

FRANZ-JÜRGEN DELVOS

WALTER SCHEMPP

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*Modélisation mathématique et analyse numérique*, tome 21, n° 2  
(1987), p. 261-267

[http://www.numdam.org/item?id=M2AN\\_1987\\_\\_21\\_2\\_261\\_0](http://www.numdam.org/item?id=M2AN_1987__21_2_261_0)

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## CONVERGENCE OF APPROXIMATE SPLINES VIA PSEUDO-INVERSES (\*)

by Franz-Jürgen DELVOS <sup>(1)</sup> and Walter SCHEMPP <sup>(1)</sup>

Communicated by P. J. LAURENT

Dedicated to the memory of Professor Dr. Ulrich Tippenhauer

*Abstract.* — *The notion of pseudo-inverse has been proved to be an efficient tool in the functional analytic theory of splines [1, 2, 3, 4, 5, 6, 7, 9]. In this paper we apply the concept of pseudo-inverse to define abstract approximate interpolating splines and extend thereby the polynomial approximate splines introduced by I. J. Schoenberg [8]. Moreover, we establish a limit characterization for abstract interpolating splines which generalizes a corresponding result of Schoenberg for natural polynomial splines.*

*Résumé.* — *La notion de pseudo-inverse a montré son efficacité comme outil dans la théorie fonctionnelle analytique des splines [1, 2, 3, 4, 5, 6, 7, 9]. Dans cet article nous appliquons le concept de pseudo-inverse pour définir un concept abstrait de splines d'interpolation et étendons de cette façon les splines polynomiaux approchés introduits par I. J. Schoenberg [8]. En outre, nous établissons une caractérisation limite pour des splines abstraites d'interpolation qui généralise le résultat correspondant de Schoenberg pour les splines polynomiaux naturels.*

### INTRODUCTION

Let  $H$ ,  $H_1$ ,  $H_2$  be real or complex Hilbert spaces and

$$T : H \rightarrow H_1, \quad S : H \rightarrow H_2$$

be surjective bounded linear operators. Let  $N(T)$  and  $N(S)$  be the kernels of  $T$  and  $S$ . If the uniqueness condition

$$N(T) \cap N(S) = \{0\}$$

is valid and  $T(N(S))$  is closed there is a unique abstract interpolating spline  $\sigma$  in  $H$  associated with  $y \in H_2$  satisfying

$$S\sigma = y, \quad \|T\sigma\| = \min \{ \|Tx\| : x \in h, Sx = y \} \quad [7].$$

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(\*) Received in February 1986.

(<sup>1</sup>) Lehrstuhl für Mathematik I, Universität Siegen, D-5900 Siegen, R.F.A.

Suppose that  $U_m$ ,  $m \in \mathbb{N}$  is an increasing sequence of closed linear subspaces of  $H$  satisfying

$$S(U_m) = H_2, \quad m \in \mathbb{N}.$$

Using the calculus of operator pseudo-inverse we will show that for any  $y \in H_2$  there is a unique  $\sigma_m \in U_m$  satisfying

$$S\sigma_m = y,$$

$$\|T\sigma_m\| = \min \{ \|Tu\| : u \in U_m, Su = y \}.$$

The element  $\sigma_m$  will be called a  $U_m$ -approximate spline (relative to  $T, S, y$ ). For the special case of natural polynomial splines ( $T = D^n$ ,  $S = \delta_{x_1} \times \dots \times \delta_{x_k}$ ,  $n \leq k$ ) Schoenberg [8] proved the existence of  $\pi_m$ -approximate splines for any  $m \geq k$  where  $\pi_m$  is the linear space of polynomials of degree less or equal  $m$ . He derived an interesting limit relation for the natural polynomial splines via the  $\pi_m$ -approximate splines. We will show that a corresponding limit relation remains valid for abstract interpolating splines. As an application we will derive a limit characterization of periodic polynomial splines by trigonometric approximate splines.

## 1. APPROXIMATE SPLINES

First we will recall the basic properties of the pseudo-inverse of a bounded linear operator  $A$  from a Hilbert space  $H$  into a Hilbert space  $G$  such that the range  $R(A)$  of  $A$  is closed.

The pseudo-inverse  $A^+$  of  $A$  is the unique bounded linear operator from  $G$  into  $H$  satisfying the Moore-Penrose equations

$$\begin{aligned} AA^+A &= A, & A^+AA^+ &= A^+ \\ (AA^+)^* &= AA^+, & (A^+A)^* &= A^+A. \end{aligned}$$

Thus,  $AA^+$  is the orthogonal projector on  $R(A)$  and  $A^+A$  is the orthogonal projector on the orthogonal complement of  $N(A)$ :

$$AA^+ = P_{R(A)}, \quad A^+A = P_{N(A)^\perp}.$$

If  $A$  is surjective the pseudo-inverse of  $A$  is given by

$$A^+ = A^*(AA^*)^{-1}.$$

It was shown in [7] that  $H$  is a Hilbert space with respect to the Golomb-Weinberger-Sard scalar product associated with  $T, S$ :

$$(x, y) = (Tx, Ty) + (Sx, Sy) \quad (x, y \in H).$$

We denote by  $\|x\| = (x, x)^{1/2}$  the associated norm which is equivalent to the original norm of  $H$  [7]. We assume for the sequel that  $H$  carries the

Golomb-Weinberger-Sard scalar product. Let  $P_m$  denote the unique orthogonal projector in  $H$  with range  $U_m$ :

$$R(P_m) = U_m .$$

THEOREM 1 : Suppose  $y \in H_2$ . Then

$$\sigma_m = (SP_m)^+ y$$

is the  $U_m$ -approximate spline relative to  $y, T, S$ , i.e.,  $\sigma_m$  is the unique element in  $U_m$  satisfying

$$S\sigma_m = y ,$$

$$\|T\sigma_m\| = \min \{ \|Tu\| : u \in U_m, Su = y \} .$$

Proof: Since  $SP_m$  is surjective we have

$$(SP_m)^+ = P_m S^*(SP_m S^*)^{-1} \tag{1}$$

which implies

$$\sigma_m \in U_m .$$

Taking into account

$$(SP_m)(SP_m)^+ = SS^+ = \text{id}_{H_2} \tag{2}$$

we get

$$S\sigma_m = S(SP_m)^+ y = SP_m(SP_m)^+ y ,$$

i.e.,

$$S\sigma_m = y .$$

Next assume

$$u \in U_m , \quad Su = y .$$

Then

$$u - \sigma_m \in N(S)$$

and we obtain by using the Moore-Penrose equations and the structure of the Golomb-Weinberger-Sard scalar product

$$\begin{aligned} \|Tu\|^2 &= \|T\sigma_m\|^2 + \|T(u - \sigma_m)\|^2 + (T\sigma_m, T(u - \sigma_m)) + (T(u - \sigma_m), T\sigma_m) \\ &= \|T\sigma_m\|^2 + \|u - \sigma_m\|^2 + (\sigma_m, u - \sigma_m) + (u - \sigma_m, \sigma_m) \\ &= \|T\sigma_m\|^2 + \|u - \sigma_m\|^2 \\ &\quad + ((SP_m)^+ SP_m \sigma_m, u - \sigma_m) + (u - \sigma_m, (SP_m)^+ SP_m \sigma_m) \\ &= \|T\sigma_m\|^2 + \|u - \sigma_m\|^2 \\ &\quad + (\sigma_m, (SP_m)^+ S(u - \sigma_m)) + ((SP_m)^+ S(u - \sigma_m), \sigma_m) , \end{aligned}$$

i.e., we have shown

$$\|Tu\|^2 = \|T\sigma_m\|^2 + \|u - \sigma_m\|^2.$$

This completes the proof of theorem 1.

*Remark* : Recall that the operator  $(SP_m)^+$  is the restricted pseudo-inverse of  $S$  with respect to  $U_m$  [2, 3].

Schoenberg's polynomial approximate splines are obtained by choosing

$$\begin{aligned} H_1 &= L_2[a, b], \quad H_2 = \mathbb{C}^k, \quad H = W_2^n[a, b], \\ T &= d^n/dx^n, \\ S &= \delta_{x_1} \times \dots \times \delta_{x_k} \quad (a \leq x_1 < x_2 < \dots < x_k \leq b, k > n), \\ U_m &= \pi_m \quad (m \geq k). \end{aligned}$$

Here  $W_2^n[a, b]$  denotes the Sobolev space of order  $n$ .

Our next example is associated with periodic polynomial splines of degree  $2n - 1$  which are obtained by specifying

$$\begin{aligned} H_1 &= L_2[0, 2\pi], \quad H_2 = \mathbb{C}^{2k+1}, \quad H = W_{2\pi}^n, \\ T &= d^n/dx^n, \\ S &= \delta_{x_1} \times \dots \times \delta_{x_{2k+1}} \\ &\quad (0 \leq x_1 < x_2 < \dots < x_{2k+1} < 2\pi, k \geq 0). \end{aligned}$$

Here  $W_{2\pi}^n$  denotes the periodic Sobolev space of order  $n$  :

$$W_{2\pi}^n = \{f \in W_2^n[0, 2\pi] : D^j f(0) = D^j f(2\pi) \quad (j = 0, \dots, n - 1)\}.$$

Let

$$U_m = \tau_m \quad (m \geq k)$$

denote the class of trigonometric polynomials of degree  $m$ . It is a well known result that for any  $(y_1, \dots, y_{2k+1}) \in \mathbb{C}^{2k+1}$  there is a unique trigonometric polynomial  $T_k \in \tau_k$  such that

$$T_k(x_j) = y_j \quad (j = 1, \dots, 2k + 1).$$

Thus, theorem 1 is applicable and we obtain the following

**COROLLARY 1.1** : *For any vector  $(y_1, \dots, y_{2k+1})$  there is a unique trigonometric polynomial  $T_m \in \tau_m$  satisfying*

$$\begin{aligned} T_m(x_j) &= y_j \quad (j = 1, \dots, 2k + 1), \\ \int_0^{2\pi} |D^n T_m(x)|^2 dx &\leq \int_0^{2\pi} |D^n T(x)|^2 dx \\ &\quad (T \in \tau_m, T(x_j) = y_j \quad (j = 1, \dots, 2k + 1)). \end{aligned}$$

2. THE LIMIT RELATION

Our next objective is to derive a limit characterization of the abstract interpolation spline  $\sigma = S^+ y$  in terms of the approximate splines  $\sigma_m = (SP_m)^+ y$ . For this purpose we need the following extremal property of  $\sigma_m$ .

**THEOREM 2:** *The approximate spline  $\sigma_m$  is the unique element in  $U_m$  satisfying*

$$S\sigma_m = y ,$$

$$\|\sigma - \sigma_m\| = \min \{ \|\sigma - u\| : u \in U_m, Su = y \} .$$

*Proof:* We proceed as in the proof of Theorem 1. Suppose that

$$u \in U_m , \quad Su = S\sigma = y .$$

Then

$$\sigma - u \in N(S)$$

and taking into account

$$\sigma \in N(S)^\perp$$

we get

$$\begin{aligned} \|Tu\|^2 &= \|T\sigma\|^2 + \|T(u - \sigma)\|^2 \\ &\quad + (T\sigma, T(u - \sigma)) + (T(u - \sigma), T\sigma) \\ &= \|T\sigma\|^2 + \|u - \sigma\|^2 + (\sigma, u - \sigma) + (u - \sigma, \sigma) , \end{aligned}$$

i.e., we have

$$\|Tu\|^2 = \|T\sigma\|^2 + \|\sigma - u\|^2 \quad (u \in U_m, Su = y) .$$

Now an application of Theorem 1 completes the proof of Theorem 2.

For the sequel we will assume that the union of the sets

$$U_m , \quad m \in \mathbb{N} ,$$

is a dense subspace of  $H$ . Then we have the limit relation

$$\lim_{m \rightarrow \infty} \|x - P_m x\| = 0 \quad (x \in H) . \tag{3}$$

**THEOREM 3 :** *Let  $\sigma$  be the abstract interpolating spline relative to  $T, S$ , and  $y$  and let  $\sigma_m$  be the  $U_m$ -approximate spline relative to  $T, S, y$ . Then the error estimate*

$$\|\sigma - \sigma_m\| \leq (1 + \|(SP_1)^+ S\|) \|\sigma - P_m \sigma\| \tag{4}$$

is valid. In particular, we have

$$\lim_{m \rightarrow \infty} \|\sigma - \sigma_m\| = 0. \quad (5)$$

*Proof:* Since  $U_m$ ,  $m \in \mathbb{N}$ , is an increasing sequence of subspaces we have in view of (1)

$$v_m = P_m \sigma + (SP_1)^+ S(\sigma - P_m \sigma) \in U_m.$$

Moreover, we obtain from (2)

$$\begin{aligned} Sv_m &= SP_m \sigma + S(SP_1)^+ S(\sigma - P_m \sigma) \\ &= SP_m \sigma + SP_1(SP_1)^+ S(\sigma - P_m \sigma) \\ &= SP_m \sigma + S(\sigma - P_m \sigma), \end{aligned}$$

i.e., we have

$$Sv_m = y, \quad v_m \in U_m.$$

Now an application of theorem 2 yields

$$\begin{aligned} \|\sigma - \sigma_m\| &\cong \|\sigma - v_m\| \\ &\cong \|\sigma - P_m \sigma\| + \|(SP_1)^+ S(\sigma - P_m \sigma)\|, \end{aligned}$$

i.e., we have

$$\|\sigma - \sigma_m\| \cong (1 + \|(SP_1)^+ S\|) \|\sigma - P_m \sigma\|.$$

Thus, Theorem 3 follows from the relation (3).

The relation (5) represents a limit characterization of the abstract interpolating spline  $\sigma$  by the approximate splines  $\sigma_m$ ,  $m \in \mathbb{N}$ . For the special case of natural polynomial splines we obtain Schoenberg's limit characterization of the natural spline  $\sigma$  via the  $\pi_m$ -approximate splines  $\sigma_m$ ,  $m \in \mathbb{N}$  [8]. It should be noted that Theorem 3 yields an explicit error estimate. For further applications of pseudo-inverses to convergence of abstract splines we refer to [6].

We conclude with applying Theorem 3 to periodic polynomial splines.

**COROLLARY 3.1:** *The sequence  $T_m$ ,  $m \in \mathbb{N}$ , of trigonometric approximate splines converges in the Sobolev norm of  $W_{2\pi}^n$  to the periodic polynomial spline  $\sigma$  of degree  $2n - 1$ .*

*Proof:* Since the trigonometric polynomials form a dense subspace of  $W_{2\pi}^n$  Theorem 3 is applicable.

## REFERENCES

- [1] R. E. CHANG, *The generalized inverse and interpolation theory*, in « Recent Applications of Generalized inverses » (S. L. Campbell, Ed.), Pitman, New York, 1983.
- [2] F.-J. DELVOS, *Splines and pseudo-inverses*, RAIRO Anal. Numer. 12 (1978), 313-324.
- [3] F.-J. DELVOS, *Pseudoinversen und Splines in Hilberträumen*, Habilitationsschrift, Universität Siegen 1979.
- [4] F.-J. DELVOS and W. SCHEMPP, *Optimal approximation and the method of least squares*, J. Approx. Theory 33 (1981), 214-223.
- [5] C. W. GROETSCH, *Generalized inverses and generalized splines*, Numer. Funct. Optim. 2 (1980), 93-97.
- [6] S. IZUMINO, *Convergence of generalized inverses and spline projectors*, J. Approx. Theory 38 (1983), 269-278.
- [7] P. LAURENT, *Approximation et optimisation*, Paris, Herrmann 1972.
- [8] I. J. SCHOENBERG, *Splines as limits of polynomials*, Linear Algebra and its Applications 52/53 (1983), 617-628.
- [9] M. TASCHE, *A unified approach to interpolation methods*, J. Integral Equations 4 (1982), 55-75.