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CONVERGENCE OF APPROXIMATE SPLINES *VIA* PSEUDO-INVERSES (*)

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Dedicated to the memory of Professor Dr. Ulrich Tippenhauer

Abstract. — *The notion of pseudo-inverse has been proved to be an efficient tool in the functional analytic theory of splines* [1, 2, 3, 4, 5, 6, 7, 9]. *In this paper we apply the concept of pseudo-inverse to define abstract approximate interpolating splines and extend thereby the polynomial approximate splines introduced by I. J. Schoenberg* [8]. *Moreover, we establish a limit characterization for abstract interpolating splines which generalizes a corresponding result of Schoenberg for natural polynomial splines.*

Résumé. — *La notion de pseudo-inverse a montré son efficacité comme outil dans la théorie fonctionnelle analytique des splines* [1, 2, 3, 4, 5, 6, 7, 9]. *Dans cet article nous appliquons le concept de pseudo-inverse pour définir un concept abstrait de splines d'interpolation et étendons de cette façon les splines polynomiaux approchés introduits par I. J. Schoenberg* [8]. *En outre, nous établissons une caractérisation limite pour des splines abstraites d'interpolation qui généralise le résultat correspondant de Schoenberg pour les splines polynomiaux naturels.*

INTRODUCTION

Let H, H_1, H_2 be real or complex Hilbert spaces and

$$T : H \to H_1 , \qquad S : H \to H_2$$

be surjective bounded linear operators. Let $N(T)$ and $N(S)$ be the kernels of T and S. If the uniqueness condition

$$N(T) \cap N(S) = \{0\}$$

is valid and $T(N(S))$ is closed there is a unique abstract interpolating spline σ in H associated with $y \in H_2$ satisfying

$$S\sigma = y , \quad \| T\sigma \| = \min \{ \| Tx \| : x \in h, Sx = y \} \quad [7] .$$

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Suppose that U_m, $m \in \mathbb{N}$ is an increasing sequence of closed linear subspaces of H satisfying

$$S(U_m) = H_2 , \qquad m \in \mathbb{N} .$$

Using the calculus of operator pseudo-inverse we will show that for any $y \in H_2$ there is a unique $\sigma_m \in U_m$ satisfying

$$S\sigma_m = y ,$$
$$\| T\sigma_m \| = \min \{ \| Tu \| : u \in U_m, Su = y \} .$$

The element σ_m will be called a U_m-approximate spline (relative to T, S, y). For the special case of natural polynomial splines ($T = D^n$, $S = \delta_{x_1} \times \cdots \times \delta_{x_k}$, $n \leq k$) Schoenberg [8] proved the existence of π_m-approximate splines for any $m \geqq k$ where π_m is the linear space of polynomials of degree less or equal m. He derived an interesting limit relation for the natural polynomial splines *via* the π_m-approximate splines. We will show that a corresponding limit relation remains valid for abstract interpolating splines. As an application we will derive a limit characterization of periodic polynomial splines by trigonometric approximate splines.

1. APPROXIMATE SPLINES

First we will recall the basic properties of the pseudo-inverse of a bounded linear operator A from a Hilbert space H into a Hilbert space G such that the range $R(A)$ of A is closed.

The pseudo-inverse A^+ of A is the unique bounded linear operator from G into H satisfying the Moore-Penrose equations

$$AA^+ A = A , \qquad A^+ AA^+ = A^+$$
$$(AA^+)^* = AA^+ , \qquad (A^+ A)^* = A^+ A .$$

Thus, AA^+ is the orthogonal projector on $R(A)$ and $A^+ A$ is the orthogonal projector on the orthogonal complement of $N(A)$:

$$AA^+ = P_{R(A)} , \qquad A^+ A = P_{N(A)^\perp} .$$

If A is surjective the pseudo-inverse of A is given by

$$A^+ = A^*(AA^*)^{-1} .$$

It was shown in [7] that H is a Hilbert space with respect to the Golomb-Weinberger-Sard scalar product associated with T, S:

$$(x, y) = (Tx, Ty) + (Sx, Sy) \qquad (x, y \in H) .$$

We denote by $\|x\| = (x, x)^{1/2}$ the associated norm which is equivalent to the original norm of H [7]. We assume for the sequel that H carries the

Golomb-Weinberger-Sard scalar product. Let P_m denote the unique orthogonal projector in H with range U_m:

$$R(P_m) = U_m .$$

THEOREM 1 : *Suppose* $y \in H_2$. *Then*

$$\sigma_m = (SP_m)^+ y$$

is the U_m*-approximate spline relative to* y, T, S, i.e., σ_m *is the unique element in* U_m *satisfying*

$$S\sigma_m = y ,$$

$$\| T\sigma_m \| = \min \{ \| Tu \| : u \in U_m, Su = y \} .$$

Proof: Since SP_m is surjective we have

$$(SP_m)^+ = P_m S^* (SP_m S^*)^{-1} \tag{1}$$

which implies

$$\sigma_m \in U_m .$$

Taking into account

$$(SP_m)(SP_m)^+ = SS^+ = \mathrm{id}_{H_2} \tag{2}$$

we get

$$S\sigma_m = S(SP_m)^+ y = SP_m(SP_m)^+ y ,$$

i.e.,

$$S\sigma_m = y .$$

Next assume

$$u \in U_m , \qquad Su = y .$$

Then

$$u - \sigma_m \in N(S)$$

and we obtain by using the Moore-Penrose equations and the structure of the Golomb-Weinberger-Sard scalar product

$$\begin{aligned} \| Tu \|^2 &= \| T\sigma_m \|^2 + \| T(u - \sigma_m) \|^2 + (T\sigma_m, T(u - \sigma_m)) + (T(u - \sigma_m), T\sigma_m) \\ &= \| T\sigma_m \|^2 + \| u - \sigma_m \|^2 + (\sigma_m, u - \sigma_m) + (u - \sigma_m, \sigma_m) \\ &= \| T\sigma_m \|^2 + \| u - \sigma_m \|^2 \\ &\quad + ((SP_m)^+ SP_m \sigma_m, u - \sigma_m) + (u - \sigma_m, (SP_m)^+ SP_m \sigma_m) \\ &= \| T\sigma_m \|^2 + \| u - \sigma_m \|^2 \\ &\quad + (\sigma_m, (SP_m)^+ S(u - \sigma_m)) + ((SP_m)^+ S(u - \sigma_m), \sigma_m) , \end{aligned}$$

i.e., we have shown

$$\| Tu \|^2 = \| T\sigma_m \|^2 + \| u - \sigma_m \|^2 .$$

This completes the proof of theorem 1.

Remark : Recall that the operator $(SP_m)^+$ is the restricted pseudo-inverse of S with respect to U_m [2, 3].

Schoenberg's polynomial approximate splines are obtained by choosing

$$\begin{aligned} &H_1 = L_2[a, b] , \quad H_2 = \mathbb{C}^k , \quad H = W_2^n[a, b] , \\ &T = d^n/dx^n , \\ &S = \delta_{x_1} \times \cdots \times \delta_{x_k} \quad (a \leqq x_1 < x_2 < \cdots < x_k \leqq b, k > n) , \\ &U_m = \pi_m \quad (m \geqq k) . \end{aligned}$$

Here $W_2^n[a, b]$ denotes the Sobolev space of order n.

Our next example is associated with periodic polynomial splines of degree $2n - 1$ which are obtained by specifying

$$\begin{aligned} &H_1 = L_2[0, 2\pi] , \quad H_2 = \mathbb{C}^{2k+1} , \quad H = W_{2\pi}^n , \\ &T = d^n/dx^n , \\ &S = \delta_{x_1} \times \cdots \times \delta_{x_{2k+1}} \\ &\qquad (0 \leqq x_1 < x_2 < \cdots < x_{2k+1} < 2\pi, k \geqq 0) . \end{aligned}$$

Here $W_{2\pi}^n$ denotes the periodic Sobolev space of order n :

$$W_{2\pi}^n = \{ f \in W_2^n[0, 2\pi] : D^j f(0) = D^j f(2\pi) \ (j = 0, \ldots, n-1) \} .$$

Let

$$U_m = \tau_m \quad (m \geqq k)$$

denote the class of trigonometric polynomials of degree m. It is a well known result that for any $(y_1, \ldots, y_{2k+1}) \in \mathbb{C}^{2k+1}$ there is a unique trigonometric polynomial $T_k \in \tau_k$ such that

$$T_k(x_j) = y_j \quad (j = 1, \ldots, 2k+1) .$$

Thus, theorem 1 is applicable and we obtain the following

COROLLARY 1.1 : *For any vector* $(y_1, \ldots, y_{2k+1})$ *there is a unique trigonometric polynomial* $T_m \in \tau_m$ *satisfying*

$$T_m(x_j) = y_j \quad (j = 1, \ldots, 2k+1) ,$$

$$\int_0^{2\pi} |D^n T_m(x)|^2 \, dx \leqq \int_0^{2\pi} |D^n T(x)|^2 \, dx$$

$$(T \in \tau_m, T(x_j) = y_j \quad (j = 1, \ldots, 2k+1)) .$$

2. THE LIMIT RELATION

Our next objective is to derive a limit characterization of the abstract interpolation spline $\sigma = S^+ y$ in terms of the approximate splines $\sigma_m = (SP_m)^+ y$. For this purpose we need the following extremal property of σ_m.

THEOREM 2 : *The approximate spline* σ_m *is the unique element in* U_m *satisfying*

$$S\sigma_m = y ,$$

$$\|\sigma - \sigma_m\| = \min \{ \|\sigma - u\| : u \in U_m, Su = y\} .$$

Proof: We proceed as in the proof of Theorem 1. Suppose that

$$u \in U_m , \qquad Su = S\sigma = y .$$

Then

$$\sigma - u \in N(S)$$

and taking into account

$$\sigma \in N(S)^\perp$$

we get

$$\begin{aligned} \|Tu\|^2 &= \|T\sigma\|^2 + \|T(u-\sigma)\|^2 \\ &\quad + (T\sigma, T(u-\sigma)) + (T(u-\sigma), T\sigma) \\ &= \|T\sigma\|^2 + \|u-\sigma\|^2 + (\sigma, u-\sigma) + (u-\sigma, \sigma) , \end{aligned}$$

i.e., we have

$$\|Tu\|^2 = \|T\sigma\|^2 + \|\sigma - u\|^2 \quad (u \in U_m, Su = y) .$$

Now an application of Theorem 1 completes the proof of Theorem 2.

For the sequel we will assume that the union of the sets

$$U_m , \qquad m \in \mathbb{N} ,$$

is a dense subspace of H. Then we have the limit relation

$$\lim_{m \to \infty} \|x - P_m x\| = 0 \quad (x \in H) . \tag{3}$$

THEOREM 3 : *Let* σ *be the abstract interpolating spline relative to* T, S, *and* y *and let* σ_m *be the* U_m*-approximate spline relative to* T, S, y. *Then the error estimate*

$$\|\sigma - \sigma_m\| \leqq (1 + \|(SP_1)^+ S\|) \|\sigma - P_m \sigma\| \tag{4}$$

is valid. In particular, we have

$$\lim_{m \to \infty} \|\sigma - \sigma_m\| = 0 . \tag{5}$$

Proof: Since U_m, $m \in \mathbb{N}$, is an increasing sequence of subspaces we have in view of (1)

$$v_m = P_m \sigma + (SP_1)^+ S(\sigma - P_m \sigma) \in U_m .$$

Moreover, we obtain from (2)

$$\begin{aligned} Sv_m &= SP_m \sigma + S(SP_1)^+ S(\sigma - P_m \sigma) \\ &= SP_m \sigma + SP_1(SP_1)^+ S(\sigma - P_m \sigma) \\ &= SP_m \sigma + S(\sigma - P_m \sigma) , \end{aligned}$$

i.e., we have

$$Sv_m = y , \qquad v_m \in U_m .$$

Now an application of theorem 2 yields

$$\begin{aligned} \|\sigma - \sigma_m\| &\leqq \|\sigma - v_m\| \\ &\leqq \|\sigma - P_m \sigma\| + \|(SP_1)^+ S(\sigma - P_m \sigma)\| , \end{aligned}$$

i.e., we have

$$\|\sigma - \sigma_m\| \leqq (1 + \|(SP_1)^+ S\|)\|\sigma - P_m \sigma\| .$$

Thus, Theorem 3 follows from the relation (3).

The relation (5) represents a limit characterization of the abstract interpolating spline σ by the approximate splines σ_m, $m \in \mathbb{N}$. For the special case of natural polynomial splines we obtain Schoenberg's limit characterization of the natural spline σ *via* the π_m-approximate splines σ_m, $m \in \mathbb{N}$ [8]. It should be noted that Theorem 3 yields an explicit error estimate. For further applications of pseudo-inverses to convergence of abstract splines we refer to [6].

We conclude with applying Theorem 3 to periodic polynomial splines.

COROLLARY 3.1 : *The sequence T_m, $m \in \mathbb{N}$, of trigonometric approximate splines converges in the Sobolev norm of $W^n_{2\pi}$ to the periodic polynomial spline σ of degree $2n - 1$.*

Proof: Since the trigonometric polynomials form a dense subspace of $W^n_{2\pi}$ Theorem 3 is applicable.

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