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MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
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FINITE ELEMENT SOLUTIONS FOR RADIATION COOLING PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS (*)

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Abstract — *We consider a finite element method for the elliptic problem*

$$\mathscr{L}u = 0 \quad in \quad \Omega,$$

with the nonlinear boundary conditions

$$\frac{\partial u}{\partial \nu} = g(x, u) \quad on \quad \Gamma$$

It is shown that the finite element solutions converge to the exact solution under some appropriate hypotheses We also give some results of numerical experiments in the two dimensional case

Résume — *Nous considérons une méthode d'elements finis pour le probleme elliptique*

$$\mathscr{L}u = 0 \quad dans \quad \Omega,$$

avec des conditions aux limites non linéaires

$$\frac{\partial u}{\partial \nu} = g(x, u) \quad sur \quad \Gamma$$

Il est montre que les solutions obtenues par éléments finis convergent vers la solution exacte sous des hypothèses convenables On donne des resultats d'expériences numériques dans le cadre bidimensionnel

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1. INTRODUCTION

This paper is concerned with the finite element solutions for the radiation cooling problem with the nonlinear boundary condition :

$$\left.\begin{aligned} \mathscr{L}u &= 0 \quad &\text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x, u) \quad &\text{on} \quad \Gamma. \end{aligned}\right\} \tag{1.1}$$

Here $x = (x_1, x_2, \dots, x_n)$, Ω is a bounded convex domain in the n-dimensional Euclidean space $\mathbb{R}^n$. The boundary Γ of Ω is assumed so smooth that the maximum principle for $\mathscr{L}$ holds [7, 16] and $\mathscr{L}$ is the uniformly elliptic self-adjoint second order operator :

$$\mathscr{L}u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)\, u\,.$$

The coefficients $a_{ij}(x) = a_{ji}(x)$, $1 \leqq i, j \leqq n$, $a_0(x)$ are sufficiently smooth,

$$a_0(x) \geqq 0\,, \qquad x \in \Omega\,, \tag{1.2}$$

and for all vectors $(\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ there exists a positive constant μ_0 such that :

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\gamma_i\,\gamma_j \geqq \mu_0 \sum_{i=1}^{n} \gamma_i^2\,, \qquad x \in \Omega\,, \tag{1.3}$$

and $\partial/\partial\nu$ is the conormal derivative

$$\frac{\partial u}{\partial \nu} \equiv \sum_{i,j=1}^{n} \tau_i(x)\, a_{ij}(x)\, \frac{\partial u}{\partial x_j}\,,$$

where $(\tau_1(x), \dots, \tau_n(x))$ is the outer unit normal to Γ at x.

The problem (1.1) arises in the theory of heat transfer between solids and gases under the nonlinear radiation boundary condition obeying *Newton's Law of Cooling* (*cf.* [6, 13, 14] and the references therein). For example, the « fourth power law » will employ $g(x, u) = -u^4 + Q(x)$ with $Q(x) > 0$ [14]. The unknown function $u(x)$ represents the absolute temperature distribution in a solid, so that $u(x)$ is required to be *positive.* In [6], Cohen established the uniqueness and existence of the positive solution of (1.1) under the following assumption.

Assumption 1 : The given function $g(x, u)$ satisfies :

(*a*) $g(x, u)$ is twice continuously differentiable in $\overline{\Omega} = \Omega \cup \Gamma$ for all u,

(*b*) $g(x, 0) > 0$ and $g(x, 1) = 0$,

(*c*) $g_u(x, u) \equiv \dfrac{\partial g}{\partial u} < 0$, in $\overline{\Omega}$ for $u > 0$,

(*d*) $g_{uu}(x, u) \equiv \dfrac{\partial^2 g}{\partial u^2} < 0$, in $\overline{\Omega}$ for $u > 0$.

In the present paper, we shall study the finite element approximation to (1.1), based upon piecewise linear polynomials and lumping operator. The monotone iterative method is considered for solving the nonlinear algebraic equations associated with the finite element approximation. Furthermore, we shall prove that the finite element solutions converge uniformly to the exact solution with a certain rate of convergence under some appropriate assumptions on discretization. Finally, some numerical results are presented to indicate the effectiveness of our theorems in the two dimensional case.

For related results on finite element approximations to the nonlinear problems with the Dirichlet boundary conditions, we refer to Ishihara [9, 10, 11, 12].

Throughout this paper, C, C_1, C_2, ... denote generic positive constants independent of the discretization parameter h, which are not necessarily the same at each occurrence.

2. NOTATION AND FINITE ELEMENT APPROXIMATION

In this section, we shall describe some notations. Let $W^{m,p}(\Omega)$ be the Sobolev space which for any integer $m \geqq 0$ and any number $p \geqq 1$, consists of real-valued functions which together with their generalized derivatives up to the m-th order belong to $L^p(\Omega)$. Here $L^p(\Omega)$ denotes the space of mesurable functions on Ω that are p-integrable. The norm in $W^{m,p}(\Omega)$ is given by :

$$\| \psi \|_{W^{m,p}(\Omega)} = \left(\sum_{|\beta| \leqq m} \| D^\beta \psi \|^p_{L^p(\Omega)} \right)^{1/p} \quad \text{if} \quad 1 \leqq p < \infty ,$$

$$\| \psi \|_{W^{m,\infty}(\Omega)} = \max_{|\beta| \leqq m} \| D^\beta \psi \|_{L^\infty(\Omega)} \quad \text{if} \quad p = \infty ,$$

where $\beta = (\beta_1, ..., \beta_n)$ is a vector of nonnegative integers,

$$| \beta | = \sum_{i=1}^{n} \beta_i , \qquad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}} ,$$

$$\| D^{\beta} \psi \|_{L^p(\Omega)} = \left(\int_{\Omega} | D^{\beta} \psi(x) |^p \, dx \right)^{1/p},$$

$$\| D^{\beta} \psi \|_{L^{\infty}(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} | D^{\beta} \psi(x) | .$$

Put :

$$(\phi, \psi)_{\Omega} = \int_{\Omega} \phi(x) \, \psi(x) \, dx ,$$

$$[\phi, \psi]_{\Gamma} = \int_{\Gamma} \phi(x) \, \psi(x) \, ds .$$

Recall that we have assumed Ω to be a convex domain in $\mathbb{R}^n$ with smooth boundary Γ. As usual, we triangulate Ω in such a way that :

$$\bar{\Omega} \supset \bigcup_{q=1}^{J} T_q ,$$

where T_q, $1 \leqq q \leqq J$ are nondegenerate closed n-simplices whose interiors are pairwise disjoint and P_i, $1 \leqq i \leqq N$ (or P_i, $N + 1 \leqq i \leqq N + M$) denote the vertices of the triangulation which belong to Ω (or Γ). See figure 1. Set :

$$h_q = \text{diameter of } T_q , \quad 1 \leqq q \leqq J ,$$

$$h = \max_{1 \leqq q \leqq J} h_q ,$$

$$\Omega_h = \text{interior of the polyhedral domain } \bigcup_{q=1}^{J} T_q ,$$

$$\Gamma_h = \text{boundary of } \Omega_h ,$$

$$\rho_q = \text{supremum of the diameter of the inscribed sphere of } T_q , \quad 1 \leqq q \leqq J .$$

Remark 1 : Let θ be the angle between Γ_h and the tangent plane to Γ at the point $P \in \Gamma$ in figure 1. Since Γ is smooth and since Γ_h is the boundary of the polyhedral domain Ω_h, there exist positive constants C_1 and C_2 such that [19, 21]

$$0 \leqq \theta \leqq C_1 \, h ,$$

$$\operatorname{dist}(x, \Gamma) \leqq C_2 \, h^2 , \qquad x \in \Gamma_h .$$

We say that a family $\{ \mathcal{T}^h \}$ of triangulations is *regular* if there exists a positive constant c_0 independent of the triangulation such that

$$h_q \leqq c_0 \, \rho_q \quad \text{for all} \quad T_q \in \mathcal{T}^h .$$

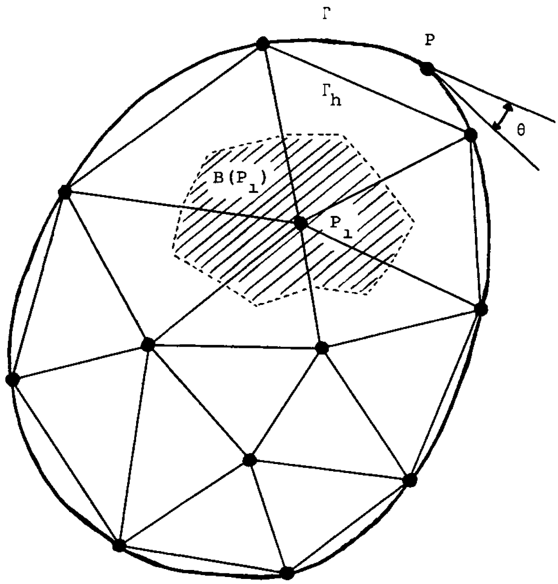

Figure 1. — Triangulation ($n = 2$).

Remark 2 : When $n = 2$, $\{ \mathcal{T}^h \}$ is regular if there exists a constant ω_0 satisfing

$$0 < \omega_0 \leqq \omega_{\min} ,$$

where $\omega_{\min}$ denotes the smallest angle of all the triangles $T_q \in \mathcal{T}^h$.

For an n-simplex $T_q \in \mathcal{T}^h$, let $P_0^{(q)} = P_i$, $P_1^{(q)} = P_{i_1}, \ldots, P_n^{(q)} = P_{i_n}$ be its vertices and let $\lambda_j^{(q)}(x)$, $0 \leqq j \leqq n$ be the barycentric coordinates of a point $x \in T_q$ with respect to $P_j^{(q)}$, $0 \leqq j \leqq n$, respectively. The barycentric subdivision B_i^q of T_q corresponding to P_i which is the vertex of T_q with the barycentric coordinate $\lambda_0^{(q)}(x)$ is given by

$$B_i^q = \bigcap_{j=1}^{n} \{ x \in T_q \; ; \; \lambda_0^{(q)}(x) \geqq \lambda_j^{(q)}(x) \} .$$

Then the lumped mass region $\mathscr{B}(P_i)$ corresponding to P_i is defined as follows (see *fig.* 1) :

$$\mathscr{B}(P_i) = \bigcup_q \{ B_i^q \; ; \; T_q \in \mathcal{T}^h \text{ such that } P_i \text{ is a vertex of } T_q \} , \quad 1 \leqq i \leqq N + M .$$

Let $\phi_{h,i}$, $\overline{\phi}_{h,i}$, $1 \leqq i \leqq N + M$ be the finite element basis such that :

$$\phi_{h,i} \in \mathscr{C}^0(\overline{\Omega}_h) \quad \text{and} \quad \phi_{h,i} \text{ is linear on each } T_q \in \mathcal{T}^h,$$

$$\phi_{h,i}(P_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\overline{\phi}_{h,i}(x) = \begin{cases} 1, & x \in \mathscr{B}(P_i), \\ 0, & x \notin \mathscr{B}(P_i), \end{cases}$$

for $1 \leqq i, j \leqq N + M$, where $\mathscr{C}^0(\overline{\Omega}_h)$ denotes the space of continuous functions on $\overline{\Omega}_h$. Define finite element spaces as follows :

$$X^h = \text{span}\,[\overline{\phi}_{h,1}, \ldots, \overline{\phi}_{h,N+M}],$$

$$Y^h = \text{span}\,[\phi_{h,1}, \ldots, \phi_{h,N+M}].$$

Let $\sim$ be the lumping operator given by

$$\sim\; : \mathscr{C}^0(\overline{\Omega}) \to X^h \quad (\text{or } \mathscr{C}^0(\overline{\Omega}_h) \to X^h),$$

$$\psi \mapsto \tilde{\psi} = \sum_{i=1}^{N+M} \psi(P_i)\, \overline{\phi}_{h,i}.$$

It is noted that $\tilde{\phi}_{h,i} = \overline{\phi}_{h,i}$. Moreover, we define a bilinear form :

$$\mathscr{A}_h(\phi\,;\psi) = \sum_{i,j=1}^{n} \left(a_{ij}(x) \frac{\partial \phi}{\partial x_j}, \frac{\partial \psi}{\partial x_i} \right)_{\Omega_h}.$$

We now formulate the finite element approximation to (1.1) in the following way :

Find $u_h \in Y^h$ *such that*

$$\mathscr{A}_h(u_h\,;\psi_h) + (\tilde{a}_0(x)\,\tilde{u}_h, \tilde{\psi}_h)_{\Omega_h} = [\tilde{g}(x, \tilde{u}_h), \tilde{\psi}_h]_{\Gamma_h} \quad \textit{for all} \quad \psi_h \in Y^h. \tag{2.1}$$

This nonlinear equation is solved by the following linear iterative method :

Find $u_{h,k} \in Y^h$ $(k = 1, 2, \ldots)$ *such that*

$$\mathscr{A}_h(u_{h,k}\,;\psi_h) + (\tilde{a}_0(x)\,\tilde{u}_{h,k}, \tilde{\psi}_h)_{\Omega_h} - [\tilde{g}_u(x, \tilde{u}_{h,k-1})\,\tilde{u}_{h,k}, \tilde{\psi}_h]_{\Gamma_h}$$
$$= [\tilde{g}(x, \tilde{u}_{h,k-1}) - \tilde{g}_u(x, \tilde{u}_{h,k-1})\,\tilde{u}_{h,k-1}, \tilde{\psi}_h]_{\Gamma_h} \quad \textit{for all} \quad \psi_h \in Y^h. \tag{2.2}$$

Here $u_{h,0} \equiv \delta > 0$ is a positive constant function, the constant δ being arbitrary. If we seek the solutions u_h and $u_{h,k}$ in the form :

$$u_h = \sum_{j=1}^{N+M} \xi_j \, \phi_{h,j} \quad \text{with} \quad \xi_j = u_h(P_j), \quad 1 \leqq j \leqq N + M,$$

$$u_{h,k} = \sum_{j=1}^{N+M} \xi_{j,k} \, \phi_{h,j} \quad \text{with} \quad \xi_{j,k} = u_{h,k}(P_j), \quad 1 \leqq j \leqq N + M, \quad k = 1, 2, \ldots,$$

then we have :

$$\left.\begin{array}{ll} \displaystyle\sum_{j=1}^{N+M} A_{ij} \, \xi_j = 0, & 1 \leqq i \leqq N, \\ \displaystyle\sum_{j=1}^{N+M} A_{ij} \, \xi_j = [g(P_i, \xi_i), \overline{\phi}_{h,i}]_{\Gamma_h}, & N + 1 \leqq i \leqq N + M, \end{array}\right\} \quad (2.3)$$

for (2.1), and :

$$\left.\begin{array}{ll} \displaystyle\sum_{j=1}^{N+M} A_{ij} \, \xi_{j,k} = 0, & 1 \leqq i \leqq N, \\ \displaystyle\sum_{j=1}^{N+M} A_{ij} \, \xi_{j,k} + D_i^{(k-1)} \, \xi_{i,k} = s_i^{(k-1)}, & N + 1 \leqq i \leqq N + M, \end{array}\right\} \quad (2.4)$$

for (2.2), $k = 1, 2, \ldots$. Here

$$A_{ij} = \mathscr{A}_h(\phi_{h,j}; \phi_{h,i}) + (\tilde{a}_0(x) \, \overline{\phi}_{h,j}, \overline{\phi}_{h,i})_{\Omega_h}, \quad 1 \leqq i, j \leqq N + M, \qquad (2.5)$$

$$D_i^{(k-1)} = - \, [\tilde{g}_u(x, \tilde{u}_{h,k-1}) \, \overline{\phi}_{h,i}, \overline{\phi}_{h,i}]_{\Gamma_h}, \qquad N + 1 \leqq i \leqq N + M, \quad k = 1, 2, \ldots,$$

$$s_i^{(k-1)} = [\tilde{g}(x, \tilde{u}_{h,k-1}) - \tilde{g}_u(x, \tilde{u}_{h,k-1}) \, \tilde{u}_{h,k-1}, \overline{\phi}_{h,i}]_{\Gamma_h}, \quad N + 1 \leqq i \leqq N + M, \quad k = 1, 2, \ldots. \qquad (2.6)$$

Following Ciarlet [3] and Ciarlet-Raviart [5], we say that a triangulation $\mathscr{C}^h$ is of *nonnegative type* if its corresponding matrix $\mathbf{A} = (A_{ij})$, $1 \leqq i, j \leqq N + M$ defined by (2.5) has the following two properties :

(i) $A_{ii} > 0$, $1 \leqq i \leqq N + M$, $A_{ij} \leqq 0$, $i \neq j$, $1 \leqq i, j \leqq N + M$,

$$\sum_{j=1}^{N+M} A_{ij} \geqq 0, \qquad 1 \leqq i \leqq N + M,$$

(ii) $\mathbf{A}$ is irreducible [22, p. 18], and $\overline{\mathbf{A}} = (A_{ij})$, $1 \leqq i, j \leqq N$ is irreducibly diagonally dominant [22, p. 23].

Remark 3 : In the case where $n = 2$ and $\mathcal{L} = -\Delta$ (Δ : Laplacian), $\mathcal{T}^h$ is of *nonnegative type* if all the angles of the triangles are less than or equal to $\pi/2$ [5].

In the sequel, we make the following assumptions on the triangulation.

Assumption 2 : $\{\mathcal{T}^h\}$ is regular.

Assumption 3 : $\mathcal{T}^h$ is of nonnegative type.

3. MONOTONE CONVERGENCE AND ERROR ESTIMATE

In this section, we show the monotone convergence of the iteration (2.2), and obtain the error estimate between the solution u of (1.1) and the solution u_h of (2.1) in the L^∞-norm. First, some lemmas are prepared without proofs.

LEMMA 1 [15, p. 103] : *An $\bar{n} \times \bar{n}$ matrix $\mathbf{K} = (K_{ij})$ is irreducible if and only if for any two distinct indices $1 \leqq i, j \leqq \bar{n}$, there exists a sequence of nonzero elements of $\mathbf{K}$ of the form*

$$\{K_{ii_1}, K_{i_1 i_2}, \ldots, K_{i_t j}\}.$$

LEMMA 2 [22, p. 85] : *Let $\mathbf{K} = (K_{ij})$ be an irreducibly diagonally dominant $\bar{n} \times \bar{n}$ matrix with $K_{ii} > 0$, $K_{ij} \leqq 0$, $i \neq j$, $1 \leqq i, j \leqq \bar{n}$. Then, the inverse $\mathbf{K}^{-1}$ exists and $\mathbf{K}^{-1} > 0$.*

The following lemma is well known.

LEMMA 3 [4, 8, 20] : *Let Assumption 2 hold. Then, we have*

$$\begin{aligned} \| w - I_h w \|_{L^\infty(\Omega_h)} &\leqq C_0 \| w - I_h w \|_{W^{1,p}(\Omega_h)} \\ &\leqq C_1 h \| w \|_{W^{2,p}(\Omega_h)}, \quad w \in W^{2,p}(\Omega_h), \quad p > n \geqq 2, \\ \| w - \widetilde{I_h w} \|_{L^p(\Omega_h)} &\leqq C_2 h \| w \|_{W^{1,p}(\Omega_h)}, \quad w \in W^{1,p}(\Omega_h), \quad p > n \geqq 2, \\ \| \psi_h - \tilde{\psi}_h \|_{L^{p'}(\Omega_h)} &\leqq C_3 h \| \psi_h \|_{W^{1,p'}(\Omega_h)}, \quad \psi_h \in Y^h, \quad 1 \leqq p' < \infty, \\ C_4 \| \psi_h \|_{L^{p'}(\Omega_h)} \leqq \| \tilde{\psi}_h \|_{L^{p'}(\Omega_h)} &\leqq C_5 \| \psi_h \|_{L^{p'}(\Omega_h)}, \quad \psi_h \in Y^h, \quad 1 \leqq p' < \infty. \end{aligned}$$

Here $I_h w$ is the interpolating function of w such that :

$$I_h w = \sum_{i=1}^{N+M} w(P_i) \phi_{h,i},$$

and C_i, $0 \leqq i \leqq 5$ are positive constants independent of h.

The following result is the trace theorem.

LEMMA 4 [1, p. 114] : *Let p' satisfy*

$$1 < p' \leqq n.$$

Then, there exists a positive constant C_1 such that

$$\| w \|_{L^{p'}(\Gamma_h)} \leqq C_1 \| w \|_{W^{1,p'}(\Omega_h)}, \quad w \in W^{1,p'}(\Omega_h).$$

We now have the following theorem concerning the uniqueness of the positive solution of (2.1).

THEOREM 1 : *Let Assumptions 1 and 3 hold. If the problem (2.1) has a positive solution, then its solution u_h is unique.*

Proof : Assume that u_h and w_h are two positive solutions. Then, from (2.1) we have :

$$\mathscr{A}_h(u_h - w_h ; \psi_h) + (\tilde{a}_0(x)(\tilde{u}_h - \tilde{w}_h), \tilde{\psi}_h)_{\Omega_h} = [\tilde{g}(x, \tilde{u}_h) - \tilde{g}(x, \tilde{w}_h), \tilde{\psi}_h]_{\Gamma_h} = [\tilde{g}_u(x, \tilde{v}_h)(\tilde{u}_h - \tilde{w}_h), \tilde{\psi}_h]_{\Gamma_h}, \quad (3.1)$$

for all $\psi_h \in Y^h$, where :

$$v_h = \varepsilon(x) u_h + (1 - \varepsilon(x)) w_h > 0, \quad \text{with} \quad 0 < \varepsilon(x) < 1.$$

Put :

$$u_h - w_h = \sum_{j=1}^{N+M} \zeta_j \phi_{h,j},$$

and define the matrix $\mathbf{K} = (K_{ij})$, $1 \leqq i, j \leqq N + M$ by :

$$\begin{cases} K_{ii} = A_{ii}, & 1 \leqq i \leqq N, \\ K_{ii} = A_{ii} + D_i, & N + 1 \leqq i \leqq N + M, \\ K_{ij} = A_{ij}, & i \neq j, \quad 1 \leqq i, j \leqq N + M, \end{cases}$$

where :

$$D_i = - [\tilde{g}_u(x, \tilde{v}_h) \overline{\phi}_{h,i}, \overline{\phi}_{h,i}]_{\Gamma_h}.$$

Then (3.1) may be written as :

$$\mathbf{K} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_{N+M} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3.2)$$

From Assumptions 1 and 3, $\mathbf{K}$ is irreducibly diagonally dominant with $K_{ii} > 0$, $K_{ij} \leqq 0$, $i \neq j$, $1 \leqq i, j \leqq N + M$, since $D_i > 0$. Thus, Lemma 2 leads to

$\mathbf{K}^{-1} > 0$. From (3.2), we obtain :

$$\zeta_1 = \cdots = \zeta_{N+M} = 0,$$

from which follows that :

$$u_h \equiv w_h .$$

Hence, the proof is complete. ■

THEOREM 2 : *Under Assumptions* 1 *and* 3, *the iteration* (2.2) *satisfies* :

$$u_{h,k} > 0, \quad k = 0, 1, 2, \ldots,$$
$$u_{h,1} \geqq u_{h,2} \geqq \cdots \geqq u_{h,k-1} \geqq u_{h,k} \geqq u_{h,k+1} \geqq \cdots,$$

and $u_h \equiv \lim_{k\to\infty} u_{h,k}$ *is a unique positive solution of* (2.1).

Proof : It is clear that $u_{h,0} \equiv \delta > 0$. Assume that :

$$u_{h,k-1} = \sum_{j=1}^{N+M} \xi_{j,k-1}\, \phi_{h,j} > 0 .$$

By Assumption 1, (2.6) and the result of [6, p. 507], we can obtain :

$$s_i^{(k-1)} = [\tilde{g}(x, \tilde{u}_{h,k-1}) - \tilde{g}_u(x, \tilde{u}_{h,k-1})\, \tilde{u}_{h,k-1}, \overline{\phi}_{h,i}]_{\Gamma_h} > 0, \quad N+1 \leqq i \leqq N+M. \tag{3.3}$$

Define the matrix $\mathbf{K} = (K_{ij})$, $1 \leqq i, j \leqq N + M$ by :

$$\begin{cases} K_{ii} = A_{ii}, & 1 \leqq i \leqq N, \\ K_{ii} = A_{ii} + D_i^{(k-1)}, & N+1 \leqq i \leqq N+M \\ K_{ij} = A_{ij}, & i \neq j, \quad 1 \leqq i, j \leqq N+M. \end{cases}$$

Then, under Assumptions 1 and 3, $\mathbf{K}$ is irreducibly diagonally dominant with $K_{ii} > 0$, $K_{ij} \leqq 0$, $i \neq j$, $1 \leqq i, j \leqq N + M$, since $D_i^{(k-1)} > 0$. An application of Lemma 2 leads to $\mathbf{K}^{-1} > 0$. From (2.4) and (3.3) we have

$$\begin{pmatrix} \xi_{1,k} \\ \vdots \\ \xi_{N+M,k} \end{pmatrix} = \mathbf{K}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ s_{N+1}^{(k-1)} \\ \vdots \\ s_{N+M}^{(k-1)} \end{pmatrix} > 0 .$$

Therefore, by mathematical induction, we get :

$$u_{h,k} = \sum_{j=1}^{N+M} \xi_{j,k} \, \phi_{h,j} > 0 , \quad k = 0, 1, 2, \dots . \tag{3.4}$$

On the other hand, from (2.2) it follows that :

$$\begin{aligned} \mathscr{A}_h(u_{h,k} - u_{h,k-1} ; \psi_h) &+ (\tilde{a}_0(x)(\tilde{u}_{h,k} - \tilde{u}_{h,k-1}), \tilde{\psi}_h)_{\Omega_h} \\ &- [\tilde{g}_u(x, \tilde{u}_{h,k-1})(\tilde{u}_{h,k} - \tilde{u}_{h,k-1}), \tilde{\psi}_h]_{\Gamma_h} \\ = [\tilde{g}(x, \tilde{u}_{h,k-1}) &- \tilde{g}(x, \tilde{u}_{h,k-2}) - \tilde{g}_u(x, \tilde{u}_{h,k-2})(\tilde{u}_{h,k-1} - \tilde{u}_{h,k-2}), \tilde{\psi}_h]_{\Gamma_h} \\ &\text{for all} \quad \psi_h \in Y^h , \quad k = 2, 3, \dots . \end{aligned} \tag{3.5}$$

The strict concavity property (*d*) in Assumption 1 implies that :

$$\tilde{g}(x, \tilde{u}_{h,k-1}) \leqq \tilde{g}(x, \tilde{u}_{h,k-2}) + \tilde{g}_u(x, \tilde{u}_{h,k-2})(\tilde{u}_{h,k-1} - \tilde{u}_{h,k-2}) , \quad k = 2, 3, \dots .$$

Since the coefficient matrix corresponding to $u_{h,k} - u_{h,k-1}$ in (3.5) is $\mathbf{K}$, we have $u_{h,k} - u_{h,k-1} \leqq 0$, $k = 2, 3, 4, \dots$, from which follows that :

$$u_{h,1} \geqq u_{h,2} \geqq \cdots \geqq u_{h,k-1} \geqq u_{h,k} \geqq u_{h,k+1} \geqq \cdots . \tag{3.6}$$

Also (3.4) and (3.6) imply that there exists a limit function :

$$u_h \equiv \lim_{k \to \infty} u_{h,k} = \sum_{j=1}^{N+M} \xi_j \, \phi_{h,j} \geqq 0 , \quad \xi_j \geqq 0 , \quad 1 \leqq i \leqq N + M ,$$

and from (2.2), its limit function u_h satisfies (2.1). This implies that $u_h \geqq 0$ is a solution of (2.1).

Next, we show that $u_h > 0$, i.e., $\xi_j > 0$, $1 \leqq j \leqq N + M$. Assume that there exists some r ($1 \leqq r \leqq N + M$) such that $\xi_r = 0$. Since $\mathbf{A}$ is irreducible, for any j ($j \neq r$, $1 \leqq j \leqq N + M$) there exists a sequence of nonzero elements of $\mathbf{A}$ of the form $\{ A_{ri_1}, A_{i_1 i_2}, \dots, A_{i_t j} \}$, from Lemma 1. From (2.3) and Assumption 1, it follows that :

$$\sum_{j=1}^{N+M} A_{rj}\, \xi_j = \begin{cases} 0\,, & \text{if} \quad 1 \leqq r \leqq N\,, \\ [g(P_r, 0), \overline{\phi}_{h,r}]_{\Gamma_h} > 0\,, & \text{if} \quad N+1 \leqq r \leqq N+M\,. \end{cases}$$

This implies that :

$$A_{rr}\, \xi_r + A_{ri_1}\, \xi_{i_1} + \sum_{\substack{j=1 \\ j \neq r \\ j \neq i_1}}^{N+M} A_{rj}\, \xi_j \geqq 0\,, \quad 1 \leqq r \leqq N+M\,.$$

Since $\xi_r = 0$ and $A_{ri_1} < 0$, we obtain :

$$\xi_{i_1} = 0\,.$$

The same arguments yield :

$$\xi_{i_2} = \cdots = \xi_{i_t} = \xi_j = 0\,.$$

Since j is arbitrary, we have $u_h \equiv 0$. However, $u_h \equiv 0$ can not be a solution of (2.1). This is a contradiction. Hence, we have $u_h > 0$. By using Theorem 1, we conclude that $u_h > 0$ is a unique solution of (2.1). This completes the proof. ∎

We are now in a position to prove the following theorem concerning the error estimate between u and u_h.

THEOREM 3 : *Let Assumptions* 1, 2 *and* 3 *hold. Let u and u_h be the positive solutions of* (1.1) *and* (2.1), *respectively. Then, there exists a positive constant C independent of h and u such that*

$$\| u - u_h \|_{L^\infty(\Omega_h)} \leqq Ch \| u \|_{W^{2,p}(\Omega)}\,,$$

provided that $u \in W^{2,p}(\Omega)$, $p > n \geqq 2$.

Proof : Integration of (1.1) over Ω_h leads to :

$$\mathscr{A}_h(u\,; \psi_h) + (a_0(x)\, u, \psi_h)_{\Omega_h} = \left[\frac{\partial u}{\partial \nu_h}, \psi_h\right]_{\Gamma_h} \quad \text{for all} \quad \psi_h \in Y^h\,, \qquad (3.7)$$

where $\partial/\partial\nu_h$ denotes the conormal derivative on Γ_h. Given $\alpha \geqq 0$, let :

$$I_h\, u = \sum_{i=1}^{N+M} u(P_i)\, \phi_{h,i}\,,$$

$$w_h = u_h - I_h\, u\,,$$

$$w_{h,\alpha}(P_i) = \begin{cases} w_h(P_i) - \alpha & \text{if} \quad w_h(P_i) > \alpha\,, \\ 0 & \text{if} \quad w_h(P_i) \leqq \alpha\,, \end{cases}$$

for $1 \leqq i \leqq N + M$. From (2.1) and (3.7), we have :

$$\begin{aligned} \mathscr{A}_h(u_h - I_h\,u\,; w_{h,\alpha}) &+ (\tilde{a}_0(x)\,(\tilde{u}_h - \widetilde{I_h\,u}),\, \tilde{w}_{h,\alpha})_{\Omega_h} \\ &- [\tilde{g}(x, \tilde{u}_h) - \tilde{g}(x, \widetilde{I_h\,u}),\, \tilde{w}_{h,\alpha}]_{\Gamma_h} \\ &= \mathscr{A}_h(w_h\,; w_{h,\alpha}) + (\tilde{a}_0(x)\,\tilde{w}_h, \tilde{w}_{h,\alpha})_{\Omega_h} - [\tilde{g}_u(x, \tilde{v}_h)\,\tilde{w}_h, \tilde{w}_{h,\alpha}]_{\Gamma_h} \\ &= \mathscr{A}_h(u - I_h\,u\,; w_{h,\alpha}) + (a_0(x)\,u, w_{h,\alpha})_{\Omega_h} - (\tilde{a}_0(x)\,\widetilde{I_h\,u}, \tilde{w}_{h,\alpha})_{\Omega_h} \\ &\quad - \left[\frac{\partial u}{\partial \nu_h}, w_{h,\alpha}\right]_{\Gamma_h} + [\tilde{g}(x, \widetilde{I_h\,u}), \tilde{w}_{h,\alpha}]_{\Gamma_h}\,, \end{aligned} \tag{3.8}$$

where

$$v_h = \varepsilon(x)\,u_h + (1 - \varepsilon(x))\,I_h\,u > 0\,, \quad \text{with} \quad 0 < \varepsilon(x) < 1\,.$$

Define p' by :

$$\frac{1}{p} + \frac{1}{p'} = 1\,.$$

We note that $p > n \geqq 2$, $1 < p' < 2$. From Lemmas 3, 4, Assumptions 1, 2, Remark 1 and Hölder's inequality, the right hand side of (3.8) is estimated as follows :

$$\begin{aligned} &\Big| \mathscr{A}_h(u - I_h\,u\,; w_{h,\alpha}) + (a_0(x)\,u, w_{h,\alpha})_{\Omega_h} - (\tilde{a}_0(x)\,\widetilde{I_h\,u}, \tilde{w}_{h,\alpha})_{\Omega_h} \\ &\qquad - \left[\frac{\partial u}{\partial \nu_h}, w_{h,\alpha}\right]_{\Gamma_h} + [\tilde{g}(x, \widetilde{I_h\,u}), \tilde{w}_{h,\alpha}]_{\Gamma_h} \Big| \\ &= \Big| \mathscr{A}_h(u - I_h\,u\,; w_{h,\alpha}) + (a_0(x)\,u - \tilde{a}_0(x)\,\widetilde{I_h\,u}, w_{h,\alpha})_{\Omega_h} \\ &\quad + (\tilde{a}_0(x)\,\widetilde{I_h\,u}, w_{h,\alpha} - \tilde{w}_{h,\alpha})_{\Omega_h} + \left[g(x, u) - \frac{\partial u}{\partial \nu_h}, w_{h,\alpha}\right]_{\Gamma_h} \\ &\quad + [\tilde{g}(x, \widetilde{I_h\,u}) - g(x, u), w_{h,\alpha}]_{\Gamma_h} + [\tilde{g}(x, \widetilde{I_h\,u}), \tilde{w}_{h,\alpha} - w_{h,\alpha}]_{\Gamma_h} \Big| \\ &\leqq C_1\,h\,\|\,u\,\|_{W^{2,p}(\Omega)}\,\|\,w_{h,\alpha}\,\|_{W^{1,p'}(\Omega_h)}\,, \end{aligned} \tag{3.9}$$

where C_1 is a positive constant independent of h.

Therefore, by using (3.8) and (3.9) and by the same arguments as used in Tabata [20, p. 348] and Ciarlet-Raviart [5], with the aid of (1.2), (1.3), and Assumption 1, we can obtain :

$$\|\,u_h - I_h\,u\,\|_{L^\infty(\Omega_h)} \leqq C_2\,h\,\|\,u\,\|_{W^{2,p}(\Omega)}\,, \tag{3.10}$$

where C_2 is a positive constant independent of h and u. From Lemma 3 and (3.10) we have the desired estimate :

$$\| u - u_h \|_{L^\infty(\Omega_h)} \leqq \| u - I_h u \|_{L^\infty(\Omega_h)} + \| I_h u - u_h \|_{L^\infty(\Omega_h)}$$
$$\leqq Ch \| u \|_{W^{2,p}(\Omega)}, \quad p > n \geqq 2,$$

where C is a positive constant independent of h and u. Therefore, the proof is complete. ■

Remark 4 : By extending u_h to $\Omega \setminus \Omega_h$ appropriately, we obtain the following error estimate :

$$\| u - u_h \|_{L^\infty(\Omega)} \leqq Ch \| u \|_{W^{2,p}(\Omega)}, \quad p > n \geqq 2.$$

For example, as one of the methods of extension, we refer to [20, Remark 1.2].

4. COMPUTATIONAL EXAMPLE

In this section, we present some numerical results in the two dimensional case to indicate the usefulness of the convergence results obtained in the preceding section. Let Ω be the interior of the unit circle defined by

$$\Omega = \{ (x_1, x_2) \in \mathbb{R}^2 ; \; x_1^2 + x_2^2 < 1 \}.$$

The boundary of Ω is denoted by Γ. We deal with the following problem.

Problem :

$$\begin{cases} - \Delta u + \dfrac{4}{x_1^2 + x_2^2 + 1} u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{8}{15}(1 - u^4), & \text{on } \Gamma. \end{cases}$$

In this case, $\partial/\partial\nu$ denotes the outward normal derivative on Γ, and Assumption 1 is satisfied. Hence, the unique positive solution for Problem is given by

$$u(x_1, x_2) = \frac{1}{4}(x_1^2 + x_2^2 + 1).$$

We triangulate Ω as shown in figures 2, 3 and 4 (7, 25 and 79 nodes). Thus, Assumptions 2 and 3 are satisfied, so that Theorems 1, 2 and 3 hold. For the initial data of the iterative method, we take $u_{h,0} \equiv 1.0$. The numerical convergence criterion for the iteration is employed as follows

$$\max_{1 \leq i \leq N+M} \left| \frac{u_{h,k}(P_i) - u_{h,k-1}(P_i)}{u_{h,k}(P_i)} \right| < 10^{-15}$$

In Table 1, we show the monotone convergence results for the iterative method. Table 2 also gives the finite element solutions, which indicate the convergence to the exact solution as h tends to zero. Thus, we can see that these numerical results demonstrate the validity of our theoretical results.

All computations were performed on the FACOM M-382 computer at Kyushu University by using double-precision arithmetic which carries about 15 significant digits. All data in Tables 1 and 2 were rounded to 5 digits.

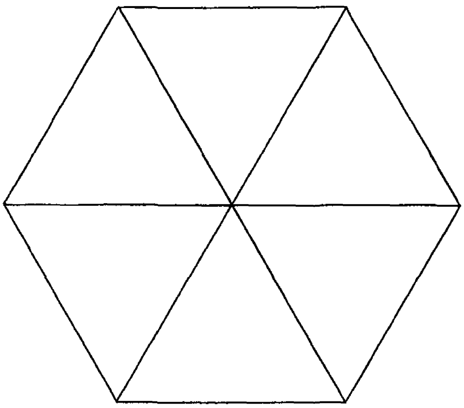

Figure 2. — 7 nodes ($h = 1$).

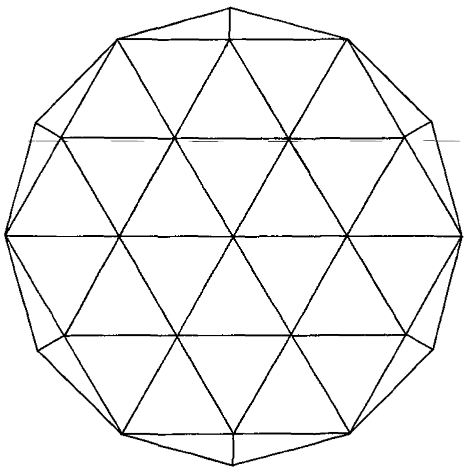

Figure 3. — 25 nodes $\left(h = \frac{\sqrt{6} - \sqrt{2}}{2}\right)$.

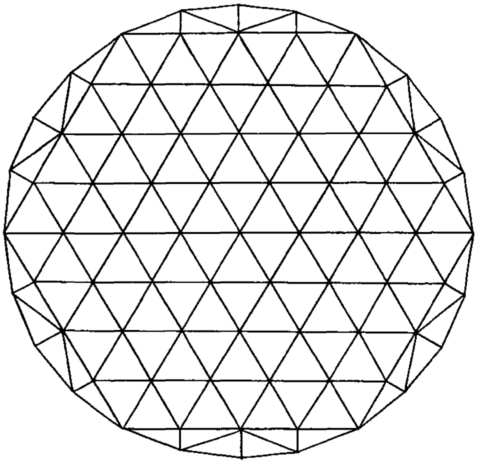

Figure 4. — 79 nodes $\left(h = \frac{3\sqrt{2} - \sqrt{10}}{4}\right)$.

TABLE 1
Monotone convergence for Problem ($u_{h,0} \equiv 1.0$)

k	$u_{h,k}(0,0)$			$u_{h,k}(1,0)$		
	7 nodes	25 nodes	79 nodes	7 nodes	25 nodes	79 nodes
1	0.35563	0.34489	0.34157	0.71126	0.68658	0.68193
2	0.28856	0.26884	0.26380	0.57725	0.53480	0.52647
3	0.27852	0.25726	0.25196	0.55703	0.51175	0.50285
4	0.27835	0.25707	0.25177	0.55669	0.51137	0.50247
5	0.27835	0.25707	0.25177	0.55669	0.51137	0.50247
6	0.27835	0.25707	0.25177	0.55669	0.51137	0.50247
7	0.27835	0.25707	0.25177	0.55669	0.51137	0.50247

TABLE 2
Finite element solutions for Problem

h	Number of nodes	$u_h(0,0)$	$u_h(1,0)$	Number of iterations for convergence
1.0000	7	0.27835	0.55669	7
$\frac{\sqrt{6}-\sqrt{2}}{2} \doteqdot 0.51764$	25	0.25707	0.51137	7
$\frac{3\sqrt{2}-\sqrt{10}}{4} \doteqdot 0.27009$	79	0.25177	0.50247	7
Exact		0.25000	0.50000	

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