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**ASYMPTOTIC TIME-BEHAVIOR FOR  
WEIGHTED SCALAR CONSERVATION LAWS (\*)**

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*Abstract.* — For weighted scalar conservation laws introduced as model equations for gas flows in axisymmetric coordinates or in a nozzle, the asymptotic time-behavior of a entropy weak solution is obtained according to the behaviors of the weight and flux functions, thanks to the explicit formula previously derived in [10].

*Résumé.* — Considérant une loi de conservation scalaire avec poids qui modélise l'évolution d'un gaz en géométrie axisymétrique ou dans une tuyère, on utilise la formule explicite obtenue dans [10] pour préciser le comportement asymptotique de la solution faible entropique suivant le comportement de la fonction-flux et de la fonction-poids de l'équation.

**1. INTRODUCTION : WEIGHTED SCALAR CONSERVATION LAWS**

We are interested in *weighted scalar nonlinear hyperbolic conservation laws* in the half space

$$(1.1) \quad \frac{\partial}{\partial t} (r(x) u(x, t)) + \frac{\partial}{\partial x} (r(x) f(u(x, t))) = 0, \quad x > 0, \quad t > 0$$

with a  $\mathcal{C}^2$  convex flux-function  $f: \mathbb{R} \rightarrow \mathbb{R}$  and a  $\mathcal{C}^2$  positive weight-function  $r: ]0, \infty[ \rightarrow ]0, \infty[$ . Such equations are considered as model equations for gas dynamics in axisymmetric coordinates where the weight-function satisfies

$$(H.1) \quad r \in L^\infty(0, 1) \quad \text{and} \quad \frac{1}{r} \in L^\infty(1, \infty)$$

(for example take  $r(x) = x^\alpha$ ,  $\alpha = 0, 1, 2$ ), or as model equations for gas

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flows in a nozzle where there exists two positive constant  $r_-$  and  $r_+$  such that

$$(H.2) \quad 0 < r_- \leq r(\cdot) \leq r_+ < +\infty .$$

For equations of the form (1.1), an uniqueness and existence result was proved in Le Floch-Nedelec [10] thanks to an explicit formula generalizing the one of Lax [7]. Moreover, a suitable formulation of boundary condition (at  $x = 0$ ) was proposed. Refer also to previous works of Whitham [17] and Schonbek [15]. In this paper, we look for the asymptotic time-behavior of the solution of (1.1). We generalize well-known results of time-behavior for equations (1.1) without weight-function (that is  $r(\cdot) \equiv 1$ ). The rate of convergence is specified according to the properties of the weight-function and the flux-function. For classical results on time-behavior of conservation laws, we refer to Lax [7], Dafermos [2], Conway [1], Liu-Pierre [13]... We present our result of time-behavior in the following section 2. Before, we detail some important properties of the mixed problem associated to the weighted scalar conservation law (1.1). The proofs can be found in [8]-[11].

We look for weak solutions (that is in the sense of distributions) of (1.1) satisfying an initial condition

$$(1.2) \quad u(x, 0) = u_0(x) , \quad x > 0$$

with an initial data  $u_0 : \mathbb{R}_+ \rightarrow \mathbb{R}$ . And for the sake of uniqueness, one adds an entropy condition (Lax [7], Oleinik [14], Kruskov [6])

$$(1.3) \quad u(x - 0, t) \geq u(x + 0, t) , \quad x > 0, t > 0 .$$

We impose a zero boundary condition at  $x = 0$ , since we are only interested in this paper in the rate of convergence and not in the nonlinear aspect of the boundary condition. Concerning boundary conditions, see [10] and also Le Floch [8], [9], Dubois-Le Floch [3, 4].

Our assumptions concerning the initial data are as follows

$$(H.3) \quad r \cdot u_0 \in L^1(\mathbb{R}_+) , \quad r \cdot f(u_0) \in L^\infty(\mathbb{R}_+) .$$

And, the flux-function satisfies for simplifications

$$(H.4) \quad f'' > 0 , \quad f(0) = f'(0) = 0 .$$

(for example, take  $f(u) = \frac{u^2}{2}$ ). Then, the problem (1.1)-(1.3) admits one and only one solution  $u(\cdot, \cdot)$  such that  $r \cdot u$  belongs to  $L^\infty(\mathbb{R}_t^+ ; L^1(\mathbb{R}_x^+))$  and  $rf(u)$  to  $L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ . It satisfies also the following *stability properties* for almost every  $t \geq 0$ : the  $L^\infty$ -stability

$$(1.4) \quad \|r(\cdot) f(u(\cdot, t))\|_{L^\infty(\mathbb{R}_x^+)} \leq \|r(\cdot) f(u_0(\cdot))\|_{L^\infty(\mathbb{R}_x^+)}$$

and the  $L^1$ -semi-group property

$$(1.5) \quad \|(u(\cdot, t) - v(\cdot, t)) \cdot r(\cdot)\|_{L^1(\mathbb{R}_x^+)} \leq \|(u_0(\cdot) - v_0(\cdot)) \cdot r(\cdot)\|_{L^1(\mathbb{R}_x^+)}$$

for two solutions  $u$  and  $v$  corresponding to two initial data  $u_0$  and  $v_0$  respectively. The property (1.5) generalizes a previous result of Keyfitz [4].

Moreover, take  $a = f'$  and let  $f_+^{-1} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and  $f_-^{-1} : \mathbb{R}_- \rightarrow \mathbb{R}_-$  be the two inverse functions of the convex function  $f$ . For each  $(x, t) \in \mathbb{R}_+ \times \mathbb{R}_+$ , we can define — thanks to the hypotheses (H.1) or (H.2) (see [10]) — the function :

$$\mathbb{R}_+ \times \{-1, +1\} \ni (c, \varepsilon) \mapsto y(c, \varepsilon) \in \mathbb{R}^+$$

by the algebraic relation

$$(1.6a) \quad t = \int_{y(c, \varepsilon)}^x \frac{d\xi}{a\left(f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right)\right)}$$

and the function  $G : \mathbb{R}_+ \times \{-1, 1\} \ni (c, \varepsilon) \rightarrow G(c, \varepsilon)$  as follows :

$$(1.6b) \quad G(c, \varepsilon) = \int_0^{y(c, \varepsilon)} u_0(\xi) r(\xi) d\xi - c \cdot t + \int_{y(c, \varepsilon)}^x f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right) r(\xi) d\xi.$$

Then the value  $u(x, t)$  of the solution  $u$  of (1.1)-(1.3) is provided by the following *explicit formula*

$$(1.7) \quad u(x, t) = f_{\varepsilon(x, t)}^{-1}\left(\frac{c(x, t)}{r(x)}\right)$$

where  $(c(x, t), \varepsilon(x, t))$  minimizes the function  $G$ .

**2. ASYMPTOTIC TIME-BEHAVIOR**

Using the explicit formula (1.6)-(1.7), we are able as in Lax [7] to get an uniform decay in power of  $t$  for the solution  $u$  of problem (1.1)-(1.3), according to the behavior of the function  $f$ . Assuming that

$$(2.1) \quad k_- \cdot |v|^{p-2} \leq f''(v) \leq k_+ \cdot |v|^{p-2}, \quad \forall v \in \mathbb{R}; p \geq 2, 0 < k_- \leq k_+$$

we have :

**THEOREM** Under hypotheses (H 1) or (H 2) and (H 3)-(H 4), the solution  $u(.,.)$  of (1.1)-(1.3), when the time  $t$  tends to infinity, decreases with the following rate

$$(2.2) \quad |u(x, t)| \leq \left( \frac{kM}{r(x)} \right)^{\frac{1}{p}} \cdot \frac{1}{t^{1/p}} \quad t > 0, x > 0$$

where the constants  $k$  and  $M$  depend only on the flux-function and the initial data respectively

$$k = 2p \frac{k_+}{k_-^2}$$

and

$$M = \|r \cdot u_0\|_{L^1(\mathbb{R}_+)} \quad \blacksquare$$

In the case of flows in axisymmetric coordinates

$$r(x) = x^\alpha, \quad \alpha \geq 0$$

the estimation (2.2) becomes

$$(2.3) \quad |u(x, t)| \leq \frac{c}{x^{\alpha/p}} \cdot \frac{1}{t^{1/p}} \quad (c > 0)$$

Note that, when  $\alpha > 0$ , this inequality (2.3) provides also the behavior of  $u$  when  $x \rightarrow 0+$  and  $x \rightarrow +\infty$

*Proof of theorem* Multiplying (2.1) by  $v \in \mathbb{R}$  and integrating lead to

$$(2.4) \quad k_- \cdot \frac{|v|^p}{p} \leq a(v)v - f(v) \leq k_+ \cdot \frac{|v|^p}{p}$$

because of

$$\frac{d}{dv} (a(v)v - f(v)) = a'(v)v = f'' * (v)v$$

Moreover, by two successive integrations of (2.1), we have also

$$(2.5) \quad k_- \cdot \frac{|v|^p}{p(p-1)} \leq f(v) \leq k_+ \cdot \frac{|v|^p}{p(p-1)}$$

Then from (2.4)-(2.5) it results that

$$\frac{k_-}{k_+} (p-1) f(v) \leq a(v)v - f(v) \leq \frac{k_+}{k_-} (p-1) f(v)$$

or, for  $v > 0$ :

$$(2.6a) \quad \frac{k_-}{k_+} (p - 1) \frac{f(v)}{a(v)} \leq v - \frac{f(v)}{a(v)} \leq \frac{k_+}{k_-} (p - 1) \frac{f(v)}{a(v)}$$

and for  $v < 0$ :

$$(2.6b) \quad \frac{k_+}{k_-} (p - 1) \frac{f(v)}{a(v)} \leq v - \frac{f(v)}{a(v)} \leq \frac{k_-}{k_+} (p - 1) \frac{f(v)}{a(v)}.$$

Henceforth, the following minoration of the function  $G$  defined by (1.6) is an immediate consequence of (2.6) (used with the value  $v = f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right)$ )

$$\begin{aligned} G(c, \varepsilon) &= \int_0^{y(c, \varepsilon)} u_0 r \, d\xi + \int_{y(c, \varepsilon)}^x \left\{ f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right) - \frac{\frac{c}{r(\xi)}}{a f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right)} \right\} r(\xi) \, d\xi \\ &\geq -M + \int_{y(c, \varepsilon)}^x \frac{k_-}{k_+} (p - 1) \frac{\frac{c}{r(\xi)}}{a f_\varepsilon^{-1}\left(\frac{c}{r(\xi)}\right)} \cdot r(\xi) \, d\xi \end{aligned}$$

where  $M = \|r \cdot u_0\|_{L^1(\mathbb{R}^+)}$ . Thus, the inequality

$$(2.7) \quad G(c, \varepsilon) \geq -M + \frac{k_-}{k_+} (p - 1) c \cdot t$$

holds for all  $c = 0, \varepsilon = \pm 1, t > 0$ .

Furthermore, for each  $(x, t)$  in  $\mathbb{R}_+ \times \mathbb{R}_+$ , the minimum value of the function  $G$  — which is by notation attained at  $(c(x, t), \varepsilon(x, t))$  — may be majorized by the value of  $G$  at  $c = 0$ :

$$(2.8) \quad G(c(x, t), \varepsilon(x, t)) \leq \int_0^x r u_0 \, d\xi \leq M.$$

Now comparing (2.7) and (2.8), it results the estimation

$$(p - 1) t \frac{k_-}{k_+} c(x, t) \leq 2M$$

or

$$0 \leq c(x, t) \leq \frac{k_+}{k_-} \frac{2M}{(p - 1) t}.$$

Thus in virtue of (1.7), we have proved that  $rf(u)$  decreases uniformly in  $x \in \mathbb{R}_+$  when  $t \rightarrow \infty$ :

$$\sup_{x \in \mathbb{R}_+} |r(x) f(u(x, t))| \leq \frac{k_+}{k_-} \frac{2M}{(p-1)t}.$$

It remains to use again (2.5) :

$$k_- \cdot \frac{|u(x, t)|^p}{p(p-1)} \leq f(u(x, t)) \leq \frac{2Mk_+}{k_- \cdot r(x)} \cdot \frac{1}{t}$$

which gives (2.2). ■

When the solution  $u$  is bounded — say the weight-function  $r$  satisfies the properties (H.2) — it suffices to assume that the inequalities (2.1) hold in the neighborhood of  $v = 0$  with one  $p_0 \geq 2$  and also that  $f''$  is uniformly bounded :

$$\text{Cte} \leq f'' \leq \text{Cte}'.$$

Namely, thanks to the theorem applied with  $p = 2$ , we know that the solution  $u$  tends to zero uniformly in  $x \in \mathbb{R}_+$  when the time goes to infinity

$$\sup_{x \in \mathbb{R}_+} |u(x, t)| \leq \left( \frac{kM}{r_-} \right)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{t}} \quad t > 0.$$

So for  $t$  sufficiently large, say  $t > T_0$ ,  $u(x, t)$  is for each  $x$  in  $\mathbb{R}_+$  in the neighborhood where (2.1) holds. And the same proof as previously gives

$$|u(x, t)| \leq \left( \frac{kM}{r_-} \right)^{\frac{1}{p_0}} \cdot \frac{1}{t^{1/p_0}}, \quad \forall x \in \mathbb{R}_+, \forall t > T_0.$$

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