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A NOTE ON THE COMPLEX AND REAL OPERATOR NORMS OF REAL MATRICES (*)

by Michel CROUZEIX [1]

Abstract — *Our purpose in this note is to show that the l_p operator norm of any real rectangular matrix is the same if we consider it as acting on complex-valued or only on real-valued vectors*

Résumé — *Cette note montre que la norme l_p d'une matrice à coefficients réels est la même, qu'on la considere comme opérant sur des vecteurs à coefficients réels ou sur des vecteurs à coefficients complexes*

We begin with a lemma.

Let us consider two vectors a and b in $\mathbb{R}^m$ with $a \neq 0$ and $b \neq 0$, and two vectors u and v in $\mathbb{R}^n$. For $1 \leqslant p \leqslant +\infty$, we use the norm :

$$\| a \|_p = \left(\sum_{i=1}^{m} | a_i |^p \right)^{1/p}$$

where $a = (a_i)$, (with the usual modification in the case $p = +\infty$).

LEMMA : *Let :*

$$\alpha = \sup_{z \in \mathbb{C}} \frac{\| u + zv \|_p}{\| a + zb \|_p} \quad \text{and} \quad \beta = \sup_{x \in \mathbb{R}} \frac{\| u + xv \|_p}{\| a + xb \|_p} ;$$

then $\alpha = \beta$.

Proof : The case when α is obtained as the limit when $| z | \to +\infty$ is obvious. In the opposite case we set, for $p < +\infty$:

$$f(x, y) = \| u + zv \|_p^p - \alpha^p \| a + zb \|_p^p \quad \text{where} \quad z = x + iy ,$$

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we have $0 = \max_{x,y \in \mathbb{R}} f(x, y)$. By an elementary computation :

$$4\, y \frac{\partial^2 f}{\partial z\, \partial \bar{z}} - ip\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar{z}}\right) = 0\,, \quad \text{for} \quad y \neq 0$$

and hence :

$$y\, \Delta f(x, y) - p \frac{\partial}{\partial y} f(x, y) = 0\,, \quad \text{for} \quad y \neq 0\,.$$

By the maximum principle, the maximum of f is attained for $y = 0$. The result in $p = +\infty$ is obtained by using a limiting argument. ■

COROLLARY : *For a matrix $A \in \mathbb{R}^{m \times n}$ we have:*

$$\sup_{\xi \in \mathbb{R}^n - \{0\}} \| A\xi \|_p / \| \xi \|_p = \sup_{\zeta \in \mathbb{C}^n - \{0\}} \| A\zeta \|_p / \| \zeta \|_p\,.$$

Proof : For $\zeta \in \mathbb{C}^n$ we write $\zeta = a + ib$ with $a, b \in \mathbb{R}^n$; set $u = Aa$, $v = Ab$ and apply the lemma to obtain :

$$\frac{\| A\zeta \|_p}{\| \zeta \|_p} = \frac{\| u + iv \|_p}{\| a + ib \|_p} \leqslant \sup_{x \in \mathbb{R}} \frac{\| u + xv \|_p}{\| a + xb \|_p} = \sup_{x \in \mathbb{R}} \frac{\| A(a + xb) \|_p}{\| a + xb \|_p}$$
$$\leqslant \sup_{\xi \in \mathbb{R}^n} \frac{\| A\xi \|_p}{\| \xi \|_p}\,.$$