equations describing elastic wave propagation in fluid-saturated porous media. In the paper denoted as Part I [6], the existence and uniqueness of the solution of such equations were analyzed. Thus we refer to Part I for the statement of Biot's model and the notation to be used in what follows.

The paper denoted here as Part II consists of two additional sections. In Section 2 the finite element spaces to be used are described and the continuous-time Galerkin method is defined and analyzed. In Section 3 the discrete-time Galerkin procedure is given and the convergence analysis is performed.

(*) Received in February 1985.
(1) Yacimientos Petrolíferos Fiscales and University of Buenos Aires, Avenida Presidente Roque Saenz Peña 777, 10° Piso, Oficina 1003, 1364 Buenos Aires, Argentina.
(2) Yacimientos Petrolíferos Fiscales.

2. THE CONTINUOUS-TIME GALERKIN METHOD

Let $k \geqslant 1$ be an integer and let $0 < h \leqslant 1$. Let $\mathcal{T}_h^s$ and $\mathcal{T}_h^f$ be quasiregular partitions of Ω into triangles or rectangles of diameter bounded by h. Boundary triangles or rectangles may have one curvilinear edge. Let $\mathcal{M}_h \subset [H^1(\Omega)]^2$ be a standard finite element space associated with $\mathcal{T}_h^s$ such that

$$\inf_{\chi \in \mathcal{M}_h} \{ \| v - \chi \|_0 + h \| v - \chi \|_1 \} \leqslant C \| v \|_r h^r , \quad 1 \leqslant r \leqslant k+1 . \tag{2.1}$$

Also, let W_h be a finite dimensional subspace of $H(\mathrm{div}, \Omega)$ associated with $\mathcal{T}_h^f$ such that

$$\begin{aligned} &\text{(i)} \inf_{\chi \in W_h} \| w - \chi \|_0 \leqslant C \| w \|_k h^k , \\ &\text{(ii)} \inf_{\chi \in W_h} \| w - \chi \|_{H(\mathrm{div},\Omega)} \leqslant C [\| w \|_k + \| \nabla . w \|_k] h^k . \end{aligned} \tag{2.2}$$

The space W_h can be taken to be the vector part of either the Raviart-Thomas-Nedelec [3], [4], [5], [7] space of index $k-1$, or the Brezzi-Douglas-Marini [1], [2], space of index k associated with $\mathcal{T}_h^f$. In the particular case in which Ω is a rectangle and $\mathcal{T}_h^f$ is related to a tensor product grid, the vector part of the Raviart-Thomas-Nedelec space of index $k-1$ can be described as follows : let $\delta = \{ y_0, \ldots, y_n \}$ and

$$\mathcal{M}_j(m, \delta) = \{ \zeta \in C^j([y_0, y_n]) : \zeta/[y_{i-1}, y_i] \in P_m \}$$

where P_m denotes the polynomials of degree not greater than m. Then if $\delta_{x_1}^f$ and $\delta_{x_2}^f$ are quasi-regular partitions of each side of Ω in subintervals of length bounded by h, we have that

$$W_h = [\mathcal{M}_0(k, \delta_{x_1}^f) \otimes \mathcal{M}_{-1}(k-1, \delta_{x_2}^f)] \times [\mathcal{M}_{-1}(k-1, \delta_{x_1}^f) \otimes \mathcal{M}_0(k, \delta_{x_2}^f)] .$$

Let $V_h = \mathcal{M}_h \times W_h$. Then it follows from (2.1)-(2.2) that

$$\begin{aligned} &\text{(i)} \inf_{v_h \in V_h} \| v - v_h \|_0 \leqslant C \| v \|_k h^k , \quad v \in [H^k(\Omega)]^4 , \\ &\text{(ii)} \inf_{v_h \in V_h} \| v - v_h \|_V \leqslant C [\| v_1 \|_{k+1} + \| v_2 \|_k + \| \nabla . v_2 \|_k] h^k , \\ &v = (v_1, v_2) , \quad v_1 \in [H^{k+1}(\Omega)]^2 , \quad v_2 \in [H^k(\Omega)]^2 , \quad \nabla . v_2 \in H^k(\Omega) . \end{aligned} \tag{2.3}$$

The continuous-time Galerkin approximation to u is defined as the twicedifferentiable map $U : J \to V_h$ such that

$$\left(\mathscr{A}\frac{\partial^2 U}{\partial t^2}, v\right) + \left(\mathscr{C}\frac{\partial U}{\partial t}, v\right) + B(U, v) =$$
$$= (F, v) + \langle \phi, v_1 \rangle + \langle v_2 . \nu, \eta \rangle , \quad v = (v_1, v_2) \in V_h , \quad t \in J . \quad (2.4)$$

The argument given in (4.7)-(4.11) of Part I implies that

$$\left\| \frac{\partial U}{\partial t} \right\|_{L^\infty(J,[L^2(\Omega)]^4)} + \| U \|_{L^\infty(J,V)} \leqslant C\left[D_0 + \| U(0) \|_V + \left\| \frac{\partial U}{\partial t}(0) \right\|_0\right],$$

so that there exists a unique solution of (2.4) for every choice of $U(x, 0)$ and $\frac{\partial U}{\partial t}(x, 0)$.

We now turn to estimate the error generated by the procedure (2.4). Set

$$E_k = \left\| \frac{\partial u_1}{\partial t} \right\|_{L^2(J,[H^{k+1}(\Omega)]^2)} + \left\| \frac{\partial u_1}{\partial t} \right\|_{L^\infty(J,[H^k(\Omega)]^2)} + \\ + \left\| \frac{\partial u_2}{\partial t} \right\|_{L^\infty(J,[H^k(\Omega)]^2)} + \left\| \nabla \cdot \frac{\partial u_2}{\partial t} \right\|_{L^2(J,H^k(\Omega))} \\ + \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(J,[H^k(\Omega)]^4)},$$

$$E_k^0 = \| u_1^0 \|_{k+1} + \| u_2^0 \|_k + \| \nabla . u_2^0 \|_k + \| v_1^0 \|_{k+1} + \\ + \| v_2^0 \|_k + \| \nabla . v_2^0 \|_k .$$

THEOREM 2.1 : *Assume that $E_k < \infty$ and $E_k^0 < \infty$ for some $k \geqslant 1$. Let $U(0) \in V_h$ and $\frac{\partial U}{\partial t}(0) \in V_h$ be defined by the projections*

$$\text{(i)} \quad B_\gamma(U(0) - u^0, v) = 0 , \qquad v \in V_h ,$$
$$\text{(ii)} \quad B_\gamma\left(\frac{\partial U}{\partial t}(0) - v^0, v\right) = 0 , \quad v \in V_h . \qquad (2.5)$$

Then the solution $U(x, t)$ of the procedure (2.4) satisfies the error estimate

$$\left\| \frac{\partial(u - U)}{\partial t} \right\|_{L^\infty(J,[L^2(\Omega)]^4)} + \| u - U \|_{L^\infty(J,V)} \leqslant C[E_k^0 + E_k]\, h^k .$$

Proof : Let $\xi = (\xi_1, \xi_2) = (u_1 - U_1, u_2 - U_2)$. Choosing $v = \xi(0) + \chi - u^0$ in (2.5 (i)), χ being an arbitrary element in V_h, and applying (4.5 (ii)) of Part I we conclude that

$$\begin{aligned} C_2 \| \xi(0) \|_V^2 &\leqslant B_\gamma(\xi(0), \xi(0)) \\ &= B_\gamma(\xi(0), \xi(0) - v) \\ &= B_\gamma(\xi(0), u^0 - \chi) \\ &\leqslant C \| \xi(0) \|_V \| u^0 - \chi \|_V . \end{aligned}$$

Thus,

$$\| \xi(0) \|_V \leqslant C \inf_{\chi \in V_h} \| u^0 - \chi \|_V \leqslant C E_k^0 h^k . \tag{2.6}$$

Similarly the choice $v = \frac{\partial \xi}{\partial t}(0) + \chi - v^0$ in (2.5 (ii)) implies that

$$\left\| \frac{\partial \xi}{\partial t}(0) \right\|_0 \leqslant C E_k^0 h^k . \tag{2.7}$$

Next recall that the true solution $u(x, t)$ satisfies the relation

$$\left(\mathscr{A} \frac{\partial^2 u}{\partial t^2}, v \right) + \left(\mathscr{C} \frac{\partial u}{\partial t}, v \right) + B(u, v) =$$
$$= (F, v) + \langle \phi, v_1 \rangle + \langle v_2 . \nu, \eta \rangle , \quad v = (v_1, v_2) \in V , \quad t \in J . \tag{2.8}$$

Combining (2.4) and the equation above we have

$$\left(\mathscr{A} \frac{\partial^2 \xi}{\partial t^2}, v \right) + \left(\mathscr{C} \frac{\partial \xi}{\partial t}, v \right) + B(\xi, v) = 0 , \quad v \in V_h , \quad t \in J .$$

Choose the test function $v = \frac{\partial}{\partial t}(\xi + \chi - u)$ where

$$\chi = (\chi_1, \chi_2) : J \to V_h .$$

Then

$$\frac{1}{2} \frac{d}{dt}\left[\left(\mathscr{A} \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t} \right) + B(\xi, \xi) \right] + \left(\mathscr{C} \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t} \right) =$$
$$= \left(\mathscr{A} \frac{\partial^2 \xi}{\partial t^2}, \frac{\partial [u - \chi]}{\partial t} \right) + B\left(\xi, \frac{\partial [u - \chi]}{\partial t} \right)$$
$$+ \left(\mathscr{C} \frac{\partial \xi}{\partial t}, \frac{\partial [u - \chi]}{\partial t} \right)$$

$$\leqslant \left(\mathscr{A}\frac{\partial^2\xi}{\partial t^2}, \frac{\partial[u-\chi]}{\partial t}\right) + \frac{1}{2}\left(\mathscr{C}\frac{\partial\xi}{\partial t}, \frac{\partial\xi}{\partial t}\right) + C\left[\|\xi_1\|_1^2 + \|\nabla.\xi_2\|_0^2 + \left\|\frac{\partial(u_1-\chi_1)}{\partial t}\right\|_1^2 + \left\|\nabla\cdot\frac{\partial(u_2-\chi_2)}{\partial t}\right\|_0^2 + \left\|\frac{\partial(u-\chi)}{\partial t}\right\|_0^2\right].$$

Adding

$$\frac{\gamma}{2}\frac{d}{dt}\|\xi\|_0^2 \leqslant \frac{\gamma}{2}\left(\|\xi\|_0^2 + \left\|\frac{\partial\xi}{\partial t}\right\|_0^2\right)$$

to the inequality above we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\left(\mathscr{A}\frac{\partial\xi}{\partial t}, \frac{\partial\xi}{\partial t}\right) + B_\gamma(\xi, \xi)\right] \leqslant \leqslant \left(\mathscr{A}\frac{\partial^2\xi}{\partial t^2}, \frac{\partial(u-\chi)}{\partial t}\right) + C\left[\|\xi\|_V^2 + \left\|\frac{\partial\xi}{\partial t}\right\|_0^2 + \left\|\frac{\partial(u-\chi)}{\partial t}\right\|_V^2\right]. \quad (2.9)$$

We shall integrate (2.9) from 0 to t, but first we shall get a bound for the integral of the first term in the right-hand side.

Using integration by parts with respect to time,

$$I_3 = \int_0^t \left(\mathscr{A}\frac{\partial^2\xi}{\partial t^2}, \frac{\partial[u-\chi]}{\partial t}\right)(s)\,ds = \left(\mathscr{A}\frac{\partial\xi}{\partial t}, \frac{\partial[u-\chi]}{\partial t}\right)\Big|_0^t - \int_0^t \left(\mathscr{A}\frac{\partial\xi}{\partial t}, \frac{\partial^2[u-\chi]}{\partial t^2}\right)(s)\,ds.$$

Hence,

$$|I_3| \leqslant \frac{1}{4}\left(\mathscr{A}\frac{\partial\xi}{\partial t}, \frac{\partial\xi}{\partial t}\right) + C\left[\left\|\frac{\partial\xi}{\partial t}(0)\right\|_0^2 + \left\|\frac{\partial[u-\chi]}{\partial t}\right\|_{L^\infty(J,[L^2(\Omega)]^4)}^2 + \left\|\frac{\partial^2[u-\chi]}{\partial t^2}\right\|_{L^2(J,[L^2(\Omega)]^4)}^2 + \int_0^t \left(\mathscr{A}\frac{\partial\xi}{\partial t}, \frac{\partial\xi}{\partial t}\right)(s)\,ds\right].$$

Then if we integrate (2.9) in time use (2.6)-(2.7) we have

$$\frac{1}{2}\left(\mathscr{A}\frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t}\right)(t) + C_2 \| \xi(t) \|_V^2 \leqslant$$

$$\leqslant C\Bigg[(E_k^0)^2 h^{2k} + \left\| \frac{\partial [u-\chi]}{\partial t} \right\|^2_{L^\infty(J,[L^2(\Omega)]^4)} + \left\| \frac{\partial [u-\chi]}{\partial t} \right\|^2_{L^2(J,V)} + \left\| \frac{\partial^2 [u-\chi]}{\partial t^2} \right\|^2_{L^2(J,[L^2(\Omega)]^4)} + \int_0^t \left(\| \xi(s) \|_V^2 + \left\| \frac{\partial \xi}{\partial t}(s) \right\|_0^2 \right) ds \Bigg].$$

It follows from the approximation hypotheses (2.3) that

$$\inf_\chi \Bigg[\left\| \frac{\partial [u-\chi]}{\partial t} \right\|_{L^\infty(J,[L^2(\Omega)]^4)} + \left\| \frac{\partial [u-\chi]}{\partial t} \right\|_{L^2(J,V)} + \left\| \frac{\partial^2 [u-\chi]}{\partial t^2} \right\|_{L^2(J,[L^2(\Omega)]^4)} \Bigg] \leqslant \leqslant C E_k h^k . \tag{2.10}$$

Thus the conclusion of the theorem follows from a Gronwall argument. This completes the proof.

3. A DISCRETE-TIME GALERKIN PROCEDURE

Let L be a positive integer, $\Delta t = T/L$, $g^n = g(n\,\Delta t)$, $n = 0, 1, \ldots, L$. Set

$$U^{n+1/2} = \frac{U^n + U^{n+1}}{2},$$

$$U^{n,1/4} = \frac{1}{4} U^{n-1} + \frac{1}{2} U^n + \frac{1}{4} U^{n+1},$$

$$d_t U^n = \frac{U^{n+1} - U^n}{\Delta t},$$

$$\partial U^n = \frac{U^{n+1} - U^{n-1}}{2\,\Delta t},$$

$$\partial^2 U^n = \frac{U^{n+1} - 2\,U^n + U^{n-1}}{(\Delta t)^2}$$

The discrete-time Galerkin approximation to u is defined as the sequence $(U^n)_{0 \leq n \leq L} \subset V_h$ such that

$$(\mathscr{A}\, \partial^2 U^n, v) + (\mathscr{C}\, \partial U^n, v) + B(U^{n,1/4}, v) = \\ = (F^{n,1/4}, v) + \langle \phi^{n,1/4}, v_1 \rangle + \langle v_2 . \nu, \eta^{n,1/4} \rangle, \quad (3.1)$$

$v = (v_1, v_2) \in V_h$, $n = 1, 2, \ldots, L - 1$. The initial values U^0 and U^1 must be specified, but we shall delay their selection until the error analysis has indicated what is needed to maintain optimal order convergence.

We now proceed to derive estimates for the error $\xi^n = u^n - U^n$. Set

$$P_n^2 = \left\| \frac{\partial^3 u}{\partial t^3} \right\|^2_{L^2(t_{n-1}, t_{n+1}, [L^2(\Omega)]^4)} + \left\| \frac{\partial^4 u}{\partial t^4} \right\|^2_{L^2(t_{n-1}, t_{n+1}, [L^2(\Omega)]^4)}$$

$$P^2 = \left\| \frac{\partial^3 u}{\partial t^3} \right\|^2_{L^2(J, [L^2(\Omega)]^4)} + \left\| \frac{\partial^4 u}{\partial t^4} \right\|^2_{L^2(J, [L^2(\Omega)]^4)}$$

THEOREM 3.1 : *Assume that $P < \infty$ and $E_k < \infty$ for some $k \geq 1$. Then the error generated by the solution $(U^n)_{0 \leq n \leq L}$ of the procedure* (3.1) *is bounded by*

$$\max_{1 \leq n \leq L-1} (\| d_t \xi^n \|_0 + \| \xi^{n+1/2} \|_V) \leq \\ \leq C[\| d_t \xi^0 \|_0 + \| \xi^{1/2} \|_V + P(\Delta t)^2 + E_k h^k].$$

Proof : First note that

$$\frac{\partial^2 u^{n,1/4}}{\partial t^2} = \partial^2 u^n - \delta_1^n$$

and

$$\frac{\partial u^{n,1/4}}{\partial t} = \partial u^n - \delta_2^n$$

where

$$\| \delta_1^n \|_0^2 + \| \delta_2^n \|_0^2 \leq C(\Delta t)^3 P_n^2 .$$

Then from (2.8) we see that

$$(\mathscr{A}\, \partial^2 u^n, v) + (\mathscr{C}\, \partial u^n, v) + B(u^{n,1/4}, v) = \\ = (F^{n,1/4}, v) + \langle \phi^{n,1/4}, v_1 \rangle + \langle v_2 . \nu, \eta^{n,1/4} \rangle \\ + (\mathscr{A} \delta_1^n, v) + (\mathscr{C}\, \delta_2^n, v),$$

for $v = (v_1, v_2) \in V$ and $n = 1, 2, \ldots, L - 1$.

Combining (3.1) and the relation above we obtain

$$(\mathscr{A}\,\partial^2\xi^n, v) + (\mathscr{C}\,\partial\xi^n, v) + B(\xi^{n,1/4}, v) =$$
$$= (\mathscr{A}\,\delta_1^n, v) + (\mathscr{C}\,\delta_2^n, v)\,, \quad \text{for} \quad v \in V_h \quad \text{and} \quad n = 1, 2, ..., L-1\,.$$

Choose the test function $v = \partial(\xi + \chi - u)^n$ where $\chi : J \to V_h$ and note that

$$\frac{\gamma}{2\,\Delta t}(\| \xi^{n+1/2} \|_0^2 - \| \xi^{n-1/2} \|_0^2) \leqslant$$
$$\leqslant \frac{\gamma}{4}(\| d_t\xi^n \|_0^2 + \| d_t\xi^{n-1} \|_0^2 + \| \xi^{n+1/2} \|_0^2 + \| \xi^{n-1/2} \|_0^2)\,.$$

Thus,

$$\begin{aligned} \frac{1}{2\,\Delta t}[\| \mathscr{A}^{1/2}\, d_t\xi^n \|_0^2 - \| \mathscr{A}^{1/2}\, d_t\xi^{n-1} \|_0^2 + B_\gamma(\xi^{n+1/2}, \xi^{n+1/2}) - \\ - B_\gamma(\xi^{n-1/2}, \xi^{n-1/2})] + (\mathscr{C}\,\partial\xi^n, \partial\xi^n) \\ \leqslant \frac{1}{2}(\mathscr{C}\,\partial\xi^n, \partial\xi^n) + C[\| d_t\xi^n \|_0^2 + \| d_t\xi^{n-1} \|_0^2 \quad (3.2) \\ + \| \xi^{n+1/2} \|_V^2 + \| \xi^{n-1/2} \|_V^2 + \| \delta_1^n \|_0^2 + \| \delta_2^n \|_0^2 \\ + \| \partial(u-\chi)^n \|_V^2] + (\mathscr{A}\,\partial^2\xi^n, \partial(u-\chi)^n)\,. \end{aligned}$$

We shall sum (3.2) from $n = 1$ to $n = m$, $1 \leqslant m \leqslant L - 1$, but first we shall bound the sum over n of the last term in the right-hand side. Using summation by parts,

$$\begin{aligned} S_1 &= \sum_{n=1}^{m} (\mathscr{A}\,\partial^2\xi^n, \partial[u-\chi]^n)\,\Delta t \\ &= \sum_{n=1}^{m} \left(\mathscr{A}\,\frac{[d_t\xi^n - d_t\xi^{n-1}]}{\Delta t}, \partial[u-\chi]^n\right)\Delta t \\ &= (\mathscr{A}\,d_t\xi^m, \partial[u-\chi]^m) - (\mathscr{A}\,d_t\xi^0, \partial[u-\chi]^0) \\ &\quad - \sum_{n=1}^{m-1} \left(\mathscr{A}\,\frac{[\partial[u-\chi]^{n+1} - \partial[u-\chi]^n]}{\Delta t}, d_t\xi^n\right)\Delta t\,. \end{aligned}$$

Since

$$\left\| \frac{\partial[u-\chi]^{n+1} - \partial[u-\chi]^n}{\Delta t} \right\|_0^2 \leqslant \frac{C}{\Delta t}\int_{t_{n-1}}^{t_{n+2}} \left\| \frac{\partial^2[u-\chi]}{\partial t^2}(s) \right\|_0^2 ds\,,$$

it follows that

$$S_1 \leqslant \frac{1}{2} \| \mathscr{A}^{1/2} d_t\xi^m \|_0^2 + C\left[\| d_t\xi^0 \|_0^2 + \left\| \frac{\partial[u-\chi]}{\partial t} \right\|^2_{L^\infty(J,[L^2(\Omega)]^4)} + \left\| \frac{\partial^2[u-\chi]}{\partial t^2} \right\|^2_{L^2(J,[L^2(\Omega)]^4)} + \sum_{n=1}^{m-1} \| d_t\xi^n \|_0^2 \Delta t \right].$$

Now we multiply (3.2) by $2\,\Delta t$ and sum from $n = 1$ to $n = m$. Applying the estimate for S_1 we obtain

$$\begin{aligned} &\frac{1}{2} \| \mathscr{A}^{1/2} d_t\xi^m \|_0^2 + C_2 \| \xi^{m+1/2} \|_V^2 + \sum_{n=1}^{m} (\mathscr{C}\, \partial\xi^n, \partial\xi^n)\,\Delta t \leqslant \\ &\leqslant C\Bigg[\| d_t\xi^0 \|_0^2 + \| \xi^{1/2} \|_V^2 + P^2(\Delta t)^4 + \\ &+ \left\| \frac{\partial[u-\chi]}{\partial t} \right\|^2_{L^\infty(J,[L^2(\Omega)]^4)} + \left\| \frac{\partial[u-\chi]}{\partial t} \right\|^2_{L^2(J,V)} \\ &+ \left\| \frac{\partial^2[u-\chi]}{\partial t^2} \right\|^2_{L^2(J,[L^2(\Omega)]^4)} \\ &+ \sum_{n=1}^{m} (\| d_t\xi^n \|_0^2 + \| \xi^{n+1/2} \|_V^2)\,\Delta t \Bigg], \quad 1 \leqslant m \leqslant L-1 . \end{aligned} \tag{3.3}$$

Then the discrete Gronwall's lemma applied to (3.3) implies that

$$\begin{aligned} &\| d_t\xi^m \|_0^2 + \| \xi^{m+1/2} \|_V^2 \leqslant \\ &\leqslant C\Bigg[\| d_t\xi^0 \|_0^2 + \| \xi^{1/2} \|_V^2 + P^2(\Delta t)^4 + \left\| \frac{\partial[u-\chi]}{\partial t} \right\|^2_{L^\infty(J,[L^2(\Omega)]^4)} \\ &+ \left\| \frac{\partial[u-\chi]}{\partial t} \right\|^2_{L^2(J,V)} + \left\| \frac{\partial^2[u-\chi]}{\partial t^2} \right\|^2_{L^2(J,[L^2(\Omega)]^4)} \Bigg], \quad 1 \leqslant m \leqslant L-1 . \end{aligned}$$

Now the conclusion of the theorem follows from (2.10) and the inequality above. Thus the proof is complete.

Finally we shall indicate a way of choosing U^0 and U^1 such that

$$\| d_t\xi^0 \| + \| \xi^{1/2} \|_V \leqslant C[(\Delta t)^2 + h^k], \tag{3.4}$$

so that the accuracy of the method (3.1) is preserved. Let $U^0 \in V_h$ be defined by the projection

$$B_\gamma(U^0 - u^0, v) = 0\,, \quad v \in V_h\,.$$

Next let

$$u^* = u^0 + v^0\,\Delta t + \mathscr{A}^{-1}[F(x,0) + \mathscr{L}(u^0) - \mathscr{C}v^0]\,\frac{(\Delta t)^2}{2}$$

and note that from (2.6)-(2.7) of Part I we have

$$u^1 = u(x, \Delta t) = u^* + \delta_3\,,$$

where

$$\| \delta_3 \|_V \leqslant C(\Delta t)^3\,.$$

Then define $U^1 \in V_h$ by the projection

$$B_\gamma(U^1 - u^*, v) = 0\,, \quad v \in V_h\,.$$

Assume that u^0, v^0, $F(x, 0)$ and the coefficients of the matrices $\mathscr{A}$ and $\mathscr{C}$ are sufficiently smooth. Then it can be seen that

$$\| \xi^0 \|_V + \| d_t \xi^0 \|_0 \leqslant C[(\Delta t)^2 + h^k]$$

and

$$\| \xi^1 \|_V \leqslant C[(\Delta t)^3 + h^k]\,,$$

so that the estimate (3.4) holds.

4. ACKNOWLEDGMENTS

We wish to thank Professor Jim Douglas, Jr. for his help and advice in the prosecution of this research.

REFERENCES

[1] F. BREZZI, J. DOUGLAS, Jr. and L. D. MARINI, *Two families of Mixed Finite Elements for Second Order Elliptic Problems*, Numer. Math. 47 (1985), pp. 217-235.

[2] F. BREZZI, J. DOUGLAS, Jr. and L. D. MARINI, *Recent Results on Mixed Finite Element Methods for Second Order Elliptic Problems*, to appear.

[3] J. DOUGLAS, Jr. and J. E. ROBERTS, *Global Estimates for Mixed Methods for Second Order Elliptic Equations*, Math. Comp. 44 (1985), pp. 39-52.

[4] J. C. NEDELEC, *Mixed Finite Elements in R^3*, Numer. Math. 35 (1980), pp. 315-341.

[5] P. A. RAVIART and J. M. THOMAS, *A Mixed Finite Element Method for 2nd Order Elliptic Problems, Mathematical Aspects of the Finite Element Method,* Lecture Notes in Mathematics 606, Springer-Verlag, Berlin, 1977.

[6] J. E. SANTOS, *Elastic Wave Propagation in Fluid-Saturated Porous Media,* Part I, *The Existence and Uniqueness Theorems,* this Journal, vol. 20, N° 1, pp. 113-128, 1986.

[7] J. M. THOMAS, *Sur l'Analyse Numérique des Méthodes d'Eléments Finis Hybrides et Mixtes,* Thèse, Université P. et M. Curie, Paris, 1977.